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A Note on Classes of Lifetime Distributions

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Abstract: Some classes of lifetime distributions based on the failure rate are considered, namely *new better than used failure rate* (NBUFR), *new better than average renewal failure rate* (NBARFR) and *generalized new better than average renewal failure rate* (GNBARFR). Moment inequalities and moment generating functions of these classes under mixture and convolution are studied. In addition, the relationship between the scaled TTT-transform and the NBARFR class is investigated.

Key Words: Moment inequality, moment generating function, closure properties, TTT-transform.

1 Introduction

The nonparametric classes of lifetime distributions are very important concepts to study the reliability of systems. In reliability analysis a large number of well-known classes are introduced, such as IFR, IFRA, NBU, NBUE, DMRL, NBUFR, and NBARFR. For detailed discussions of properties and applications, we refer to Bryson and Siddiqui [13], Rolski [17], Barlow and Proschan [9], Klefsjo [15, 16],Ahmad [3] and Mugdadi and Ahmad [4], Abouammoh and Ahmed [1, 2] and Alzaid et al. [5].

Any random lifetime may be modelled by a nonnegative random variable $X \geq 0$ with cumulative distribution function F and survival function $\overline{F} = 1 - F$. For any random variable X, let

$$X_t = [X - t | X > t] \quad \text{for } t \in \{x : F(x) < 1\} \tag{1}$$

denote the conditional random variable X on condition $X > t$. When X is the lifetime of a device, X_t represents the residual lifetime of the device at time t, given that the device has survived up to time t. Its survival function is

$$\overline{F_t}(x) = \frac{\overline{F}(t+x)}{\overline{F}(t)} \quad \text{for } \overline{F}(t) > 0, \tag{2}$$

where $\overline{F}(x)$ is the survival function of X.

Let X be a non-negative random variable representing the lifetime of an equipment with absolutely continuous distribution function $F(x) = P(X \leq x)$, survival function $\overline{F}(x) = 1 - F(x)$ and density function $f(x) = \frac{dF(x)}{dx}$. The corresponding failure rate $r(t)$ is defined by

$$r(t) = \frac{f(t)}{\overline{F}(t)} \tag{3}$$

Then the following nonparametric classes of lifetime distributions have been introduced for $t \geq 0$:

increasing failure rate (IFR)	$\Leftrightarrow$	$r(t)$ is increasing	(4)
increasing failure rate average (IFRA)	$\Leftrightarrow$	$\frac{1}{t}\int_0^t r(x)dx$ is increasing	(5)
new better than used in failure rate (NBUFR)	$\Leftrightarrow$	$\begin{cases} \lim_{x\to 0^+} \frac{F(x)}{x} \text{ exists} \\ \lim_{t\to 0^+} t^{-1}\int_0^t r(x)dx \le r(t) \end{cases}$	(6)
new better (worse) than average renewal failure rate NBARFR (NWARFR)	$\Leftrightarrow$	$\int_t^\infty \overline{F}(u)du \le (\ge)\mu\ e^{-r(0)t}$	(7)
generalized new better (worse) than average renewal failure rate GNBARFR (GNWARFR)	$\Leftrightarrow$	$\int_0^\infty\int_t^\infty \overline{F}(u)dudt \le (\ge)\frac{\mu}{r(0)}$	(8)

Abouammoh and Ahmed [1] proved the following characterization of the NBUFR property:

Theorem 1:
A distribution function F has the new better than used in failure rate (NBUFR) property, if

$$\overline{F}(x+t) \le \overline{F}(x)\ e^{-r(0)t} \quad \text{for } x,t \ge 0 \tag{9}$$

This paper is built up as follows:

- Section 2 introduces moment inequalities for the NBUFR class.
- The upper bounds of the moment generating function are given in the Section 3.
- Preservation results under some reliability operations such as convolution and mixtures are discussed in the Section 4.
- Finally, in the Section 5, a characterization of the NBARFR class is given in terms of the scaled TTT-transforms.

2 Moment Inequalities of NBUFR (NWUFR)

In this section we established some inequalities for the NBUFR(NWUFR) class. The following notion is used:

$$\mu_{(r)} = E[X^r] = \int_0^\infty x^r dF(x) \quad \text{for } r = 0, 1, 2, \dots \tag{10}$$

or

$$\mu_{(r)} = r\int_0^\infty x^{r-1}\overline{F}(x)dx \quad \text{and} \quad \lambda_{(r)} = \frac{\mu_{(r)}}{\Gamma(r+1)}. \tag{11}$$

Theorem 2:
If F is NBUFR (NWUFR) then for some integers $r, s \geq 0$ the inequality

$$\lambda_{(r+s+2)} \leq (\geq)\lambda_{(1+r)}\left(\frac{1}{r(0)}\right)^{s+1} \tag{12}$$

holds.

