MA20033 - Solution Sheet Three

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1. **If $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ random quantities, then an unbiased estimator of σ^2 is**

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

This estimator has variance $2\sigma^4/(n-1)$.

(a) **Write down the estimator's MSE.**

As S^2 is an unbiased estimator for σ^2 then

$$MSE(S^2) = Var(S^2|\mu, \sigma^2) = \frac{2\sigma^4}{(n-1)}.$$

(b) **Consider a second estimator constructed as αS^2 where α is any positive constant.**

i. **Find the expectation and variance of αS^2, and hence its MSE.**

$$\begin{aligned} E(\alpha S^2|\mu, \sigma^2) &= \alpha E(S^2|\mu, \sigma^2) = \alpha\sigma^2, \\ Var(\alpha S^2|\mu, \sigma^2) &= \alpha^2 Var(S^2|\mu, \sigma^2) = \frac{2\alpha^2\sigma^4}{(n-1)}. \end{aligned}$$

Thus,

$$b(\alpha S^2) = E(\alpha S^2|\mu, \sigma^2) - \sigma^2 = (\alpha - 1)\sigma^2,$$

and

$$\begin{aligned} MSE(\alpha S^2) &= Var(\alpha S^2|\mu, \sigma^2) + b^2(\alpha S^2) \\ &= \frac{2\alpha^2\sigma^4}{(n-1)} + (\alpha-1)^2\sigma^4. \end{aligned}$$

ii. **Find the value of α which minimises this mean square error.**

To find the value of α which minimises this mean square error, we first differentiate $MSE(\alpha S^2)$ with respect to α.

$$\frac{dMSE(\alpha S^2)}{d\alpha} = 2\sigma^4\left\{\frac{2\alpha}{n-1} + (\alpha - 1)\right\}$$

Solving $\frac{dMSE(\alpha S^2)}{d\alpha} = 0$ gives $\alpha = \frac{n-1}{n+1}$ which is a minimum since

$$\frac{d^2 MSE(\alpha S^2)}{d\alpha^2} \;=\; 2\sigma^4\left(\frac{2}{n-1}+1\right) \;>\; 0 \text{ for all } \alpha.$$

The optimal (in the sense of minimising the mean square error) estimator, of the form αS^2, is thus

$$T \;=\; \frac{n-1}{n+1}S^2 \;=\; \frac{1}{n+1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

iii. **What is the efficiency of S^2 relative to the smallest MSE αS^2?**

Note that

$$b(T) \;=\; \left(\frac{n-1}{n+1}-1\right)\sigma^2 \;=\; -\frac{2\sigma^2}{n+1}$$

so that

$$\begin{aligned} MSE(T) &= \frac{2(n-1)\sigma^4}{(n+1)^2} + \frac{4\sigma^4}{(n+1)^2} \\ &= \frac{2\sigma^4}{(n+1)^2}\{(n-1)+2\} \;=\; \frac{2\sigma^2}{n+1}. \end{aligned}$$

Hence,

$$Rel.Eff(S^2, T) \;=\; \frac{MSE(T)}{MSE(S^2)} \;=\; \frac{n-1}{n+1},$$

so that $Rel.Eff(S^2, T) < 1$: T is more efficient than S^2. This is, of course, not unexpected as S^2 is in the class of estimators over which T has the minimum mean square error.

2. **Let $X_1, \ldots, X_n$ be iid uniform random quantities on the interval $(\theta, \theta+1)$.**

   (a) **Show that $T_1 = \overline{X} - \frac{1}{2}$ is an unbiased estimator of θ and find its MSE.**

If $X \sim U(\theta, \theta+1)$ then (it is straightforward to show that)

$$E(X|\theta) \;=\; \theta + \frac{1}{2}; \;\; Var(X|\theta) \;=\; \frac{1}{12}.$$

Thus, for each i, $E(X_i|\theta) = \theta + \frac{1}{2}$ and $Var(X_i|\theta) = \frac{1}{12}$. Hence,

$$\begin{aligned} E(T_1|\theta) &= E(\overline{X}|\theta) - \frac{1}{2} \\ &= \frac{1}{n}\sum_{i=1}^{n}E(X_i|\theta) - \frac{1}{2} \;=\; \theta. \end{aligned}$$

T_1 is an unbiased estimator of θ. Thus,

$$\begin{aligned} MSE(T_1) &= Var(T_1|\theta) \;=\; Var(\overline{X}|\theta) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i|\theta) \;=\; \frac{1}{12n} \end{aligned}$$

as the individuals are independent.

