A Note on Classes of Life Distributions

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Abstract

Some classes of life distributions defined based on the failure rate are considered.These are the new better than used failure rate(NBUFR), the new better than average renewal failure rate(NBARFR) and generalized new better than average renewal failure rate(GNBARFR).Moment inequalities and moment generating functions behavior.closure of these classes under mixture and convolution are studied.In addition we studied the relationship between the scaled TTT-transform and NBARFR class.

Key Words: $NBUFR, NBARFR, GNBARFR$,Moment enequalitry,Moment generating function,Closure properties,TTT-transform.

1 Introduction

The classes of life distributions are one of the most important concepts in study reliability and queue.In performing reliability analysis ,there are many well-known classes of life distributions,such as IFR, $IFRA$, NBU, $NBUE$, , $DMRL$, $NBUFR$, and $NBARFR$ .For more detailed discussions on properties and for some possible applications, we refer to *Bryson* and *Siddiqui* [13], *Rolski* [17], *Barlow* and *Proschan* [9], *Klefsjo* [15] , [16],Ahmad[3] and Mugdadi and Ahmad [4],Abouammoh and Ahmed[1] , [2]and Alzaid et al [5].

In reliability theory , ageing life is usually characterized by a nonnegative random variable $X \geq 0$ with cumulative distribution function (cdf) F and survival function (sf) $\overline{F}$ =$1 - F$. For any random variable X, let

$$X_t = [X - t|X > t], \quad t \in \{x : F(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t. Its survival function is

$$\overline{F}_t(x) = \frac{\overline{F}(t+x)}{\overline{F}(t)}, \qquad \overline{F}(t) > 0,$$

where $\overline{F}(x)$ is the survival function of X.

Let X be a non-negative random variable representing equipment failure time with failure distribution $F(x) = P(X \leq x)$,survival function $\overline{F}(x) = 1 - F(x)$ and density function f (if it exists).The corresponding failure rate $r(t)$when the distribution is absolutely continuous ,is defined by

$$r(t) = \underset{\triangle t \longrightarrow o}{Lim} \frac{1}{t} P(X \leq t + \triangle t \mid X > t).$$

Precisely we have the following definitions:

Definition1.1

(i) The random variable X is said to have increasing failure rate (IFR) or increasing failure rate average$(IFRA)$if

$$r(t) \text{ or } t^{-1} \int_0^t r(x)dx \quad \text{is increasing.}$$

(ii) An absolutely continuous probability distribution Fon $[0, \infty)$ for which $\frac{F(x)}{x}$ has a limit as x$\rightarrow 0^+$ is said to be NBUFR class if there exists a version r of the failure rate function such that

$$r(0) \leq r(t) \tag{1.1}$$

where

$$r(0) = \underset{t \longrightarrow o^+}{Lim} t^{-1} \int_0^t r(x)dx \ .$$

The next definition is equivalent to the a bove definition due to *Abouammoh and Ahmed* [1].

Definition1.2

The distribution function F is said to be new better than used in failure rate (NBUFR) if

$$\overline{F}(x+t) \leq \overline{F}(x) \ e^{-r(0)t} \quad \text{for all } x, t \geq 0. \tag{1.2}$$

Definition1.3

The distribution function F is said to be new better (worse) than average renewal failure rate $NBARFR$ $(NWARFR)$ if

$$\int_t^\infty \overline{F}(u)du \leq (\geq) \mu \ e^{-r(0)t}.$$

Definition1.4

The distribution function F is said to be generalized new better (worse) than average renewal failure rate $GNBARFR$ ($GNWARFR$) if

$$\int_0^\infty \int_t^\infty \overline{F}(u)dudt \leq (\geq)\frac{\mu}{r(0)} \quad \text{for all } t \geq 0.$$

The composition of this work is as follows..In *Section* 2 we give the moment inequality of $NBUFR$ class.The upper bounds of the moment generating function is discussed in *Section* 3.Preservation results under some reliability operations such as convolution and mixtures are discussed in *Section* 4.Finally,*In section* 5 we give the characterization of NBARFR class in terms of scaled TTT-transforms.

