Analysis Prelim – August 2001

Time allowed is four hours. Work on as many problems as possible, but do at least three problems from each section.

Real Analysis

Here m denotes Lebesgue measure on $\mathbb{R}^n$.

1. Find a sequence of real-valued nonnegative functions $\{f_k\}$ on [0,1] so that

$$(a)\ \limsup f_k = +\infty\ \forall x,\ \ (b) \int_{[0,1]} f_k\ dx \to 0.$$

Adapt this argument to the case where the domain is $\mathbb{R}^n$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function such that

$$m\{x : |f(x)| > \lambda\} \leq C\lambda^{-2},\ \ \lambda > 0.$$

Prove that there is a constant C_1 such that for any Borel set $E \subset \mathbb{R}^n$ of finite and positive measure

$$\int_E |f(x)| dx \leq C_1 \sqrt{m(E)}.$$

3. For $g \in L^1(\mathbb{R})$ and any $f \in C^1(\mathbb{R})$, suppose that

$$g * f'(x) = f(x+h) - f(x-h)$$

Show that $g \equiv \chi_{[-h,h]}$.

4. Let $B_{m,1}$ denote the ball of radius one centered at the origin in $\mathbb{R}^m$.

a) Show that there exists a function $f : \mathbb{R} \to [0,1]$ such that

$$m\{B_{n+1,1}\} = m\{B_{n,1}\} \int [f(t)]^n dt$$

b) Show that $\int [f(t)]^n dt \to 0$ as $n \to \infty$.

c) Show that for any finite positive number A, $A^n m\{B_{n,1}\} \to 0$ as $n \to \infty$.

5. Let $\{f_n\}$ be a sequence of real-valued functions in L^2 such that $||f_n||_{L^2} = 1$, and for $n < m$

$$|\int f_n f_m dx| \;\leq\; 2^{-m} \to 0,\; m \to \infty$$

Let $\{e_n\}$ be a Gram-Schmidt orthoginalization for this sequence with $\|e_n\| = 1$.

a) Show that

$$||f_m - e_m||^2 \leq 2^{-m}.$$

b) For any $g \in L^2$, show that as $m \to \infty$, $\int g\, f_m dx \to 0$.

6. Consider a real-valued nonnegative function $g \in L^1[0,1]$. Suppose that $\int gf dx < A$ whenever $\int e^f dx \leq 1$.

a) What can you say about the measure of the set $\{g > \lambda\}$ for large λ? (try $f = c\chi_E$)

b) Can you say that $g \in L^2$? Why or why not?

Complex Analysis

Here $\mathbb{C}$ denotes the complex plane, Ω denotes a region in $\mathbb{C}$, $dxdy$ denotes Lebesgue measure on $\mathbb{C}$.

1. Suppose that $\{f_n\}$ is a sequence of analytic functions on a region Ω, and F a continuous function on Ω such that for any disk $D \subset \Omega$,

$$\lim_{n\to\infty} \int_D |f_n(z) - F(z)| dxdy = 0.$$

Show that F is analytic on Ω, and that $f_n \to F$ and $f_n' \to F'$ uniformly on compact subsets of Ω.

2. Let $p(z)$ be a polynomial of degree $N \geq 1$ and define

$$L = \{z \in \mathbb{C} : |p(z)| = 1\}.$$

Prove that the open set $\mathbb{C} - L$ has at most $N + 1$ components.

3. Suppose f is a complex-valued function defined on $[0, \infty)$ that belongs to $L^p([0, \infty))$ for some p, $1 < p < \infty$. Define G on the right half-plane by

$$G(z) = \int_0^\infty f(t) e^{-tz} dt.$$

Prove that $G(z)$ is analytic for $\Re z > 0$. What additional assumption on f would ensure that G is entire?

4. Use a contour integral to compute the value of one of the following integrals:

a) the improper Riemann integral

$$\int_0^\infty e^{ix^2}\,dx$$

b) the Fourier transform of a Gaussian function on $\mathbb{R}$

$$\int_{-\infty}^{\infty} e^{ixy} e^{-x^2}\,dx.$$

5. Consider a compact set $K \subset \mathbb{R}$ with positive Lebesgue measure. On $\mathbb{C} - K$ define the function

$$g(z) = \int_K \frac{1}{t-z}\,dt.$$

a) Prove that g is analytic on $\mathbb{C} - K$;

b) Prove that g cannot be extended by analytic continuation to an entire function;

c) Show that $\lim_{z\to\infty}[zf(z)]$ exists and determine its value.

6. a) Construct a one-to-one conformal mapping of the upper half-plane $H = \{z \in \mathbb{C} : \Im z > 0\}$ onto the angle region with an interval removed

$$A = \{z \in \mathbb{C} : z \neq 0, \arg z \in (-\pi/4, \pi/4)\} - \{z : \Im z = 0, 0 \leq \Re z \leq 1\}.$$

b) Describe all the one-to-one conformal mappings of H onto A.