(b) **Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$.**

i. **Show that the distribution of $X_{(1)}$ is**

$$f_{X_{(1)}}(x|\theta) = \begin{cases} n(\theta+1-x)^{n-1} & \theta \le x \le \theta+1, \\ 0 & \text{otherwise.} \end{cases}$$

For $X \sim U(\theta, \theta+1)$,

$$P(X \ge x|\theta) = \begin{cases} 1 & x < \theta \\ 1+\theta-x & \theta \le x \le \theta+1 \\ 0 & x > \theta. \end{cases}$$

Now,

$$\begin{aligned} P(X_{(1)} \ge x|\theta) &= P(X_1 \ge x, \ldots, X_n \ge x|\theta) \\ &= \prod_{i=1}^{n} P(X_i \ge x|\theta) \\ &= \begin{cases} 1 & x < \theta \\ (1+\theta-x)^n & \theta \le x \le \theta+1 \\ 0 & x > \theta, \end{cases} \end{aligned}$$

so that

$$f_{X_{(1)}}(x|\theta) = -\frac{d}{dx}P(X_{(1)} \ge x|\theta) = \begin{cases} n(\theta+1-x)^{n-1} & \theta \le x \le \theta+1, \\ 0 & \text{otherwise.} \end{cases}$$

ii. **Show that $T_2 = X_{(1)} - \frac{1}{n+1}$ is an unbiased estimator of θ and find its MSE.**

Consider

$$\begin{aligned} E(X_{(1)}|\theta) &= \int_{\theta}^{\theta+1} xn(\theta+1-x)^{n-1}\,dx \\ &= [-x(\theta+1-x)^n]_{\theta}^{\theta+1} + \int_{\theta}^{\theta+1} (\theta+1-x)^n\,dx \\ &= \theta + \left[-\frac{(\theta+1-x)^{n+1}}{n+1}\right]_{\theta}^{\theta+1} = \theta + \frac{1}{n+1}. \end{aligned}$$

Thus,

$$b(T_2) = E\left(X_{(1)} - \frac{1}{n+1}\bigg|\,\theta\right) - \theta = 0$$

so that T_2 is an unbiased estimator of θ. Now,

$$\begin{aligned} E(X_{(1)}^2|\theta) &= \int_{\theta}^{\theta+1} x^2 n(\theta+1-x)^{n-1}\,dx \\ &= \left[-x^2(\theta+1-x)^n\right]_{\theta}^{\theta+1} + \int_{\theta}^{\theta+1} 2x(\theta+1-x)^n\,dx \\ &= \theta^2 + \frac{2}{n+1}\int_{\theta}^{\theta+1} x(n+1)(\theta+1-x)^{(n+1)-1}\,dx \\ &= \theta^2 + \frac{2}{n+1}\left(\theta + \frac{1}{n+2}\right), \end{aligned}$$

so that

$$
\begin{aligned}
Var(X_{(1)}|\theta) &= \theta^2 + \frac{2}{n+1}\left(\theta + \frac{1}{n+2}\right) - \left(\theta + \frac{1}{n+1}\right)^2 \\
&= \frac{2}{(n+1)(n+2)} - \frac{1}{(n+1)^2} = \frac{n}{(n+1)^2(n+2)}
\end{aligned}
$$

which is the mean square error of the unbiased estimator of T_2 since $Var(T_2|\theta) = Var(X_{(1)}|\theta)$.

iii. **What is the efficiency of T_1 relative to T_2?**

$$
Rel.Eff(T_1, T_2) = \frac{MSE(T_2)}{MSE(T_1)} = \frac{12n^2}{(n+1)^2(n+2)}.
$$

3. **The independent observations $x_1, \ldots, x_{10}$ are assumed to come from a $N(\mu, \sigma^2)$ distribution, and $x_{11}, \ldots, x_{15}$ from a $N(2\mu, \sigma^2/2)$ distribution, where σ^2 is assumed known.**

   (a) **Write down the joint probability density function of $X_1, \ldots, X_{10}$, and the joint probability density function of $X_{11}, \ldots, X_{15}$. Hence write down the likelihood function for μ based on all 15 observations.**

   We use the independence of the X_i to write down the joint densities of all the variables (the fact that they have two different distributions does not affect this construction).