Proof:
If F is NBUFR (NWUFR) then

$$\overline{F}(x+t) \leq (\geq)\overline{F}(x)\ e^{-r(0)t} \tag{13}$$

Multiplying both sides of (13) by $x^r t^s$ and integrating,we get

$$\int_0^\infty\int_0^\infty x^r t^s \overline{F}(x+t)dxdt \leq (\geq)\int_0^\infty x^r\overline{F}(x)dx\int_0^\infty t^s e^{-r(0)t}dt \tag{14}$$

The left hand side of (13) is equal to

$$\begin{aligned}\int_0^\infty\int_0^v (v-w)^r w^s\overline{F}(v)dwdv &= \int_0^\infty \overline{F}(v)\ v^{r+s+1}dv\int_0^1 z^s(1-z)^r dz \\ &= \frac{\mu_{(r+s+2)}}{r+s+2}\beta(s+1,r+1) \\ &= \frac{\mu_{(r+s+2)}r!s!}{(r+s+2)(r+s+1)!} \end{aligned} \tag{15}$$

The right hand side of (13) is equal to

$$\int_0^\infty x^r\overline{F}(x)dx\int_0^\infty t^s e^{-r(0)t}dt = \frac{\mu_{(r+1)}}{r+1}\left(\frac{1}{r(0)}\right)^{s+1} \tag{16}$$

From (15) and (16) we obtain:

$$\frac{\mu_{(r+s+2)}r!s!}{(r+s+2)(r+s+1)!} \leq (\geq)\frac{\mu_{(r+1)}}{r+1}\left(\frac{1}{r(0)}\right)^{s+1}. \tag{17}$$

which completes the proof of Theorem 2.

From Theorem 2 the following corollary is obtained:

Corollary:
(a) If $r = s = 0$, then $\mu_{(2)} \leq \frac{2\mu_{(1)}}{r(0)}$

(b) If $r = 0$, then $\mu_{(s+2)} \leq \frac{(s+2)!}{(r(0))^{s+1}}$

(c) If $s = 0$, then $\mu_{(r+2)} \leq \frac{(r+2)\mu_{(r+1)}}{r(0)}$

3 Moment Generating Functions of the NBUFR Class

In this section we shall show for the NBUFR class that the moment generating function of X exists and is finite, if $\mu_{(1)}$ exists. Actually, upper bounds of the moment generating function are given. Precisely, we have

Theorem 3:
If F is NBUFR and $E[X] = \mu < \infty$, then for all nonnegative $t \neq r(0)$, we have

$$\phi(t) \leq 1 + \frac{\mu t}{1 - \frac{t}{r(0)}} \tag{18}$$

where $\phi(t) = E\left[e^{Xt}\right]$.

Proof:
We notice that

$$\phi(t) = 1 + t \int_0^\infty e^{tx} \overline{F}(x) dx \tag{19}$$

for

$$\begin{aligned} 1 + t \int_0^\infty e^{tx} \overline{F}(x) dx &= 1 + t\ E\left[\int_0^X e^{tx} dx\right] \\ &= 1 + E\left[e^{tx} - 1\right] = \phi(t) \end{aligned} \tag{20}$$

The distribution F is NBUFR, thus:

$$\int_0^\infty \int_0^\infty e^{tx} \overline{F}(x+y) dx dy \leq (\geq) \int_0^\infty e^{tx} \overline{F}(x) dx \int_0^\infty e^{-r(0)y} dy \tag{21}$$

The left hand side of (21) is equal to

$$\begin{aligned} \int_0^\infty \int_0^u e^{t(u-v)} \overline{F}(u) dv du &= \frac{1}{t}\left[\int_0^\infty e^{tu} \overline{F}(u) du - \mu\right] \\ &= \frac{1}{t^2}\left(\phi(t) - 1\right) - \frac{\mu}{t} \end{aligned} \tag{22}$$

The right hand side of (21) is equal to

$$\begin{aligned} \int_0^\infty e^{tx} \overline{F}(x) dx \int_0^\infty e^{-r(0)y} dy &= \frac{\phi(t) - 1}{t} . \frac{1}{r(0)} \\ &= \frac{\phi(t) - 1}{t r(0)} \end{aligned} \tag{23}$$

With (22) and (23) the proof is complete.

4 Preservation Results

In this section, we discuss closure properties of the NBARFR and the GNWARFR classes under some reliability operations.