2 Moment Inequalities of NBUFR(NWUFR)

In this sections we established some inequalities for the NBUFR(NWUFR) distributions.We use the following notion for r=0,1,2,....

$$\mu_{(r)} = E(X^r) = \int_0^\infty x^r dF(x)$$

or

$$\mu_{(r)} = r\int_0^\infty x^{r-1}\overline{F}(x)dx \quad \text{and} \quad \lambda_{(r)} = \frac{\mu_{(r)}}{\Gamma(r+1)}.$$

Theorem 2.1

If F is $NBUFR(NWUFR)$ then for some integers$r, s \geq 0$,

$$\lambda_{(r+s+2)} \leq (\geq)\lambda_{(1+r)}\left(\frac{1}{r(0)}\right)^{s+1} \tag{2.1}$$

Proof:

If F is $NBUFR(NWUFR)$ then

$$\overline{F}(x+t) \leq (\geq)\overline{F}(x)\ e^{-r(0)t} \tag{2.2}$$

Multiplying both sides of (2.2) by $x^r t^s$ and integrating,we get

$$\int_0^\infty \int_0^\infty x^r t^s \overline{F}(x+t)dxdt \leq (\geq)\int_0^\infty x^r\overline{F}(x)dx \int_0^\infty t^s e^{-r(0)t}dt \tag{2.3}$$

The left hand side of (2.2) is equal to

$$\begin{aligned}\int_0^\infty \int_0^v (v-w)^r w^s \overline{F}(v)dwdv &= \int_0^\infty \overline{F}(v)\ v^{r+s+1}dv \int_0^1 z^s(1-z)^r dz \\ &= \frac{\mu_{(r+s+2)}}{r+s+2}\beta(s+1, r+1) \\ &= \frac{\mu_{(r+s+2)}r!s!}{(r+s+2)(r+s+1)!} \qquad (2.4)\end{aligned}$$

Also,the right hand side of (2.2) is equal to

$$\int_0^\infty x^r \overline{F}(x)dx \int_0^\infty t^s e^{-r(0)t}dt = \frac{\mu_{(r+1)}}{r+1}\left(\frac{1}{r(0)}\right)^{s+1} \tag{2.5}$$

From (2.4)and (2.5),it follows that

$$\frac{\mu_{(r+s+2)}r!s!}{(r+s+2)(r+s+1)!} \leq (\geq)\frac{\mu_{(r+1)}}{r+1}\left(\frac{1}{r(0)}\right)^{s+1}.$$

Thus the proof of theorem is completed.

Remark 2.2: The above theorem 2.1 can be extended as follows.Let $x_1, x_2, ..., x_{k+1}$ be nonnegative,then F is $NBUFR(NWUFR)$ if and only if

$$\begin{aligned} \overline{F}(\sum_{i=1}^{k+1} x_i \quad &\leq \quad (\geq)\overline{F}(x_1)e^{-r(0)\sum_{i=2}^{k+1} x_i} \\ &\Longrightarrow \lambda_{(\sum r_i+k+1)} \leq (\geq)\lambda_{(r+1)}\left(\frac{1}{r(0)}\right)^{\sum_{i=2}^{k+1} x_i} \end{aligned}$$

Corollary 2.3

(a) If r=s=0,then $\mu_{(2)} \leq \frac{2\mu_{(1)}}{r(0)}$

(b) If r=0, then $\mu_{(s+2)} \leq \frac{(s+2)!}{(r(0))^{s+1}}$

(c) If s=0,then $\mu_{(r+2)} \leq \frac{(r+2)\mu_{(r+1)}}{r(0)}$

3 Existence of the moment generating functions of life distribution

In this section we show that the moment generating function(mgf) of X exists and is finite for the $NBUFR$ class.If $\mu_{(1)}$ exists.Actually,upper bounds of the moment generating function are given.Precisely,we have

Theorem 3.1

If F is $NBUFR$ and $E(X) = \mu < \infty$,then for all nonnegative $t \neq r(0)$,we have

$$\phi(t) \leq 1 + \frac{\mu^t}{1 - \frac{t}{r(0)}} \tag{3.1}$$

where $\phi(t) = E(e^{Xt})$.

Proof:

We notice that

$$\phi(t) = 1 + t\int_0^\infty e^{tx}\overline{F}(x)dx \tag{3.2}$$

for

$$1+t\int_0^\infty e^{tx}\overline{F}(x)dx = 1+t\ E\int_0^X e^{tx}dx$$
$$= 1+E(e^{tx}-1)=\phi(t).$$

Since F is $NBUFR$,then

$$\int_0^\infty\int_0^\infty e^{tx}\overline{F}(x+y)dxdy \leq (\geq)\int_0^\infty e^{tx}\overline{F}(x)dx\int_0^\infty e^{-r(0)y}dy \tag{3.3}$$

the lift hand side of (3.3) is equal to

$$\int_0^\infty\int_0^u e^{t(u-v)}\overline{F}(u)dvdu = \frac{1}{t}\left[\int_0^\infty e^{tu}\overline{F}(u)du-\mu_{(1)}\right]$$
$$= -\frac{\mu_{(1)}}{t} \tag{3.4}$$

Also the right hand side of (3.3) is equal to

$$\int_0^\infty e^{tx}\overline{F}(x)dx\int_0^\infty e^{-r(0)y}dy = \frac{\phi(t)-1}{t}.\frac{1}{r(0)}$$
$$= \frac{\phi(t)-1}{tr(0)} \tag{3.5}$$

From (3.4) and (3.5) the proof is complete.

