MA20033 - Question Sheet Three

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Attempt questions 1-2; question 3 is for tutorial discussion. If you also wish to attempt this, I'm happy to mark it. Hand in by 5.00pm Friday 5 November 2004 either to me, or the envelope on my door, 1W4.8.

1. If $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ random quantities, then an unbiased estimator of σ^2 is

$$S^2 \;=\; \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

   This estimator has variance $2\sigma^4/(n-1)$.

   (a) Write down the estimator's MSE.

   (b) Consider a second estimator constructed as αS^2 where α is any positive constant.

      i. Find the expectation and variance of αS^2, and hence its MSE.

      ii. Find the value of α which minimises this mean square error.

      iii. What is the efficiency of S^2 relative to the smallest MSE αS^2?

2. Let $X_1, \ldots, X_n$ be iid uniform random quantities on the interval $(\theta, \theta + 1)$.

   (a) Show that $T_1 = \overline{X} - \frac{1}{2}$ is an unbiased estimator of θ and find its MSE.

   (b) Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$.

      i. Show that the distribution of $X_{(1)}$ is

$$f_{X_{(1)}}(x|\theta) \;=\; \begin{cases} n(\theta + 1 - x)^{n-1} & \theta \leq x \leq \theta + 1, \\ 0 & \text{otherwise.} \end{cases}$$

      [Hint: Note that $P(X_{(1)} \geq x|\theta) = P(X_1 \geq x, \ldots, X_n \geq x|\theta)$]

      ii. Show that $T_2 = X_{(1)} - \frac{1}{n+1}$ is an unbiased estimator of θ and find its MSE.

      iii. What is the efficiency of T_1 relative to T_2?

3. The independent observations $x_1, \ldots, x_{10}$ are assumed to come from a $N(\mu, \sigma^2)$ distribution, and $x_{11}, \ldots, x_{15}$ from a $N(2\mu, \sigma^2/2)$ distribution, where σ^2 is assumed known.

   (a) Write down the joint probability density function of $X_1, \ldots, X_{10}$, and the joint probability density function of $X_{11}, \ldots, X_{15}$. Hence write down the likelihood function for μ based on all 15 observations.

   (b) Find the maximum likelihood estimator of μ and determine its bias and its mean square error.