CHAPTER 5

Introduction to Functional Analysis

1. Normed Linear and Banach Spaces

Recall that a normed linear space is a vector space X with a function $\|\cdot\| : X \longrightarrow \mathbb{R}$ called a norm that satisfies the conditions of Definition 15. If $(X, \|\cdot\|)$ is a normed linear space, then the norm induces a metric ρ on X defined by

$$\rho(x, y) = \|x - y\|$$

and hence a topology. We will always assume that X is equipped with this topology.

Definition 25. *A* Banach space *is a complete normed linear space.*

Example 21. *The spaces* $\left(L^p(X, \mu), \|\cdot\|_p\right)$, $1 \leq p \leq \infty$, *are Banach spaces.*

Example 22. *The space* $(C([a, b]), \|\cdot\|_\infty)$ *is a Banach space. Note that putting different norms on a function space will give rise to different normed spaces. For example,* $(C([0, 1]), \|\cdot\|_\infty)$ *is a Banach space while* $(C([0, 1]), \|\cdot\|_1)$ *is not a Banach space.*

Proposition 5. *Let* (X, ρ) *be a metric space and* $\{x_n\}_{n=1}^\infty \subset X$ *a Cauchy sequence. Let* $\{x_{n_k}\}_{k=1}^\infty$ *be any subsequence of* $\{x_n\}_{n=1}^\infty$. *Then,* $\{x_{n_k}\}_{k=1}^\infty$ *converges to* x *if, and only if,* $\{x_n\}_{n=1}^\infty$ *converges to* x.

Proof. Suppose that $\{x_{n_k}\}_{k=1}^\infty$ converges to x. Fix $\epsilon > 0$. There is an N such that for $n, m \geq N$, $\rho(x_n, x_m) < \frac{\epsilon}{2}$. There is a $K \geq N$ such that $\rho(x_{n_K}, x) < \frac{\epsilon}{2}$. Then, for all $n > N$,

$$\rho(x_n, x) \leq \rho(x_n, x_{n_K}) + \rho(x_{n_K}, x) < \epsilon.$$

Thus, $\{x_n\}_{n=1}^\infty$ converges to x. The proof of the other implication is obvious. □

Definition 26. *A sequence* $\{x_n\}_{n=1}^\infty \subset X$ *is called* summable *if the sequence of partial sums* $\left\{\sum_{n=1}^j x_n\right\}_{j=1}^\infty$ *converges to* $x \in X$. *A sequence is called* absolutely summable *if* $\sum_{n=1}^\infty \|x_n\| < \infty$.

Proposition 6. *A normed linear space is complete if, and only if, every absolutely summable sequence is summable.*

Proof. Let X be a complete space and assume that $\sum_{n=1}^\infty \|x_n\| < \infty$. Given $\epsilon > 0$, there is an N such that $\sum_{n=N}^\infty \|x_n\| < \epsilon$. Let $S_k = \sum_{n=1}^k x_n \in X$ be the k^{th} partial sum of $\{x_n\}_{n=1}^\infty$. If $k > l > N$, then

$$\|S_k - S_l\| = \left\|\sum_{n=l+1}^k x_n\right\| \leq \sum_{n=l+1}^k \|x_n\| < \epsilon.$$

Therefore, $\{S_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X and, since X is complete, converges to $x \in X$. Thus, $\{x_n\}_{n=1}^{\infty}$ is summable.

Suppose next that every absolutely summable sequence is summable. Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy. We want to show that $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$. For all $k \in \mathbb{N}$, there is an $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $\|x_n - x_m\| < 2^{-k}$ for all $n, m \geq n_k$. Set $g_1 = x_{n_1}$ and, for $k > 1$ set $g_k = x_{n_k} - x_{n_{k-1}}$. Then, $\sum_{i=1}^{k} g_i = x_{n_k}$ and $\|g_k\| = \left\|x_{n_k} - x_{n_{k-1}}\right\| < 2^{-k+1}$ for all $k > 1$. Thus,

$$\sum_{k=1}^{\infty} \|g_k\| \leq \|g_1\| + \sum_{k=2}^{\infty} 2^{-k+1} = \|g_1\| + 1.$$

Therefore, $\{g_k\}_{k=1}^{\infty}$ is absolutely summable and hence summable, so that the sequence $\left\{\sum_{i=1}^{k} g_i\right\}_{i=1}^{\infty}$ converges to $x \in X$. This implies that $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x. By the previous proposition, $\{x_n\}_{n=1}^{\infty}$ converges to x and X is complete. □

EXAMPLE 23. *We can use this result to show that $(C([0,1]), \|\cdot\|_1)$ is not complete. Recall Example 13. Let $g_0 \equiv 0$ and set $f_n = g_n - g_{n-1}$ for $n \geq 1$. The support of f_n is contained in $\left[\frac{1}{2} - \frac{2}{2^n}, \frac{1}{2}\right]$ and $|f_n| \leq 1$ for all n. It follows that $\|f_n\|_1 \leq 2^{1-n}$ which implies that $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$ so that $\{f_n\}_{n=1}^{\infty}$ is absolutely summable. But,*

$$\sum_{n=1}^{N} f_n = \sum_{n=1}^{N} [g_n - g_{n-1}] = g_n,$$

which converges to $\chi_{\left[\frac{1}{2},1\right]} \notin C([0,1])$. Therefore, the space is not complete.

DEFINITION 27. *A* subspace *(or* linear manifold*) S of a vector space X is a subset of X which is closed under addition and scalar multiplication. That is, for all $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$, $\alpha \cdot x + \beta \cdot y \in X$.*

Suppose that $(X, \|\cdot\|)$ is a normed space and S is a subspace of X. Then, $(S, \|\cdot\|)$ is a normed space, since it is a linear space with a norm.

Let M be a subspace of a normed linear space X. Define an equivalence relation on X by saying $x \sim y$ if, and only if, $x - y \in M$. The quotient space X/M consists of the equivalence classes of this relation. For $[x] \in X/M$, define $\|[x]\|_{X/M}$ by

$$\|[x]\|_{X/M} = \inf\{\|y\| : y \in [x]\} = \inf\{\|x + m\| : m \in M\}.$$

The next result shows that when M is a closed subspace of X then $\|\cdot\|_{X/M}$ is a norm and $\left(X/M, \|\cdot\|_{X/M}\right)$ is a Banach space.

LEMMA 8. *Let M be a closed subspace of a Banach space X. Then, the quotient space $\left(X/M, \|\cdot\|_{X/M}\right)$ is a Banach space.*

PROOF. First show that $\|\cdot\|_{X/M}$ is a norm. Clearly, it is nonnegative. Next, $\|[x]\|_{X/M} = 0$ if, and only if, the distance from x to M is 0. Since M is closed, this is true if, and only if, $x \in M$, so that $[x] = 0$. Since M is a linear space, $m \in M$ implies $\alpha \cdot m \in M$ for all $\alpha \in \mathbb{F}$. Thus, if $\alpha \neq 0$,

$$\begin{aligned} \|\alpha [x]\|_{X/M} &= \|[\alpha \cdot x]\|_{X/M} = \inf\{\|\alpha \cdot x + m\| : m \in M\} \\ &= |\alpha| \inf\left\{\left\|x + \frac{1}{\alpha} \cdot m\right\| : m \in M\right\} = |\alpha| \|[x]\|_{X/M} \end{aligned}$$

Finally,

$$\begin{aligned} \|[x]+[y]\|_{X/M} &= \|[x+y]\|_{X/M} \\ &= \inf\{\|x+y+m\| : m \in M\} \\ &= \inf\{\|x+m+y+m'\| : m, m' \in M\} \\ &\leq \inf\{\|x+m\| : m \in M\} + \inf\{\|y+m'\| : m' \in M\} \\ &= \|[x]\|_{X/M} + \|[y]\|_{X/M}. \end{aligned}$$

Therefore, $\|\cdot\|_{X/M}$ is a norm on X/M.