4 Preservation Results:

In this section we discuss the closure properties of the $NBARFR$ and $GNWARFR$ under some reliability operations.

Let $\{F_\alpha\}$ be a family of life distributions where α is a random variable with cumulative distribution function $G(\alpha)$,then the mixture F of F_α according to G is defined by

$$F(x)=\int_0^\infty F_\alpha(x)dG(\alpha) \tag{4.1}$$

The following theorem demonestrates that the $NBARFR$ class is closed under mixtures if every F_α in equation (4.1) is $NBARFR$.

Theorem 4.1

Let $\overline{F}$ be an arbitrary mixture of $\overline{F}_\alpha$ where each $\overline{F}_\alpha$ is $NBARFR$. for all α,then $\overline{F}$ is $NBARFR$. if

$$\begin{array}{rrcll} (i) & r_\alpha(0) & = & r(0) & \text{for all } \alpha \quad \text{or} \\ (ii) & \mu_\alpha & = & \mu & \text{for all } \alpha \end{array}$$

Proof:

Observe that

$$\overline{F}(x) = \int_{all \ \ \alpha} \overline{F_\alpha}(x) dG(\alpha)$$

It follows that

$$\begin{aligned} \int_t^\infty \int_{all \ \ \alpha} \overline{F_\alpha}(x) dG(\alpha) dx &= \int_{all \ \ \alpha} \left[\int_t^\infty \overline{F_\alpha}(x) dx \right] dG(\alpha) \\ &\leq \int_{all \ \ \alpha} \mu_\alpha e^{-r_\alpha(0)t} dG(\alpha), \qquad \text{using definition(4.1)} \end{aligned}$$

(i) Suppose $r_\alpha(0) = r(0)$,therfore

$$\begin{aligned} \int_t^\infty \overline{F}(x) dx &\leq e^{-r(0)t} \int_{all \ \ \alpha} \mu_\alpha dG(\alpha) \\ &= \mu e^{-r(0)t}. \end{aligned} \tag{4.2}$$

(ii) suppose $\mu_\alpha = \mu$ for all α then

$$\begin{aligned} \int_t^\infty \overline{F}(x) dx &\leq \mu \int_{all \ \ \alpha} e^{-r_\alpha(0)t} dG(\alpha) \\ &\leq \mu e^{-r(0)t} \end{aligned} \tag{4.3}$$

This completes the proof.

Next,in the following theorem we prove that the $GNWARFR$ is closed under convolution.

Theorem 4.2

Let X_1 and X_2 be two random variables having the $GNWARFR$,with mean μ_i and initial failure rate $r_i(0)$, for i=1,2. Let $r(0)$ =max$[r_1(0), r_2(0)]$, then $X_1 + X_2$ has the $GNWARFR$ property with initial failure rate $r(0)$.

Proof:

To prove theorem 4.2 ,we need the following Lemma.

Lemma 4.3

$$E(X^2) = 2\int_0^\infty \int_x^\infty \overline{F}(t)dtdx$$

Proof:

$$E(X^2) = \int_0^\infty \left(2\int_0^x udu\right) f(x)dx = 2\int_0^\infty \int_x^\infty \overline{F}(t)dtdx$$

Proof of Theorem 4.2.
If X_1 and X_2 are $GNWARFR$ then

$$E(X_1^2) = 2\int_0^\infty \int_t^\infty \overline{F}_1(u)dudt \geq 2\frac{\mu_1}{r_1(0)} \tag{4.4}$$

and

$$E(X_2^2) = 2\int_0^\infty \int_t^\infty \overline{F}_2(u)dudt \geq 2\frac{\mu_2}{r_2(0)} \tag{4.5}$$

Now $X = X_1 + X_2$ implies that,

$$\begin{aligned} E(X^2) &= E(X_1^2) + E(X_2^2) + 2E(X_1X_2) \geq E(X_1^2) + E(X_2^2) \\ &\geq 2\frac{\mu_1}{r_1(0)} + 2\frac{\mu_2}{r_2(0)} \geq 2\frac{\mu_1}{r(0)} + 2\frac{\mu_2}{r(0)} \\ &= \frac{2}{r(0)}\left(\mu_1 + \mu_2\right) \end{aligned}$$

Hence X has the $GNWARFR$ property with initial failure rate $r(0)$.

