Exercises-3 for Applied Functional Analysis

1. Calculation of inner product

   Define an inner product as follows:

$$(f,g)=\int_0^\infty e^{-x}f(x)g(x)dx \tag{1}$$

   Determine the constants A, B and C so that A (a constant-valued function) and $Bx+C$ are normalized and mutually orthogonal with respect to the above inner product. (*i.e.*, $\|A\|=1$, $\|Bx+C\|=1$, $(A,Bx+C)=0$)

   Ans.

   $\|A\|^2=A^2\int_0^\infty e^{-x}dx=1$, therefore, $A=\pm 1$.

$$(A,Bx+C)=\pm\int_0^\infty e^{-x}(Bx+C)dx=B\int_0^\infty e^{-x}xdx+C\int_0^\infty e^{-x}dx=B+C$$

$$\text{because} \int_0^\infty e^{-x}dx=1 \int_0^\infty e^{-x}xdx=[-e^{-x}x]_0^\infty+\int_0^\infty e^{-x}dx=1$$

   $B+C=0$

$$\|Bx+C\|^2=B^2\int_0^\infty e^{-x}(x-1)^2dx=B^2e^{-1}\int_{-1}^\infty e^{-x}x^2dx$$

$$\begin{aligned}\int_{-1}^\infty e^{-x}x^2dx &= [-e^{-x}x^2]_{-1}^\infty+2\int_{-1}^\infty e^{-x}xdx\\ &= e+2\{[-e^{-x}x]_{-1}^\infty+\int_{-1}^\infty e^{-x}dx\}=e+2\{-e+e\}=e\end{aligned}$$

   Consequently, $B=\pm 1, C=\mp 1$

2. A discrete measure

   Define a measure μ as follows:

   (1) If m is integer, $\mu(\{m\})=|m|$

   (2) $\mu((a,b))=0$ so long as the interval (a,b) does not include integer.

   Calculate the following integral:

$$\int_0^\infty e^{-x}d\mu(x) \tag{2}$$

   Ans.

$$\int_0^\infty e^{-x}d\mu(x)=\sum_{m=1}^\infty e^{-m}m=-[\frac{\partial}{\partial t}\sum_{m=1}^\infty e^{-mt}]_{t\to 1}=-[\frac{\partial}{\partial t}\{\frac{e^{-t}}{1-e^{-t}}\}]_{t\to 1}$$

$$=-[\frac{\partial}{\partial t}\{\frac{1}{1-e^{-t}}-1\}]_{t\to 1}=[\frac{e^{-t}}{(1-e^{-t})^2}]_{t\to 1}=\frac{e^{-1}}{(1-e^{-1})^2}=-\frac{e}{(e-1)^2}$$

3. one-dimensional diffusion equation

Consider the following wave equation about the unknown variable $u = u(x,t)$ extending over $(\infty, -\infty)$:

$$\kappa \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \kappa > 0 \tag{3}$$

with initial conditions: $u(x,0) = f(x)$

(1) Calculate the Fourier transform of u.

(2) Calculate u.

Ans. (1)

$$\begin{aligned} \kappa \frac{d\hat{u}}{dt} &= -\xi^2, \\ \hat{u} &= const.e^{-\xi^2 \tau}, \quad \tau = \frac{t}{\kappa} \end{aligned}$$

Consider the Fourier transform of the initial condition, and

$$\hat{u} = \hat{f}(\xi) e^{-\xi^2 \tau}$$

(2) Let $h(x;t) = \frac{1}{\sqrt{2\pi}} F^*[e^{-\xi^2 \tau}]$, and $\hat{u} = \sqrt{2\pi} \hat{f} \hat{h}$, *i.e.,* $u = h * f$

$$h(x;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\xi^2 \tau + ix\xi} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\tau \xi^2 + ix\xi} d\xi$$

Because $-\tau\xi^2 + ix\xi = -\tau(\xi - i\frac{x}{2\tau})^2 - \frac{x^2}{4\tau}$,

$$\int_{-\infty}^{\infty} e^{-\tau\xi^2 + ix\xi} d\xi = e^{-\frac{x^2}{4\tau}} \int_{-\infty}^{\infty} e^{-\tau(\xi - i\frac{x}{2\tau})^2} d\xi$$

$$\int_{-\infty}^{\infty} e^{-\tau(\xi - i\frac{x}{2\tau})^2} d\xi = \int_C e^{-\tau\xi^2} d\xi = \int_{-\infty}^{\infty} e^{-\tau\xi^2} d\xi = \frac{\sqrt{\pi}}{\sqrt{\tau}}$$

in which path C is from $(-\infty - i\frac{x}{2\tau})$ to $(+\infty - i\frac{x}{2\tau})$

$$\text{Eventually,} \quad h(x;t) = \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{x^2}{4\tau}} \tag{4}$$