To show that X/M is complete, let $\{[x_n]\}_{n=1}^{\infty}$ be a Cauchy sequence in X/M. For all $k \in \mathbb{N}$, there is an $N_k \in \mathbb{N}$ such that $N_k > N_{k-1}$ and $n, m \geq N_k$ implies that $\|[x_n] - [x_m]\|_{X/M} < 2^{-k}$. Set $u_k = x_{N_k}$. Since $\|[u_{k+1}] - [u_k]\|_{X/M} < 2^{-k}$, there is an $m_k \in M$ such that $\|u_{k+1} - u_k + m_k\| < 2^{-k}$. Set

$$w_n = \sum_{k=1}^{n} (u_{k+1} - u_k + m_k) = u_{n+1} - u_1 + \sum_{k=1}^{n} m_k.$$

If $n > m$, then

$$\|w_n - w_m\| \leq \sum_{k=m+1}^{n} \|u_{k+1} - u_k + m_k\| < \sum_{k=m+1}^{n} 2^{-k}$$

which implies that $\{w_n\}_{n=1}^{\infty}$ is Cauchy in X. Since X is complete, $\{w_n\}_{n=1}^{\infty}$ converges to $w \in X$. Thus,

$$\|[u_n] - [w+u_1]\|_{X/M} \leq \left\| u_n - w - u_1 + \sum_{k=1}^{n-1} m_k \right\| = \|w_{n-1} - w\|$$

which implies that $\{[u_n]\}_{n=1}^{\infty}$ converges to $[w+u_1]$ in X/M. By Proposition 5, $\{[x_n]\}_{n=1}^{\infty}$ converges to $[w+u_1]$, completing the proof. □

Proposition 7. *Let $\|\cdot\|$ be a semi-norm on a linear space X. Set $M_0 = \{x \in X : \|x\| = 0\}$. Then, M_0 is a closed subspace of X and the mapping from X to X/M_0 given by $x \longrightarrow [x]$ is norm preserving.*

Proof. Suppose that $\{x_n\}_{n=1}^{\infty} \subset M_0$ converges to $x \in X$. Since

$$0 \leq \|x\| \leq \|x - x_n\| + \|x_n\| = \|x - x_n\|$$

and this last term approaches 0 as $n \longrightarrow \infty$, it follows that $x \in M_0$ and M_0 is closed. To see that the mapping is norm preserving, note that $\|m\| = 0$ for all $m \in M_0$. Thus, for any $m \in M_0$ and $x \in X$,

$$\|x\| = \|x\| - \|m\| \leq \|x+m\| \leq \|x\| + \|m\| = \|x\|$$

so that $\|x\| = \|x+m\|$. Therefore,

$$\|x\| = \inf\{\|x+m\| : m \in M_0\} = \|[x]\|_{X/M}.$$

□

Example 24. *If $X = \mathcal{L}^p(X,\mu)$ then $M_0 = \{f : X \longrightarrow \mathbb{R} : f = 0$ μ-a.e.$\}$ and $X/M_0 = L^p(X,\mu)$.*

2. Linear Operators

Let X and Y be normed linear spaces over a field $\mathbb{F}$. Let $\mathcal{D}(T) \subset X$ be a linear subspace.

DEFINITION 28. *An operator $T : \mathcal{D}(T) \subset X \longrightarrow Y$ is called a* linear operator *if for all $x_1, x_2 \in X$ and $\alpha \in \mathbb{F}$:*

(1) $T(x_1 + x_2) = T(x_1) + T(x_2)$*;*
(2) $T(\alpha \cdot x) = \alpha \cdot Tx$*.*

The operator T is called bounded *if there is an $M \geq 0$ such that $\|Tx\|_Y \leq M\|x\|_X$ for all $x \in \mathcal{D}(T)$.*

Note that if T is a linear operator, we always have $T(0) = 0$ and $\mathcal{R}(T) = \{y \in Y : y = Tx \text{ for some } x \in \mathcal{D}(T)\}$ is a subspace of Y. We also have that if $(X, \|\cdot\|_X)$ is a normed space then $(\mathcal{D}(T), \|\cdot\|_X)$ is also a normed space. Thus, we can think of $T : \mathcal{D}(T) \subset X \longrightarrow Y$ as a operator between normed spaces.

THEOREM 20. *Let X and Y be normed spaces and $T : X \longrightarrow Y$ a linear operator. The following are equivalent:*

(1) *T is continuous at every $x \in X$;*
(2) *T is continuous at some $x \in X$;*
(3) *T maps bounded set in X to bounded sets in Y;*
(4) *T is a bounded operator.*

PROOF. Clearly, 1 implies 2. Suppose that 2 is true and 3 fails. There is an $x_0 \in X$ such that T is continuous at x_0. Further, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\|Tx_n\|_Y > n\|x_n\|_X$. Let

$$z_n = \frac{x_n}{n\|x_n\|_X} + x_0.$$

Then, $\|z_n\|_X \leq \|x_0\|_X + 1$, $\{z_n\}_{n=1}^{\infty}$ converges to x_0, and since T is continuous at x_0, $\lim_{n\to\infty} Tz_n = Tx_0$. Since T$z_n = \frac{Tx_n}{n\|x_n\|_X} + Tx_0$, this implies that $\left\{\frac{Tx_n}{n\|x_n\|_X}\right\}_{n=1}^{\infty}$ converges to 0. But, $\frac{\|Tx_n\|_Y}{n\|x_n\|_X} > 1$, which contradicts the assumption that 3 fails. Thus, 2 implies 3.

Suppose that 3 is true. Then, there is an M such that $\|Tx\|_Y \leq M$ for all $\|x\|_X \leq 1$. If $x \in X$ and $x \neq 0$, then

$$\frac{\|Tx\|_Y}{\|x\|_X} = \left\|T\left(\frac{x}{\|x\|_X}\right)\right\|_Y \leq M,$$

so that $\|Tx\|_Y \leq M\|x\|_X$ and T is bounded, proving 4.

Finally, suppose that 4 holds. Fix $\epsilon > 0$ and set $\delta = \frac{\epsilon}{M+1}$. Then, $\|x - y\|_X < \delta$ implies

$$\|Tx - Ty\|_Y = \|T(x-y)\|_Y \leq M\|x-y\|_X < M\delta = \epsilon.$$

Thus, T is continuous, completing the proof. □

EXAMPLE 25. *We list several examples of linear operators.*

(1) *Let $g \in L^{\infty}$ and define $T : L^1 \longrightarrow \mathbb{R}$ by $Tf = \int_X fg d\mu$. This is a bounded linear operator by the Riesz Representation Theorem.*

(2) *Consider the space* $BV([0,1])$ *with the norm* $\|f\|_{BV} = |f(a)| + V(f;[a,b])$. *For* $x \in BV([0,1])$, *define a sequence of mappings* $\{n_k\}_{k=1}^{\infty}$ *by*

$$n_k(x) = \int_0^1 s^k dx(s).$$

The operator $T : BV([0,1]) \longrightarrow \ell^{\infty}$ *defined by* $Tx = \{n_k(x)\}_{k=1}^{\infty}$ *defines a bounded linear operator.*

(3) *Any linear operator* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is given by a matrix. To see this, consider what* T *does to a basis of* $\mathbb{R}^n$. *The operator is necessarily bounded.*

(4) *Let* $C^1([0,1]) = \{f : [0,1] \longrightarrow \mathbb{R} : f' \in C([0,1])\} \subset C([0,1])$ *inherit the norm of* $C([0,1])$. *The operator* $T : C^1([0,1]) \longrightarrow C([0,1])$ *defined by* $Tf = f'$ *is linear but not bounded. To see it is not bounded, consider the functions* $f_n(t) = t^n$.

(5) *Let* $K \in C([a,b] \times [a,b])$ *and define* $T : C([a,b]) \longrightarrow C([a,b])$ *by*

$$Tf(x) = \int_a^n K(x,s) f(s) ds.$$

Such an operator is called an integral operator of Fredholm type of the first kind. *Since* $|K(x,s)| \le M$,

$$\|Tf\|_{\infty} = \sup_{x \in [a,b]} \left| \int_a^b K(x,s) f(s) ds \right| \le M \|f\|_{\infty} (b-a)$$

and T *is bounded and linear.*

Note that this proof would not show that T *is continuous if* $[a,b]$ *were replaced by* $\mathbb{R}$. *What kind of condition on* K *would be required to make* T *continuous on* $\mathbb{R}$*?*

Let X and Y be normed linear spaces. Denote the set of all continuous, linear operators T from X to Y by $L[X,Y]$. For $A, B \in L[X,Y]$, $x \in X$ and $\alpha \in \mathbb{F}$, set $(A+B)x = Ax + Bx$ and $(\alpha \cdot A)x = \alpha \cdot (A)x$. These definitions make $L[X,Y]$ a vector space. Analogously to the case of linear functionals on the L^p spaces, the function

$$\begin{aligned} \|T\| &= \sup\left\{ \frac{\|Tx\|_Y}{\|x\|_X} : \|x\|_X \neq 0 \right\} \\ &= \sup\{\|Tx\|_Y : \|x\|_X \le 1\} = \sup\{\|Tx\|_Y : \|x\|_X = 1\} \end{aligned}$$

defines a norm on $L[X,Y]$. Thus, we have

LEMMA 9. *For any pair of normed linear spaces* X *and* Y, $L[X,Y]$ *is a normed linear space.*

We would like conditions that guarantee $L[X,Y]$ is a Banach space. Interestingly, this depends only on Y.

