MATHEMATICS COMPREHENSIVE EXAMINATION
CORE 1-ANALYSIS
AUGUST 2007

DIRECTIONS: This test consists of two parts (A) and (B). You must do *two problems from Part* (A) and *three problems from Part* (B). Please answer the problems in the order in which they appear and turn in only those that you wish to have graded. If you start each problem on a fresh sheet of paper, it will be easy to organize the solutions in order at the end. Be sure to refer to all theorems you apply and check hypotheses are satisfied. You have two and one half hours for this test. We wish you well.

TERMINOLOGY: The integrals that appear in this examination are to be understood as Lebesgue integrals. Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^d$ is denoted by m.

PART A

A.1 Let f be a real valued function on $\mathbb{R}^d$ such that the set of x at which f is discontinuous has measure 0. Show f is Lebesgue measurable.

A.2 Show every real valued bounded Lebesgue measurable function on $\mathbb{R}$ is a uniform limit of simple functions.

A.3 Prove that $\lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx$ exists but that the function $\frac{\sin x}{x}$ is not Lebesgue integrable over $(0, \infty)$.

A.4 Let $f : [a, b] \to \mathbb{R}$ be monotonically increasing. Show the set of points in $[a, b]$ where f is discontinuous is countable.

PART B

B.1 Prove that, if f is a real-valued Lebesgue-integrable function on $\mathbb{R}$, then

$$\lim_{x\to 0} \int |f(x+t) - f(t)|\, dt = 0.$$

B.2 A real-valued function f on an interval I for which there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in I is called a *Lipschitz function.*

(a) Show that a Lipschitz function is absolutely continuous.

(b) Show that an absolutely continuous function f is Lipschitz if and only if f' is bounded.

B.3 Determine

$$\lim_{n\to\infty} \int_0^1 \sin\left(\frac{x}{n}\right) \frac{n^3}{1+n^2x^3}\, dx.$$

B.4 Let X be the space of all real valued differentiable functions f on the interval $0 < x < 1$ such that $f'(x)$ is bounded for $-1 < x < 1$.

(a) Show if $f \in X$, then f is bounded.

(b) Define a norm on X by $|f| = \sup_{x\in(0,1)} |f(x)| + \sup_{x\in(0,1)} |f'(x)|$. Show X with metric $d(f,g) = |f - g|$ is complete.

B.5 Suppose f is integrable on $\mathbb{R}^d$. Prove

$$H(\alpha) = m\{x \mid f(x) > \alpha\} - m\{x \mid f(x) < -\alpha\}$$

is integrable for $\alpha \geq 0$ and show

$$\int_{[0,\infty)} H\, dm_1 = \int_{\mathbb{R}^d} f\, dm.$$

Here m_1 is Lebesgue measure on $\mathbb{R}^1$ and m is Lebesgue measure on $\mathbb{R}^d$.