$$
\begin{aligned}
f_{X_i}(x_i|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\} \quad i = 1, \ldots, 10 \\
f_{X_1,\ldots,X_{10}}(x_1, \ldots, x_{10}|\mu, \sigma^2) &= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\} \\
f_{X_i}(x_i|\mu, \sigma^2) &= \frac{1}{\sqrt{\pi\sigma^2}} \exp\left\{-\frac{1}{\sigma^2}(x_i - 2\mu)^2\right\} \quad i = 11, \ldots, 15 \\
f_{X_{11},\ldots,X_{15}}(x_{11}, \ldots, x_{15}|\mu, \sigma^2) &= \prod_{i=11}^{15} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left\{-\frac{1}{\sigma^2}(x_i - 2\mu)^2\right\} \\
f_{X_1,\ldots,X_{15}}(x_1, \ldots, x_{15}|\mu, \sigma^2) &= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\} \times \\
&\quad \prod_{i=11}^{15} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left\{-\frac{1}{\sigma^2}(x_i - 2\mu)^2\right\}
\end{aligned}
$$

   (b) **Find the maximum likelihood estimator of μ and determine its bias and its mean square error.**

   The likelihood function is the joint density function considered as a function of μ:

$$
\begin{aligned}
L(\mu) &= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\} \prod_{i=11}^{15} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left\{-\frac{1}{\sigma^2}(x_i - 2\mu)^2\right\} \\
&= \frac{1}{32}\left(\frac{1}{\sqrt{\pi\sigma^2}}\right)^{15} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{10}(x_i - \mu)^2 - \frac{1}{\sigma^2}\sum_{i=11}^{15}(x_i - 2\mu)^2\right\}
\end{aligned}
$$

The log-likelihood is then

$$l(\mu) \;=\; -\log 32 - 15\log\sqrt{\pi\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{10}(x_i-\mu)^2 - \frac{1}{\sigma^2}\sum_{i=11}^{15}(x_i-2\mu)^2.$$

Differentiating with respect to μ gives

$$l'(\mu) \;=\; \frac{1}{\sigma^2}\sum_{i=1}^{10}(x_i-\mu) + \frac{4}{\sigma^2}\sum_{i=11}^{15}(x_i-2\mu).$$

Solving $l'(\mu)=0$, gives us that

$$\sum_{i=1}^{10}(x_i-\mu) + 4\sum_{i=11}^{15}(x_i-2\mu) \;=\; \sum_{i=1}^{10}x_i + 4\sum_{i=11}^{15}x_i - 10\mu - 40\mu \;=\; 0$$

so that

$$\hat{\mu} \;=\; \frac{1}{50}\left(\sum_{i=1}^{10}x_i + 4\sum_{i=11}^{15}x_i\right) \;=\; \frac{1}{5}\overline{x}_1 + \frac{2}{5}\overline{x}_2$$

where $\overline{x}_1$ is the average of the first 10 observations, and $\overline{x}_2$ is the average of the last 5 observations. Note that $l''(\mu) = -50/\sigma^2 < 0$ so that $\hat{\mu}$ is the maximum likelihood estimate. The maximum likelihood estimator is $T = \frac{1}{5}\overline{X}_1 + \frac{2}{5}\overline{X}_2$ where $\overline{X}_1 = \frac{1}{10}\sum_{i=1}^{10}X_i$ and $\overline{X}_2 = \frac{1}{5}\sum_{i=11}^{15}X_i$ with $X_i \sim N(\mu,\sigma^2)$ for $i=1,\ldots,10$ and $X_i \sim N(2\mu,\sigma^2/2)$ for $i=11,\ldots,15$. Hence,

$$E(\overline{X}_1|\mu,\sigma^2) \;=\; \mu;\; Var(\overline{X}_1|\mu,\sigma^2) \;=\; \frac{\sigma^2}{10}$$
$$E(\overline{X}_2|\mu,\sigma^2) \;=\; 2\mu;\; Var(\overline{X}_2|\mu,\sigma^2) \;=\; \frac{\sigma^2}{10}$$

Thus,

$$E(T|\mu,\sigma^2) \;=\; \frac{1}{5}E(\overline{X}_1|\mu,\sigma^2) + \frac{2}{5}E(\overline{X}_2|\mu,\sigma^2) \;=\; \mu,$$

so that T is an unbiased estimator of μ with

$$MSE(T) \;=\; Var(T|\mu,\sigma^2) \;=\; \frac{1}{25}Var(\overline{X}_1|\mu,\sigma^2) + \frac{4}{25}Var(\overline{X}_2|\mu,\sigma^2) \;=\; \frac{1}{50}\sigma^2.$$