THEOREM 21. *If* Y *is a Banach space then* $L[X,Y]$ *is a Banach space.*

PROOF. Let $\{T_n\}_{n=1}^{\infty} \subset L[X,Y]$ be a Cauchy sequence. For all $\epsilon > 0$, there is an N such that $\|T_n - T_m\| < \epsilon$ for $n, m \ge N$. If $x \in X$ and $x \neq 0$, then $\|T_n x - T_m x\|_Y < \epsilon \|x\|_X$. Thus, $\{T_n x\}_{n=1}^{\infty}$ is Cauchy in Y. Since Y is complete, there is a $Tx \in Y$ such that $\lim_{n \to \infty} T_n x = Tx$. This defines an operator $T : X \longrightarrow$

Y which is linear since each T_n is. Since $\{T_n\}_{n=1}^{\infty}$ is Cauchy, there is an M such that $\|T_n\| \leq M$ for all n. Thus,

$$\|Tx\|_Y = \lim_{n\to\infty} \|T_n x\|_Y \leq M \|x\|_X$$

and $T \in L[X,Y]$. To see that $\{T_n\}_{n=1}^{\infty}$ converges to T in $L[X,Y]$, fix $n \geq N$ and notice that $\|T_n x - T_m x\|_Y < \epsilon \|x\|_X$ for all $m \geq N$ implies $\|T_n x - Tx\|_Y \leq \epsilon \|x\|_X$ for all $n \geq N$. Therefore, $\|T_n - T\| \leq \epsilon$ for all $n \geq N$, which implies that $\{T_n\}_{n=1}^{\infty}$ converges to T in $L[X,Y]$. □

Definition 29. *Let X be a linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A linear operator $T : X \longrightarrow \mathbb{F}$ is called a* linear functional*. Denote $L[X,\mathbb{F}]$ by X^* (or X').*

With the norm

$$\|x'\|_{X^*} = \sup\{|x'(x)| : \|x\|_X \leq 1\} = \sup\{|x'(x)| : \|x\|_X = 1\},$$

X^* becomes a normed space. Since $\mathbb{F}$ is complete, X^* is always a Banach space. This space is often called the *dual space*, the *conjugate space*, or the *adjoint*. One often uses the notation $\langle x', x\rangle$ to stand for $x'(x)$.

2.1. Exercises.

Exercise 40. *The* kernel *of an operator A is the set $\{x \in X : Ax = 0\}$. Prove that the kernel of a linear operator is a subspace and that the kernel of a continuous linear operator is closed.*

Exercise 41. *Let X be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $T : X \longrightarrow F$ be a linear functional. If T is not continuous on X, prove that $\{Tx : \|x\| \leq 1\} = \mathbb{F}$.*

Exercise 42. *Let X be a normed linear space and suppose T is a bounded linear functional on X with norm 1. Given $\epsilon > 0$, show there is an $x_\epsilon \in X$ such that $\|x_\epsilon\| = 1$ and $Tx_\epsilon > 1-\epsilon$. Show by example that there need not exist an $x \in X$ such that $\|x\| = 1$ and $Tx = 1$.*

Exercise 43. *Let X be a normed space. Suppose that $T, S \in L[X,X]$. Prove that $TS \in\in L[X,X]$ and $\|TS\| \leq \|T\|\|S\|$. Give an example in which $\|TS\| < \|T\|\|S\|$.*

3. Uniform Boundedness Principle

Theorem 22 (Principle of Uniform Boundedness). *Let X be a Banach space and Y a normed space. Suppose that $W \subset L[X,Y]$ is such that for each $x \in X$ there is an $M(x)$ satisfying $\|Tx\|_Y \leq M(x)$ for all $T \in W$. Then, there is an M such that*

$$\sup\left\{\|T\|_{L[X,Y]} : T \in W\right\} \leq M.$$

Proof. Let

$$X_m = \{x \in X : \|Tx\|_Y \leq m \text{ for all } T \in W\} = \cap_{T\in W}\{x \in X : \|Tx\|_Y \leq m\}.$$

Since each term in the intersection is closed, X_m is closed. By assumption, $X = \cup_{m=1}^{\infty} X_m$ and since X is a Banach space, X is a complete metric space. Thus, there is an N such that X_N contains an open ball; that is, there is an $x_0 \in X$ and an $r > 0$ such that $B(x_0, r) \subset X_N$. Let $x \in X$ and set $z = x_0 + \dfrac{rx}{2\|x\|_X}$. Then, $z \in B(x_0, r)$

which implies that $\|Tz\|_Y \leq N$ for all $T \in W$. Since $x = \frac{2\|x\|_X}{r}(z - x_0)$ and each T is linear,

$$\|Tx\|_Y = \frac{2\|x\|_X}{r}\|Tz - Tx_0\|_Y \leq \frac{4N}{r}\|x\|_X .$$

Therefore, $\sup\left\{\|T\|_{L[X,Y]} : T \in W\right\} \leq \frac{4N}{r}$. □

This statement is actually one about equicontinuity. Suppose that $\|T\|_{L[X,Y]} \leq M$ for all $T \in W$. Fix $\epsilon > 0$ and set $\delta = \frac{\epsilon}{M}$. If $\|x - y\|_X < \delta$, then $\|Tx - Ty\|_Y < \epsilon$.

Corollary 11. *Let X and Y be Banach spaces and suppose that $\{T_n\}_{n=1}^{\infty} \subset L[X,Y]$. Then, $\{T_n x\}_{n=1}^{\infty}$ converges for all $x \in X$ if, and only if:*
(1) *$\{T_n x\}_{n=1}^{\infty}$ converges for all x in a dense subset $S \subset X$;*
(2) *$\|T_n\|_{L[X,Y]} \leq M$ for all n.*

Proof. If $\{T_n x\}_{n=1}^{\infty}$ converges for all $x \in X$, then 1 is clear and 2 follows from the Principle of Uniform Boundedness. So, assume 1 and 2 hold. Fix $x \in X$ and $\epsilon > 0$. Choose $s \in S$ such that $\|x - s\|_X < \frac{\epsilon}{3M}$. There is an N such that $\|T_n s - T_m s\|_Y < \frac{\epsilon}{3}$ for all $n, m \geq N$. Then, if $n, m \geq N$,

$$\begin{aligned}\|T_n x - T_m x\|_Y &\leq \|T_n x - T_n s\|_Y + \|T_n s - T_m s\|_Y + \|T_m s - T_m x\|_Y \\ &\leq 2M\|x - s\|_X + \tfrac{\epsilon}{3} \\ &< \epsilon.\end{aligned}$$

Therefore, $\{T_n x\}_{n=1}^{\infty}$ is Cauchy in Y. Since Y is complete, $\{T_n x\}_{n=1}^{\infty}$ converges. This completes the proof. □

Corollary 12. *Let X be a Banach space and Y a normed space. If $\{T_n\}_{n=1}^{\infty} \subset L[X,Y]$ and $\{T_n x\}_{n=1}^{\infty}$ converges to $Tx \in Y$ for all $x \in X$, then $T \in L[X,Y]$.*

Proof. The mapping T is linear since each T_n is. The Principle of Uniform Boundedness implies that the T_n's are uniformly bounded; let M be such a bound. Since

$$\|Tx\|_Y \leq \|T_n x\|_Y + \|Tx - T_n x\|_Y \leq M\|x\|_X + \|Tx - T_n x\|_Y$$

and $\{T_n x\}_{n=1}^{\infty}$ converges to Tx, it follows that $\|Tx\|_Y \leq M\|x\|_X$, so T has norm at most M and $T \in L[X,Y]$. □