Let $\{F_\alpha\}$ be a family of lifetime distributions where α is a random variable with cumulative distribution function $G(\alpha)$, then the mixture F of F_α according to G is defined by:

$$F(x) = \int_0^\infty F_\alpha(x)dG(\alpha) \tag{24}$$

The following theorem demonstrates that the NBARFR class is closed under mixtures.

Theorem 4:
Let $\overline{F}$ be an arbitrary mixture of $\overline{F}_\alpha$, where each $\overline{F}_\alpha$ is NBARFR for any α. Then $\overline{F}$ is NBARFR, if

$$\begin{array}{llll} \text{(i)} & r_\alpha(0) = r(0) & \text{for all } \alpha \\ \text{or} & & \\ \text{(ii)} & \mu_\alpha = \mu & \text{for all } \alpha \end{array} \tag{25}$$

Proof:
Observe that

$$\overline{F}(x) = \int_{\text{all } \alpha} \overline{F}_\alpha(x)dG(\alpha) \tag{26}$$

implying

$$\begin{aligned} \int_t^\infty \int_{\text{all } \alpha} \overline{F}_\alpha(x)dG(\alpha)dx &= \int_{\text{all } \alpha} \left(\int_t^\infty \overline{F}_\alpha(x)dx \right) dG(\alpha) \\ &\leq \int_\alpha \mu_\alpha e^{-r_\alpha(0)t} dG(\alpha) \qquad \text{by (24)} \end{aligned} \tag{27}$$

(a) Suppose $r_\alpha(0) = r(0)$, thus:

$$\begin{aligned} \int_t^\infty \overline{F}(x)dx &\leq e^{-r(0)t} \int_{all \;\; \alpha} \mu_\alpha dG(\alpha) \\ &= \mu e^{-r(0)t} \end{aligned} \tag{28}$$

(b) Suppose $\mu_\alpha = \mu$ for all α then:

$$\begin{aligned}\int_t^\infty \overline{F}(x)dx &\leq \mu \int_{all\ \ \alpha} e^{-r_\alpha(0)t} dG(\alpha) \\ &\leq \mu e^{-r(0)t} \end{aligned} \tag{29}$$

which completes the proof.

Next, we prove that the GNWARFR class is closed under convolution.

Theorem 5:
Let X_1 and X_2 be two random variables having the GNWARFR property with mean μ_i and initial failure rate $r_i(0)$, for $i = 1, 2$. Let

$$r(0) = \max\left(r_1(0), r_2(0)\right) \tag{30}$$

then $X_1 + X_2$ has the GNWARFR property with initial failure rate $r(0)$.

Proof:
To prove Theorem 5, we need the following representation of the second moment:

$$E\left[X^2\right] = \int_0^\infty \left(2\int_0^x udu\right) f(x)dx = 2\int_0^\infty \int_x^\infty \overline{F}(t)dtdx \tag{31}$$

If X_1 and X_2 are GNWARFR then

$$E\left[X_1^2\right] = 2\int_0^\infty \int_t^\infty \overline{F}_1(u)dudt \geq 2\frac{\mu_1}{r_1(0)} \tag{32}$$

and

$$E\left[X_2^2\right] = 2\int_0^\infty \int_t^\infty \overline{F}_2(u)dudt \geq 2\frac{\mu_2}{r_2(0)} \tag{33}$$

Now $X = X_1 + X_2$ implies that:

$$\begin{aligned} E\left[X^2\right] &= E\left[X_1^2\right] + E\left[X_2^2\right] + 2E\left[X_1X_2\right] \geq E\left[X_1^2\right] + E\left[X_2^2\right] \\ &\geq 2\frac{\mu_1}{r_1(0)} + 2\frac{\mu_2}{r_2(0)} \geq 2\frac{\mu_1}{r(0)} + 2\frac{\mu_2}{r(0)} \\ &= \frac{2}{r(0)}\left(\mu_1 + \mu_2\right) \end{aligned} \tag{34}$$

Hence X has the GNWARFR property with initial failure rate $r(0)$.

5 A Characterization of the NBARFR Property

In this section the relationship between the scaled TTT-transform and the NBARFR class is presented. The related problem has been considered by different authors for different classes of distribution functions. Barlow [6], Bickel and Doksum [12] and Klefsjo [15] considered this problem for the IFR class. Barlow and Campo [8], Bergman [11] and Klefsjo [16] for the HNBUE class, Holander and Proschan [14] for DMRL class, Basu and Ebrahimi [10] and Singh and Kochar [18] have considered the problem for the HNBUE class. Alzaid at al [5] for the NBAFR class.