5 Characterization of NBARFR in terms of scaled TTT-transforms.

We shall present in this section the relationship between the scaled TTT-transform and the NBARFR class. This problem has been considered by different authors for different classes of life distribution. *Barlow* [6], *Bickel* and *Doksum* [12] and *Klefsjo* [15]considered this problem for IFR class. *Barlow and Campo* [8] *Bergman* [11]and

Klefsjo [16] for $HNBUE$ class, *Holander* and *Proschan* [14]for DMRL class,*Basu* and *Ibrahimi* [10] and *Singh* and *Kochar* [18] have considered the problem for $HNBUE$. Alzaid at al [5] for NBAFR class. Here we present the characterizationof aging via the scaled total time transform (TTT) concept.The TTT-transform is denoted by $H_{F^{-1}(t)}$ and the scaled TTT-transform by $\phi_F(t)$.These transforms have been introduced by *Barlow and Campo*[8] and *Barlow* [7].

Next we give the following definitions.

Definition 5.1

Let F be a continuous life distribution(i.e.,a distribution function for which $F(t) = 0$,for $t \leq 0$),with survival function $\overline{F}$ and finite mean μ_F.The TTT-transform $H_{F^{-1}(t)}$ of F is defined by

$$H_{F^{-1}(t)} = \int_0^{F^{-1}(t)} \overline{F}(u)du \ ,0 \leq t \leq 1, \tag{5.1}$$

where $F^{-1}(t) = \inf\{x : F(x) \geq t\}$ is the inverse function of F.

Definition 5.2

Let F be a continuous life distribution with finite mean $\mu_{_F} = \int_0^\infty \overline{F}(u)du$.The scxaled TTT-transform $\phi_F(t)$

of F is defined by

$$\phi_F(t) = \frac{1}{\mu} \int_0^{F^{-1}(t)} \overline{F}(x)dx, \ 0 \leq t \leq 1 \tag{5.2}$$

or

$$\phi_F(t) = \frac{H_{F^{-1}(t)}}{H_{F^{-1}(1)}}, \ 0 \leq t \leq 1$$

where by equation(5.1) we have

$$H_{F^{-1}(1)} = \mu_{_F}.$$

Remark:

If $F = 1 - e^{-\lambda t}$, for $t \geq 0, \lambda > 0$,then the scaled TTT-transformt is given by $\phi(t) = t, \ 0 \leq t \leq 1$.

Now ,the empirical TTT-transformt is obtained as follows: Let $t_{(1)} \leq t_{(2)} \leq ... \leq t_{(n)}$ be an ordered sample from a life distribution F and $t_{(0)} = 0$.Further,let

$$Z_j = (n - j + 1)(t_{(j)} - t_{(j-1)}) \ \ , j = 1, 2,...$$

denote the scaled spacing ,then $D_j = \sum_{k=1}^{j} Z_k$, for $j = 1, 2$, denotes the TTT-transform at $t_{(j)}$where $D_0 = 0$. Not that

$U_j = \frac{D_j}{D_n}$ is an estimate of the scaled TTT-transform.The TTT-plot is usually obtained by plotting$(U_j\ ,\frac{j}{n})$ and joining these points by straight lines.Since the TTT-plot U_j converges to the scaled TTT-transform $\phi(t)$ as $t \to \infty$ and $\frac{j}{n} \longrightarrow t$,then the TTT-plot based on an ordered sample $t_{(1)} \leq t_{(2)} \leq ... \leq t_{(n)}$ behaves as $\phi(t)$ does.

Now we give the following characterization of $NBARFR$ class.

Theorem5.2

Let F be a life distribution and $\phi_F(t)$ be the corresponding scaled TTT-transform.Let F have a finite mean $\mu = \int_0^\infty \overline{F}(u)du$ then F is $NBARFR$ if

$$Log(1 - \phi(t)) \leq -r(0)F^{-1}(t)$$

Proof:

The life distribution F is $NBARFR$ if

$$\int_x^\infty \overline{F}(u)du \leq \mu\ e^{-r(0)x}.$$

this implies that

$$\begin{aligned} \int_0^\infty \overline{F}(u)du - \int_0^x \overline{F}(u)du \quad &\leq \quad \mu\ e^{-r(0)x} \\ &\Longrightarrow \quad \mu - \int_0^x \overline{F}(u)du \leq \mu\ e^{-r(0)x} \end{aligned}$$

this can be written by substituting with $x = F^{-1}(t)$,as

$$\begin{aligned} \mu - \int_0^{F^{-1}(t)} \overline{F}(u)du \quad &\leq \quad \mu\ e^{-r(0)F^{-1}(t)} \\ &\Longrightarrow \quad \mu - \mu\phi_F(t) \leq \mu\ e^{-r(0)F^{-1}(t)} \\ &\Longrightarrow \quad 1 - \phi_F(t) \leq \ e^{-r(0)F^{-1}(t)} \qquad (5.3) \end{aligned}$$

taking logarithm of both sides of (5.3) we get

$$Log(1 - \phi_F(t)) \leq -r(0)F^{-1}(t)$$

hence F is strictly increasing and NBARFR if

$$Log(1 - \phi_F(t)) \leq -r(0)F^{-1}(t).$$

This completes the proof.

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