We actually have a much stronger statement than the Principle of Uniform Boundedness. This is known as the *Banach-Steinhaus Theorem.*

Theorem 23 (Banach-Seinhaus Theorem). *Let X be a Banach space, Y be a normed space, and suppose that $W \subset L[X,Y]$. Either* $\sup\left\{\|T\|_{L[X,Y]} : T \in W\right\} < \infty$ *or there is a dense $\mathcal{G}_\delta$ set $G \subset X$ such that* $\sup\{\|Tx\|_Y : T \in W\} = \infty$ *for all $x \in G$.*

Proof. For each $x \in X$, let $\varphi(x) = \sup\{\|Tx\|_Y : T \in W\}$. It follows that the set $V_n = \{x \in X : \varphi(x) > n\}$ is open in X. If there is an N such that V_N is not dense in X, then there is an $x_0 \in X$ and a $\delta > 0$ such that $B(x_0, \delta) \cap V_n = \emptyset$; that is, $\|Tx\|_Y \leq \varphi(x) \leq N$ for all $x \in B(x_0, \delta)$. Arguing as above, there is an M such that $\sup\{\|Tx\|_Y : T \in W\} \leq M$.

The other possibility is that every V_n is dense in X. Then, $G = \cap_{n=1}^{\infty} V_n$ is dense in X and G is a $\mathcal{G}_\delta$ set. If $x \in G = \cap_{n=1}^{\infty} V_n$, then $\varphi(x) > n$ for all n. That is, $\varphi(x) = \infty$ and $\sup\{\|Tx\|_Y : T \in W\} = \infty$ for all $x \in G$. □

Note that this theorem implies Theorem 3 of Section 5.3 of [Swartz, page 179] by setting $W = \{T_k\}$ and observing that by the Uniform Boundedness Principle implies that T is bounded.

4. The Open Mapping Theorem

DEFINITION 30. *An operator* $T : \mathcal{D}(T) \subset X \longrightarrow Y$ *is called an* open mapping *if* $T(\mathcal{D}(T) \cap U)$ *is an open set in* Y *for every open set* $U \subset X$.

If T is an open mapping and T^{-1} exists, then T^{-1} is continuous. Thus, a one-to-one continuous open mapping is a homeomorphism. The next result is known as the *Open Mapping Theorem.*

THEOREM 24 (Open Mapping Theorem). *Let* X *and* Y *be Banach spaces. Suppose* $T \in L[X, Y]$ *maps onto* Y. *Then,* T *is an open mapping.*

COROLLARY 13. *Let* X *and* Y *be Banach spaces. Suppose* $T \in L[X, Y]$ *maps onto* Y. *If* T *is one-to-one then* T *is an homeomorphism.*

The proof of the Open Mapping Theorem is based on the following lemma. We use the notation $E - y = \{x \in X : x = e - y, e \in E\}$ for the *translation* of E by y and $E \setminus F = \{x \in X : x = e - f, e \in E, f \in F\}$ for the *set difference.*

LEMMA 10. *Let* X *and* Y *be Banach spaces and* $T \in L[X, Y]$ *map onto* Y. *Then, the image of the unit sphere in* X *contains a sphere about the origin in* Y.

PROOF. Set $B_n = B(0, 2^{-n}) \subset X$ and note that $X = \cup_{k=1}^{\infty} k \cdot B_1$. Since T is onto, $Y = \cup_{k=1}^{\infty} k \cdot T(B_1)$. Since Y is complete, and hence of the second category, $T(B_1)$ cannot be nowhere dense in Y. Thus, there is a $y_0 \in Y$ and a $\delta > 0$ such that $B(y_0, \delta) \subset \overline{T(B_1)}$. Thus, $B(0, \delta) \subset \overline{T(B_1)} - y_0$. Since

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_1)} \setminus \overline{T(B_1)} \subset \overline{T(B_1 \setminus B_1)} \subset 2 \cdot \overline{T(B_1)} = \overline{T(B_0)},$$

it follows that $\overline{T(B_0)}$ contains a sphere about 0 or radius δ.

It remains to show that $B\left(0, \frac{\delta}{2}\right) \subset T(B_0)$. By the linearity of T, $B(0, \delta) \subset \overline{T(B_0)}$ implies $B(0, \delta 2^{-n}) \subset \overline{T(B_n)}$. Fix $y \in Y$ such that $\|y\|_Y < \frac{\delta}{2}$. Since $y \in \overline{T(B_1)}$, there is an $x_1 \in B_1$ such that $\|y - Tx_1\|_Y < \frac{\delta}{4}$. Thus, $y - Tx_1 \in B\left(0, \delta 2^{-2}\right) \subset \overline{T(B_2)}$ and there is an $x_2 \in B_2$ such that $\|y - Tx_1 - Tx_2\|_Y < \frac{\delta}{8} = \delta 2^{-3}$. Continuing in this manner, there are $x_k \in B_k$ such that $\|y - \sum_{k=1}^{n} Tx_k\|_Y < \delta 2^{-(n+1)}$. Since $\|x_k\|_X < 2^{-k}$, $\sum_{k=1}^{\infty} x_k$ is absolutely summable. Since X is complete $x_0 = \sum_{k=1}^{\infty} x_k$ exists and $x_0 \in B_0$. By the continuity of T,

$$Tx_0 = T\left(\sum_{k=1}^{\infty} x_k\right) = \sum_{k=1}^{\infty} T(x_k) = y.$$

Therefore, $y \in T(B_0)$ which shows that $B\left(0, \frac{\delta}{2}\right) \subset T(B_0)$. □

We can now prove the Open Mapping Theorem.

PROOF. Let U be an open set in X and fix $y \in T(U)$. Choose $x \in U$ and a $\delta > 0$ such that $y = Tx$ and $B = B(x, \delta) \subset U$. The lemma implies that the image of the unit sphere about the origin in X contains a sphere about the origin in Y; i.e., there is an $\eta > 0$ such that $T(B_X(0, 1)) \supset B_Y(0, \eta)$. By linearity, this implies that $B_Y(0, \eta\delta) \subset T(B_X(0, \delta))$ so that $B_Y(0, \eta\delta) \subset T(B - x)$. Therefore,

$T(B) \supset B_Y(0,\eta\delta) + Tx = B(y,\eta\delta)$. Thus, $B_Y(y,\eta\delta) \subset T(U)$ and T is an open mapping. □

Let $T : X \longrightarrow Y$ be one-to-one and onto. Define $T^{-1} : Y \longrightarrow X$ by $T^{-1}y = x$ if, and only if, $Tx = y$. Notice that T is linear if, and only if, T^{-1} is linear.

COROLLARY 14. *Let X and Y be Banach spaces and suppose that $T \in L[X,Y]$ is one-to-one and onto. Then, there is a $\delta > 0$ such that $\|Tx\|_Y \geq \delta \|x\|_X$ for all $x \in X$. In other words, $T^{-1} \in L[Y,X]$ and $\|T\|_{L[Y,X]} \leq \delta^{-1}$.*

PROOF. There is a $\delta > 0$ so that $B_Y(0,\delta) \subset T(B_X(0,1))$. Since T is one-to-one, $\|Tx\|_Y < \delta$ implies that $\|x\|_X < 1$. Thus, if $\|x\|_X \geq 1$, then $\|Tx\|_Y \geq \delta$.But, if $\|x\|_X > 0$, then $\frac{x}{\|x\|_X}$ has norm 1 so that $\left\|T\left(\frac{x}{\|x\|_X}\right)\right\|_Y \geq \delta$, or $\|Tx\|_Y \geq \delta\|x\|_X$ for all x. (Note that the inequality is trivially true for $x = 0$.)

Finally, $\|Tx\|_Y \geq \delta \|x\|_X$ if, and only if, $\|y\|_Y \geq \delta \left\|T^{-1}y\right\|_X$ or $\left\|T^{-1}y\right\|_X \leq \frac{1}{\delta}\|y\|_Y$ for all y. This completes the proof. □

COROLLARY 15. *Let X be a vector space that is complete in both of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Suppose there is a C such that $\|x\|_1 \leq C\|x\|_2$. Then, the norms are equivalent, and there is a c such that for all $x \in X$,*

$$c\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2 .$$

PROOF. Observe that the identity mapping $Id : (X, \|\cdot\|_2) \longrightarrow (X, \|\cdot\|_1)$ satisfies the previous corollary. □

5. The Closed Graph Theorem

The next major result is the *Closed Graph Theorem*. Before introducing the idea of a closed operator, consider the following example.