Here we present the characterization of aging by means of the scaled total time transform (TTT) concept. The TTT-transform is denoted by $H_{F^{-1}}(t)$ and the scaled TTT-transform by $\phi_F(t)$. These transforms have been introduced by Barlow and Campo [8] and Barlow [7].

Let F be a absolutely continuous lifetime distribution function with $F(0) = 0$, survival function $\overline{F}$ and finite mean μ_F given by:

$$\mu_F = \int_0^\infty \overline{F}(u)du \tag{35}$$

The TTT-transform $H_{F^{-1}}(t)$ of F is defined by

$$H_{F^{-1}}(t) = \int_0^{F^{-1}(t)} \overline{F}(u)du \qquad \text{for } 0 \leq t \leq 1 \tag{36}$$

where $F^{-1}(t) = \inf\{x \,|\, F(x) \geq t\}$ is the inverse function of F.

The scaled TTT-transform $\phi_F(t)$ of F is defined by:

$$\phi_F(t) = \frac{1}{\mu} \int_0^{F^{-1}(t)} \overline{F}(x)dx \qquad \text{for } 0 \leq t \leq 1 \tag{37}$$

Definition (37) is equivalent to the following:

$$\phi_F(t) = \frac{H_{F^{-1}}(t)}{H_{F^{-1}}(1)} \qquad \text{for } 0 \leq t \leq 1 \tag{38}$$

where by (36) we have

$$H_{F^{-1}}(1) = \mu_F \tag{39}$$

Remarks:

1. If $F = 1 - e^{-\lambda t}$ for $t \geq 0$ and $\lambda > 0$, then the scaled TTT-transform is given by $\phi(t) = t$, $0 \leq t \leq 1$.

2. The empirical TTT-transform is obtained as follows:
Let $t_{(1)} \leq t_{(2)} \leq ... \leq t_{(n)}$ be an ordered sample from the lifetime distribution F and $t_{(0)} = 0$. Further, let

$$Z_j = (n-j+1)(t_{(j)} - t_{(j-1)}) \quad \text{for } j = 1, 2, \dots \tag{40}$$

denote the scaled spacing, then

$$D_j = \sum_{k=1}^{j} Z_k \quad \text{for } j = 1, 2, \dots \tag{41}$$

defines the empirical TTT-transform at $t_{(j)}$ with $D_0 = 0$.

3. The ratio $U_j = \frac{D_j}{D_n}$ is an estimator of the scaled TTT-transform.

4. The TTT-plot is usually obtained by plotting $(U_j, \frac{j}{n})$ and joining these points by straight lines. The TTT-plot U_j converges to the scaled TTT-transform $\phi(t)$ as $t \to \infty$ and $\frac{j}{n} \longrightarrow t$ and, hence, the TTT-plot based on an ordered sample $t_{(1)} \leq t_{(2)} \leq \ldots \leq t_{(n)}$ behaves as $\phi(t)$ does.

Next, we give a characterization of the NBARFR class:

Theorem 6:
Let F be a lifetime distribution function with and $\phi_F(t)$ be the corresponding scaled TTT-transform. Then F is NBARFR, if the following holds:

$$\log(1 - \phi(t)) \leq -r(0)F^{-1}(t) \tag{42}$$

Proof:
A lifetime distribution is NBARFR, if its distribution function meets the condition:

$$\int_x^\infty \overline{F}(u)du \leq \mu\, e^{-r(0)x} \tag{43}$$

The NBARFR property implies that

$$\int_0^\infty \overline{F}(u)du - \int_0^x \overline{F}(u)du \leq \mu\ e^{-r(0)x} \quad \Longrightarrow \quad \mu - \int_0^x \overline{F}(u)du \leq \mu\ e^{-r(0)x} \tag{44}$$

with (44) and setting $x = F^{-1}(t)$ the right hand side of (44) implies:

$$\begin{aligned} \mu - \int_0^{F^{-1}(t)} \overline{F}(u)du \quad &\leq \quad \mu\ e^{-r(0)F^{-1}(t)} \\ &\Longrightarrow \quad \mu - \mu\phi_F(t) \leq \mu\ e^{-r(0)F^{-1}(t)} \\ &\Longrightarrow \quad 1 - \phi_F(t) \leq\ e^{-r(0)F^{-1}(t)} \end{aligned} \tag{45}$$

Taking logarithm of both sides of (45) we get

$$\log(1-\phi_F(t)) \le -r(0)F^{-1}(t) \quad (46)$$

Hence F is strictly increasing and NBARFR, if

$$\log(1-\phi_F(t)) \le -r(0)F^{-1}(t). \quad (47)$$

This completes the proof.

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