EXAMPLE 26. *Define an operator $T : \mathcal{D}(T) = C^1([0,1]) \subset C([0,1]) \longrightarrow C([0,1])$ by $Tx = x'$. It is clear that T is linear. We also see that T is unbounded since*

$$\|T(x^n)\|_{C([0,1])} = \|T(x^n)\|_\infty = \left\|nx^{n-1}\right\|_\infty = n = n\|x^n\|_\infty .$$

Hence, T is unbounded.

Let X and Y be normed spaces. Consider the product space $X \times Y$ as a norm space. Any of several norms can be used, such as $\|\cdot\|_X + \|\cdot\|_Y$ or $\max(\|\cdot\|_X, \|\cdot\|_Y)$. The most common to use is $\sqrt{\|\cdot\|_X^2 + \|\cdot\|_Y^2}$.

DEFINITION 31. *Let X and Y be normed spaces and $T : \mathcal{D}(T) \subset X \longrightarrow Y$ be a linear operator. Then, T is called* closed *if the graph of T, $G(T) = \{(x,Tx) : x \in \mathcal{D}(T)\}$ is a closed set in $X \times Y$.*

THEOREM 25. *Let X and Y be normed spaces and suppose $T : \mathcal{D}(T) \subset X \longrightarrow Y$ is linear. Then, T is closed if, and only if, $\{x_n\}_{n=1}^\infty \subset \mathcal{D}(T)$ converges to x and $\{Tx_n\}_{n=1}^\infty$ converges to y implies that $x \in \mathcal{D}(T)$ and $y = Tx$.*

PROOF. If T is a closed operator then $G(T)$ is a closed set. Suppose $\{x_n\}_{n=1}^\infty \subset \mathcal{D}(T)$ converges to x and $\{Tx_n\}_{n=1}^\infty$ converges to y. Then, $\{(x_n,Tx_n)\}_{n=1}^\infty \subset G(T)$ converges to (x,y). Since $G(T)$ is a closed set, $(x,y) \in G(T)$. Therefore, $x \in \mathcal{D}(T)$ and $y = Tx$.

The other implication is easy since if the condition is true then $G(T)$ is a closed set. Hence, T is closed. □

Using the metric $\rho = \sqrt{\rho_X^2 + \rho_Y^2}$, one sees that this result is true for metric spaces.

EXAMPLE 27. *The operator T in Example 26 is closed. Let $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}(T)$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x and $\{Tx_n\}_{n=1}^{\infty} = \{x_n'\}_{n=1}^{\infty}$ converges to y. Since the convergence is uniform on $[0,1]$, it follows that $x \in C^1([0,1])$ and $x' = y$.*

The next result is called the *Closed Graph Theorem*. The importance of the theorem is that many questions about continuity of linear operators can be reduced to simpler problems about closed operators.

THEOREM 26 (Closed Graph Theorem). *Let X and Y be Banach spaces. Suppose $T : X \longrightarrow Y$ is a closed linear operator. Then, T is continuous.*

Note that one hypothesis of the theorem is that $\mathcal{D}(T) = X$.

PROOF. Let $G(T) = \{(x, Tx) : x \in X\}$ and let $\|(x, Tx)\| = \|x\|_X + \|Tx\|_Y$ be the norm on $G(T)$. Since T is a closed operator, $G(T)$ is a closed set. Thus, $G(T)$ is a Banach space. Define $A : G(T) \longrightarrow X$ by $A(x, Tx) = x$. Then, A is linear, one-to-one, onto and

$$\|x\|_X \leq \|(x, Tx)\|.$$

Therefore, $A \in L[G(T), X]$. By Corollary 14, the operator $A^{-1} : X \longrightarrow G(T)$ defined by $A^{-1}(x) = (x, Tx)$ exists and is continuous. Thus,

$$\|Tx\|_Y \leq \|(x, Tx)\| = \left\|A^{-1}x\right\|_{X \times Y} \leq M \|x\|_X$$

which shows that T is bounded and hence continuous. □

6. Hahn-Banach Theorems

The *Hahn-Banach Theorem* and its extensions and corollaries deal with extending a linear operator from a subspace of a vector space to the entire space. The proof below relies on *Zorn's Lemma* (which is equivalent to the *Axiom of Choice*).

DEFINITION 32. *A set P is called* partially ordered *if there is a relation $\preceq$ defined for some pairs of elements of P, such that for all $a, b, c \in P$:*

(1) *If $a \preceq b$ and $b \preceq c$ then $a \preceq c$;*
(2) *$a \preceq a$;*
(3) *If $a \preceq b$ and $b \preceq a$ then $a = b$.*

The set P is called totally ordered *if for every pair of elements of P are related by $\preceq$; that is, if $a, b \in P$, then either $a \preceq b$ or $b \preceq a$. An element $m \in P$ is called* maximal *if $a \in P$ and $m \preceq a$ implies that $m = a$. Let S be a subset of P. An element $u \in P$ is called an upper bound of S if $s \preceq u$ for all $s \in S$.*

One writes $(X, \preceq)$ to denote a set and an order relation.

EXAMPLE 28. *The set $(\mathbb{R}, \leq)$ is totally ordered. Given $z, w \in \mathbb{C}$, say $z \preceq w$ if $\operatorname{Re}(z) \leq \operatorname{Re}(w)$ and $\operatorname{Im}(z) \leq \operatorname{Im}(w)$. The relation $\preceq$ is a partial order on $\mathbb{C}$. Define a relation $\preceq$ on $\mathbb{R}$ by $x \preceq y$ if $|x| \leq |y|$. This is not a partial order on $\mathbb{R}$.*

LEMMA 11 (Zorn's Lemma). *Suppose that P is a nonempty partially ordered set such that each totally ordered subset of P has an upper bound. Then, P has a maximal element.*

Given a set $S \subset X$, the *span* of S in X is the smallest vector space contained in X that contains S.

LEMMA 12. *Let X and Y be vector spaces and $M \subset X$ a proper subspace. Let $f : M \longrightarrow Y$ be a linear operator. There is a linear operator $F : X \longrightarrow T$ such that $F(x) = f(x)$ for all $x \in M$.*

PROOF. We first show that f has some extension. Let $x_0 \in X \setminus M$ and let $M_0 = span\{x_0, M\}$. If $x \in M_0$, then there is a unique pair $m \in M$ and $\alpha \in \mathbb{F}$ such that $x = m + \alpha \cdot x_0$. Fix $y_0 \in Y$. Define $F_0 : M_0 \longrightarrow Y$ by

$$F_0(x) = F_0(m + \alpha \cdot x_0) = f(m) + \alpha \cdot y_0.$$

It is easy to check that the operator F_0 is linear. If $x \in M$, then $x = x + 0 \cdot x_0$ so that $F_0(x) = f(x)$. Thus, F_0 is an extension of f.

Let P be the set of all linear extensions of f. Since $F_0 \in P$, $P \neq \emptyset$. Define a partial order $\preceq$ on P by $g \preceq h$ if, and only if, $\mathcal{D}(g) \subset \mathcal{D}(h)$ and $g(x) = h(x)$ for all $x \in \mathcal{D}(g)$.

Let Q be a totally ordered subset of P. We want to show that Q has an upper bound $G \in P$. Set $\mathcal{D}(G) = \cup_{g \in Q} \mathcal{D}(g)$. If $x_1, x_2 \in \mathcal{D}(G)$ then there are functions $g_1, g_2 \in Q$ such that $x_i \in \mathcal{D}(g_i)$, $i = 1, 2$. Without loss of generality, we may assume that $\mathcal{D}(g_1) \subset \mathcal{D}(g_2)$. Then, $x_1, x_2 \in \mathcal{D}(g_2)$ so that $x_1 + x_2 \in \mathcal{D}(g_2) \subset \mathcal{D}(G)$. If $\alpha \in \mathbb{F}$, then $\alpha \cdot x_1 \in \mathcal{D}(g_1) \subset \mathcal{D}(G)$. Thus, $\mathcal{D}(G)$ is a subspace. Let $x \in \mathcal{D}(G)$. There is a $g \in Q$ such that $x \in \mathcal{D}(g)$. Set $G(x) = g(x)$. Since Q is totally ordered, G is well defined. Clearly, G extends f, since each $g \in Q$ does. It follows that G is an upper bound of Q since $G \in P$ and $g \preceq G$ for all $g \in Q$.

By Zorn's Lemma, P has a maximal element. Call this element F. Since F is an extension of f, we are done if we can show that $\mathcal{D}(F) = X$. If not, then there is an $x_0 \in X \setminus \mathcal{D}(F)$. Since $\mathcal{D}(F)$ is a subspace of X, we could extend F as in the first part of the proof. But this would contradict the maximality of F. Thus, it must be the case that $\mathcal{D}(F) = X$. □

Note that the proof has two main steps: show that f has some extension; use Zorn's Lemma to get a maximal extension. Further, this lemma says only that f has some extension. It may not be unique.

EXAMPLE 29. *Let $f : \mathbb{R} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x, 0) = x$. All functions of the form*

$$F_\alpha(x, y) = x + \alpha \cdot y,$$

with $\alpha \in \mathbb{R}$, are extensions of f defined on all of $X = \mathbb{R}^2$.

In addition to extending a linear functional, Hahn-Banach theorems pass restrictions on the original function to the extension.

DEFINITION 33. *A functional $p : X \longrightarrow \mathbb{R}$ is called* sublinear *if:*

(1) $p(x + y) \leq p(x) + p(y)$;
(2) $p(\alpha \cdot x) = \alpha \cdot p(x)$, *for all* $\alpha \geq 0$.

The next theorem is our first Hahn-Banach result.

THEOREM 27 (Hahn-Banach Theorem). *Let X be a real vector space and $M \subset X$ a proper subspace. Let p be a sublinear functional on X. Suppose that f is a linear functional on M such that $f(x) \leq p(x)$ for all $x \in M$. Then, there is a linear* *functional* $F : X \longrightarrow \mathbb{R}$ *such that* $F(x) \le p(x)$ *for all* $x \in X$ *and* $F(x) = f(x)$ *for all* $x \in M$.

PROOF. Using the previous proof, it is enough to show that there is some extension of f that is bounded by p; the proof then follows from Zorn's Lemma. Fix $x_0 \in X \setminus M$ and let $M_0 = \text{span}\{x_0, M\}$. If $x \in M_0$, then $x = m + \alpha \cdot x_0$. Define $F_0 : M_0 \longrightarrow \mathbb{R}$ by

$$F_0(x) = F_0(m + \alpha \cdot x_0) = f(m) + \alpha \cdot y.$$

The proof of the theorem follows if we can choose $y \in \mathbb{R}$ so that $F_0(x) \le p(x)$ for all $x \in M_0$.

Let $x, t \in M$. Then

$$f(x) + f(t) = f(x+t) \le p(x+t) \le p(x - x_0) + p(x_0 + t),$$

which implies that

$$-p(x - x_0) + f(x) \le p(x_0 + t) - f(t).$$

It now follows that

$$A = \sup\{-p(x - x_0) + f(x) : x \in M\} \le \inf\{p(x_0 + t) - f(t) : t \in M\} = B.$$

Choose y such that $A \le y \le B$. We are done if we can show that

$$F_0(m + \alpha \cdot x_0) = f(m) + \alpha \cdot y \le p(m + \alpha \cdot x_0).$$

If $\alpha = 0$, then $F_0(m) = f(m) \le p(m)$. If $\alpha > 0$, then

$$\begin{aligned} F_0(m + \alpha \cdot x_0) &= f(m) + \alpha \cdot y \\ &= \alpha\left(f\left(\frac{m}{\alpha}\right) + y\right) \\ &\le \alpha\left(f\left(\frac{m}{\alpha}\right) + p\left(x_0 + \frac{m}{\alpha}\right) - f\left(\frac{m}{\alpha}\right)\right) \\ &= \alpha \cdot p\left(x_0 + \frac{m}{\alpha}\right) \\ &= p(m + \alpha \cdot x_0) \end{aligned}$$

Finally, suppose that $\alpha < 0$. Then,

$$\begin{aligned} F_0(m + \alpha \cdot x_0) &= f(m) + \alpha \cdot y \\ &= (-\alpha)\left(f\left(\frac{m}{-\alpha}\right) - y\right) \\ &\le (-\alpha)\left(f\left(\frac{m}{-\alpha}\right) + p\left(\frac{m}{-\alpha} - x_0\right) - f\left(\frac{m}{-\alpha}\right)\right) \\ &= (-\alpha) \cdot p\left(\frac{m}{-\alpha} - x_0\right) \\ &= p(m + \alpha \cdot x_0). \end{aligned}$$

This shows that f has an extension F satisfying $F(x) \le p(x)$, completing the proof. □

Suppose that f is a linear functional and $f(x) \le p(x)$. This implies that $f(-x) \le p(-x)$ so that $f(x) \ge -p(-x)$. Therefore, if $f(x) \le p(x)$ for all $x \in M$, then $-p(-x) \le f(x) \le p(x)$ for all $x \in M$. If F is an extension guaranteed by the Hahn-Banach Theorem, then $-p(-x) \le F(x) \le p(x)$ for all $x \in X$. If, in addition, $p(-x) = p(x)$, then $|f(x)| \le p(x)$ and so $|F(x)| \le p(x)$.

Suppose that X is a complex vector space. Suppose that F is any complex-linear functional on a subspace M of X, $F : M \longrightarrow \mathbb{C}$. Since $F(i \cdot x) = i \cdot F(x)$, we see that

$$F(x) = \operatorname{Re} F(x) - i \cdot \operatorname{Re} F(i \cdot x).$$

Thus, the study of F can be reduced to the study of $\operatorname{Re} F$. From the Hahn-Banach Theorem, we get:

THEOREM 28. *Let M be a subspace of a complex vector space X and let p be a sublinear functional on X such that $p(\alpha \cdot x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{C}$. Let $f : M \longrightarrow \mathbb{C}$ be a linear functional such that $|f(x)| \leq p(x)$ for all $x \in M$. There is a linear functional $F : M \longrightarrow \mathbb{C}$ such that $|F(x)| \leq p(x)$ for all $x \in X$ and $F(x) = f(x)$ for all $x \in M$.*

6.1. Applications of the Hahn-Banach Theorem. This section contains applications of the Hahn-Banach Theorem to normed spaces. Let X be a nonempty normed linear space. The first result is a Hahn-Banach theorem for normed spaces.

THEOREM 29. *Let X be a normed space and $M \subset X$ a proper subspace. If $f \in M^*$, then there is an $F \in X^*$ such that $F(x) = f(x)$ for all $x \in M$ and $\|F\|_{X^*} = \|f\|_{M^*}$.*

PROOF. Define a sublinear functional on X by $p(x) = \|f\|_{M^*} \|x\|_X$. Then, p is a norm on M and $|f(x)| \leq p(x)$ for all $x \in M$. Therefore, there is a linear functional F on X such that $|F(x)| \leq p(x) = \|f\|_{M^*} \|x\|_X$ for all $x \in X$. Therefore, $\|F\|_{X^*} \leq \|f\|_{M^*}$. Since $F = f$ on M, it follows that $\|F\|_{X^*} \geq \|f\|_{M^*}$, so the two are equal. □

Let $M \subset X$ be a proper subspace and $f \in M^*$. Then, there is an $F \in X^*$ such that $F|_M = f$. If f and g are two elements of M^* and $f \neq g$, then the corresponding extensions F and G in X^* are also unequal (since they differ on M). On the other hand, suppose that $F \in X^*$. Then, $F|_M \in M^*$. In this case, if we start with $F \neq G$, we cannot conclude that $F|_M \neq G|_M$. (See Example 29.) In this sense, X^* is bigger than M^*.

COROLLARY 16. *The dual space of L^∞ is not L^1.*

PROOF. Consider $C([0,1])$ as a closed subspace of L^∞. Define $T \in (C([0,1]))^*$ by $Tf = f(0)$; i.e., T is the map defined by evaluation at 0. By the Hahn-Banach Theorem, T has an extension $\mathcal{T} \in (L^\infty([0,1]))^*$. The proof will be completed by showing that there is no function $g \in L^1$ such that $\mathcal{T} f = \int_0^1 fg dx$ for all $f \in L^\infty$. Let $\{f_n\}_{n=1}^\infty \subset C([0,1])$ be such that $\|f_n\|_\infty = 1$, $f_n(0) = 1$, and $\lim_{n\to\infty} f_n(t) = 0$ for all $t > 0$. Then, $\mathcal{T} f_n = 1$ for all n. Let g be an arbitrary function in L^1. Since $|f_n g| \leq |g|$ and g is finite a.e., it follows that $\{f_n g\}_{n=1}^\infty \subset L^1$ and converges to 0 a.e.. By the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} \int_0^1 f_n g dx = 0$ while $\mathcal{T} f_n = 1$ for all n. Therefore, $\mathcal{T} f \neq \int_0^1 fg dx$ for any $g \in L^1$. □

EXAMPLE 30. *Define $f : \mathbb{R} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x, 0) = x$, as in Example 29. All the functions $F_\alpha(x, y) = x + \alpha \cdot y$, $\alpha \in \mathbb{R}$, are extensions of f to $\mathbb{R}^2$. We can think of F_α as being given by the matrix $(1, \alpha)$. If we set $M = \mathbb{R} \times \{0\}$, then $\|f\|_{M^*} = 1$.*

On the other hand,

$$\|F_\alpha\|_{(\mathbb{R}^2)^*} = \sup_{\|(x,y)\|\neq 0} \frac{|x+\alpha\cdot y|}{\sqrt{x^2+y^2}} = \sup_{x^2+y^2=1} |x+\alpha\cdot y| = \sqrt{1+\alpha^2}.$$

(To see this, think of the Lagrange multiplier problem of maximizing the function $H(x,y,\lambda) = x+\alpha\cdot y-\lambda\left(x^2+y^2-1\right)$*.) The only extension of* f *that does not increase the norm is* $F_0(x,y) = x$*. For the seminorm in this problem, we can consider* $p(x,y) = |x|$ *or* $p(x,y) = \sqrt{x^2+y^2}$*.*

We say two normed spaces are *congruent* or *isometrically isomorphic* if there is a one-to-one, onto mapping from one space to the other, which is norm preserving. We often use the notation $\cong$ to indicate that two spaces are congruent.

EXAMPLE 31. *By the Riesz Representation Theorem,* $(L^p)^* \cong L^{p'}$ *for* $1 \le p < \infty$*.*

THEOREM 30. *Let* X *be a normed linear space and* M *a subspace of* X*. Let* $M^0 = \{x' \in X^* : x'(x) = 0 \text{ for all } x \in M\} \subset X^*$*. Then,* M^0*is a closed subspace of* X^* *and* $M^* \cong X^*/M^0$*.*

PROOF. We first show that M^0 is a closed subspace of X^*. It is easy to see that M^0 is a subspace of X^*. To see that it is closed, suppose that $\{x'_n\}_{n=1}^\infty \subset M^0$ and there is an $x' \in X^*$ such that $\lim_{n\to\infty} \|x'_n - x'\|_{X^*} = 0$. Then, for all $m \in M$,

$$|x'(m)| = |x'(m) - x'_n(m)| \le \|x'_n - x'\|_{X^*} \|m\|_X .$$

Since the right hand side approaches 0 as n approaches ∞, $x'(m) = 0$ for all $m \in M$. Thus, $x' \in M^0$ and M^0 is closed.

It remains to show that $M^* \cong X^*/M^0$. Let $[x'] \in X^*/M^0$. Then, $y' \in [x']$ if, and only if, $y' - x' \in M^0$, so that $y'(m) - x'(m) = 0$ for all $m \in M$. Define m' by $m'(m) = x'(m)$ for all $m \in M$. Note that m' depends only on the equivalence class $[x']$ and m' is linear since x' is. Since $|m'(m)| \le \|x'\|_{X^*} \|m\|_X$, we see that $\|m'\|_{M^*} \le \|x'\|_{X^*}$, so that m' is bounded and, hence, $m' \in M^*$. Further, $\|m'\|_{M^*} \le \|y'\|_{X^*}$ for all $y' \in [x']$, which shows that

$$\|m'\|_{M^*} \le \inf\{\|y'\|_{X^*} : y' \in [x']\} = \|[x']\|_{X^*/M^0} .$$

By the Hahn-Banach Theorem, there is a $y' \in [x']$ such that $\|y'\|_{X^*} = \|m'\|_{M^*}$, which implies that $\|m'\|_{M^*} \ge \|[x']\|_{X^*/M^0}$. Thus, $\|m'\|_{M^*} = \|[x']\|_{X^*/M^0}$.

Define $T : X^*/M^0 \longrightarrow M^*$ by $T([x']) = m'$. By the definition of the equivalence classes, T is linear and, as we showed above, T is an isometry. Thus, T is one-to-one. Let $m' \in M^*$. By the , there is an $x' \in X^*$ such that $x'(m) = m'(m)$ for all $m \in M$. Thus, $T([x']) = m'$ and T is onto. □

The next six results are direct consequences of the Hahn-Banach Theorem.

COROLLARY 17. *Let* $x_0 \in X$*. There is an* $x' \in X^*$ *such that* $\|x'\|_{X^*} = 1$ *and* $x'(x_0) = \|x_0\|_X$*.*

PROOF. Assume that $x_0 \neq 0$ and set $M = \text{span}\{x_0\}$. For $x \in M$, set $x'(x) = x'(\alpha\cdot x_0) = \alpha\|x_0\|_X$. It follows that $\|x'\|_{M^*} = 1$ and $x'(x_0) = \|x_0\|_X$. Extend x' to all of X by the Hahn-Banach Theorem. This functional also works for $x_0 = 0$, since $0 \in M$ and $x'(0) = 0 = \|0\|_X$. □

Corollary 18. *If X is a nonempty normed space then X^* contains nontrivial linear functionals.*

The next corollary shows that we can use the Hahn-Banach Theorem to find a functional that separates a point from a closed subspace.

Corollary 19. *Let M be a subspace of X. Suppose that $x_0 \in X$ and*

$$d = distance\,(x_0, M) = \inf \{\|x_0 - m\|_X : m \in M\} > 0.$$

Then, there is an $x' \in X^*$ such that $\|x'\|_{X^*} = 1$, $x'(x_0) = d$, and $x'(m) = 0$ for all $m \in M$.

Proof. Let M_0 =span$\{x_0, M\}$ and define x' by $x'(m + \alpha \cdot x_0) = \alpha d$. Clearly, x' is linear, $x'(x_0) = d$, and $x'(m) = 0$ for all $m \in M$. If $m \in M$ and $\alpha \neq 0$, then

$$|x'(m + \alpha \cdot x_0)| = |\alpha d| = |\alpha|\, d \leq |\alpha| \left\| x_0 + \frac{m}{\alpha} \right\|_X = \|\alpha \cdot x_0 + m\|_X$$

so that $\|x'\|_{(M_0)^*} \leq 1$ and $x' \in (M_0)^*$. Fix $\epsilon > 0$. There is an $m \in M$ such that $\|x_0 - m\|_X < d + \epsilon$. Set $y = \dfrac{x_0 - m}{\|x_0 - m\|_X}$. Then, $y \in M_0$ and $\|y\|_X = 1$. Since

$$|x'(y)| = \frac{d}{\|x_0 - m\|_X} > \frac{d}{d + \epsilon},$$

it follows that $\|x'\|_{(M_0)^*} > \frac{d}{d+\epsilon}$ for all $\epsilon > 0$. Thus, $\|x'\|_{(M_0)^*} \geq 1$ which implies that $\|x'\|_{(M_0)^*} = 1$. The proof is completed by the Hahn-Banach Theorem. □

Corollary 20. *For all $x \in X$,*

$$\|x\|_X = \sup \left\{ \frac{|x'(x)|}{\|x'\|_{X^*}} : 0 \neq x' \in X^* \right\} = \sup \{|x'(x)| : \|x'\|_{X^*} = 1\}.$$

Proof. For any $x' \in X^*$, $|x'(x)| \leq \|x'\|_{X^*} \|x\|_X$, so that

$$\sup \left\{ \frac{|x'(x)|}{\|x'\|_{X^*}} : 0 \neq x' \in X^* \right\} \leq \|x\|_X.$$

By Corollary 17, there is a $y' \in X^*$ such that

$$\|x\|_X = |y'(x)| = \frac{|y'(x)|}{\|y'\|_{X^*}} \leq \sup \left\{ \frac{|x'(x)|}{\|x'\|_{X^*}} : 0 \neq x' \in X^* \right\}.$$

This completes the proof. □

We saw this result earlier when we studied the converse to Hölder's inequality. As a consequence of this corollary, we get

Corollary 21. *Suppose that $x \in X$ and $x'(x) = 0$ for all $x' \in X^*$. Then, $x = 0$.*

Corollary 22. *X^* separates points in X.*

Proof. If $x, y \in X$ and $x \neq y$, then there is an $x' \in X^*$ such that

$$\frac{|x'(x - y)|}{\|x'\|_{X^*}} > \frac{1}{2} \|x - y\|_X > 0.$$

Therefore, $|x'(x) - x'(y)| > 0$ and X^* separates points. □

Theorem 31. *If X^* is separable then X is separable.*

PROOF. Let $\{x_n'\}_{n=1}^{\infty}$ be a countable dense subset of $\{x' \in X^* : \|x'\|_{X^*} = 1\}$. Choose $x_n \in X$ such that $\|x_n\|_X = 1$ and $|x_n'(x_n)| \geq \frac{3}{4}$. Let $M = \overline{\text{span}\,\{x_n\}_{n=1}^{\infty}}$. If $M \neq X$, then there is an $x_0 \in X \setminus M$. There is an $x' \in X^*$ such that $\|x'\|_{X^*} = 1$, $x'(x_0) \neq 0$, and $x'(m) = 0$ for all $m \in M$. In particular, $x'(x_n) = 0$ for all n. Thus,

$$\frac{3}{4} \leq |x_n'(x_n)| \leq |x_n'(x_n) - x'(x_n)| + |x'(x_n)| = |x_n'(x_n) - x'(x_n)|.$$

Thus, $\frac{3}{4} \leq |x_n'(x_n) - x'(x_n)|$. This contradicts the assumption that $\{x_n'\}_{n=1}^{\infty}$ is dense in $\{x' \in X^* : \|x'\|_{X^*} = 1\}$. Thus, $M = X$ and linear combinations of rational multiples of the x_n's are dense in X. Hence, X is separable. □

EXAMPLE 32. *Since $L^1(\mathbb{R}^n)$ is separable, if $L^1 = (L^\infty)^*$ then L^∞ would be separable. Therefore, L^1 is not the dual of L^∞. Since the dual of L^1 is L^∞, this shows that X may be separable while X^* is not.*

We have defined the dual space of X, $X^* = L[X, \mathbb{R}]$. Since $\mathbb{R}$ is complete, X^* is a Banach space. Consider the *second dual* of X, $X^{**} = (X^*)^*$. This is the vector space of bounded linear functionals defined on X^*.

EXAMPLE 33. *If $1 < p < \infty$, then $(L^p)^* \cong L^{p'}$ and $(L^p)^{**} \cong L^p$.*

Given $x \in X$, define φ_x to be the linear functional on X^* defined by $\varphi_x(x') = x'(x)$; i.e., φ_x is defined by evaluation at x. Define a map $\varphi : X \longrightarrow X^{**}$ by $\varphi(x) = \varphi_x$. The operator φ is called the *canonical map* from X to X^{**}. Note that φ is linear since given $x, y \in X$,

$$\varphi(x+y)(x') = \varphi_{(x+y)}(x') = x'(x+y) = x'(x) + x'(y) = \varphi(x)(x') + \varphi(y)(x'),$$

for all $x' \in X^*$. This shows that $\varphi(x+y) = \varphi(x) + \varphi(y)$. Similarly, $\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x)$, so that φ is linear. To see that φ is bounded, for any $x \in X$ and $x' \in X^*$, $|\varphi_x(x')| = |x'(x)| \leq \|x'\|_{X^*} \|x\|_X$, which implies that $\|\varphi_x\|_{X^{**}} \leq \|x\|_X$. Thus, φ is bounded (with norm at most 1). Moreover, since there is an $x_0' \in X^*$ such that $\|x_0'\|_{X^*} = 1$ and $x_0'(x) = \|x\|_X$, it follows that $|\varphi_x(x_0')| = \|x\|_X$. Therefore, $\|\varphi(x)\|_{X^{**}} = \|x\|_X$.

From the previous paragraph, $\varphi \in L[X, X^{**}]$ and φ is an isometry. Therefore, φ defines a congruence between X and $\varphi(X) \subset X^{**}$. We call φ the *natural isomorphism* of X into X^{**}.

DEFINITION 34. *A normed vector space X is called* (norm) reflexive *if $\varphi(X) = X^{**}$.*

EXAMPLE 34. *The L^p spaces are reflexive for $1 < p < \infty$. The spaces L^1 and L^∞ are not reflexive.*

Since $X^{**} = (X^*)^*$ is always a Banach space, if X is not a Banach space then X cannot be reflexive.

THEOREM 32. *If X is reflexive then X^* is reflexive.*

PROOF. Let $\varphi_0 : X \longrightarrow X^{**}$ and $\varphi_1 : X^* \longrightarrow X^{***}$ be the canonical maps and assume that $\varphi_0(X) = X^{**}$. We want to show that $\varphi_1(X^*) = X^{***}$. Fix $x''' \in X^{***}$. Define x' by $x'(x) = \langle x, x' \rangle = \langle \varphi_0(x), x''' \rangle$ for all $x \in X$. Since φ_0 and x''' are linear, it follows that x' is linear. Further,

$$|\langle x, x' \rangle| = |\langle \varphi_0(x), x''' \rangle| \leq \|x'''\|_{X^{***}} \|\varphi_0(x)\|_{X^{**}} = \|x'''\|_{X^{***}} \|x\|_X,$$

so that x' is bounded and $x' \in X^*$.

We are done if we can show that $\varphi_1(x') = x'''$. Let $x'' \in X^{**}$. Since X is reflexive, there is an $x \in X$ such that $\varphi_0(x) = x''$. Therefore,

$$\begin{aligned} x'''(x'') &= \langle x'', x''' \rangle = \langle \varphi_0(x), x''' \rangle = \langle x, x' \rangle \\ &= x'(x) = \varphi_0(x)(x') = x''(x') = \varphi_1(x')(x'') . \end{aligned}$$

Therefore, $x''' = \varphi_1(x')$ which shows that φ_1 is onto and X^* is reflexive. □

THEOREM 33. *Let X be a Banach space. Then, X is reflexive if, and only if, X^* is reflexive.*

PROOF. By the previous result, it is enough to show that X^* reflexive implies that X is reflexive. Assume that φ_0 and φ_1 are defined as in the previous proof and X is not reflexive. Since φ_0 is an isometry and X is complete, $\varphi_0(X)$ is a proper, closed subspace of X^{**}. By Corollary 19, there is an $x''' \in X^{***}$ such that $\|x'''\|_{X^{***}} = 1$ and $x'''(x'') = 0$ for all $x'' \in \varphi_0(X)$. Since X^* is reflexive, there is an $x' \in X^*$ such that $\varphi_1(x') = x'''$ and $\|x'\|_{X^*} = \|x'''\|_{X^{***}}$. Given any $x \in X$,

$$0 = x'''(x'') = x'''(\varphi_0(x)) = x'(x)$$

which says that x' is the 0 functional on X. But $\|x'\|_{X^*} = 1$. Therefore, X is reflexive. □

7. Exercises

EXERCISE 44. *Let X and Y be normed spaces and $x \in X$ and $y \in Y$ be arbitrary points with $x \neq 0$. Prove there is a $T \in L[X, Y]$ such that $Tx = y$.*

EXERCISE 45. *Let X and Y be normed spaces with $X \neq \{0\}$. Prove that if $L[X, Y]$ is complete, then Y is complete.*

EXERCISE 46. *Give an example of a linear functional on ℓ^∞ which is not given by an ℓ^1 sequence.*

EXERCISE 47. *Let X be a normed linear space and M a subspace of X. Suppose that $T \in X^*$ and $Tm = 0$ for all $m \in M$ imply that $T = 0$ in X^*. Prove that M is dense in X.*