Flux
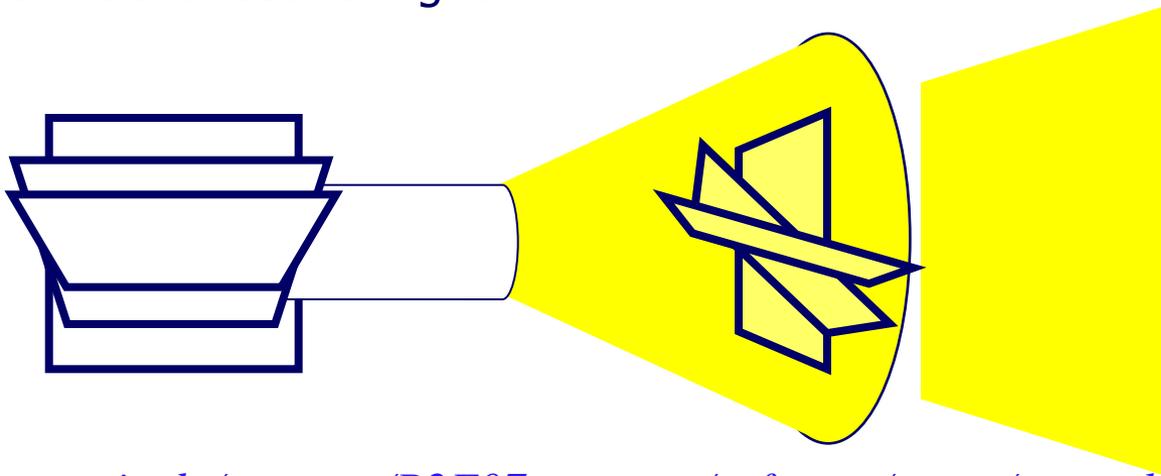
- Flux in Physics is used to two distinct ways.
- The first meaning is the rate of flow, such as the amount of water flowing in a river, i.e. volume per unit area per unit time. Or, for light, it is the amount of energy per unit area per unit time.
- Let's look at the case for light:

http://www.rpi.edu/~persap/P2F07_persans/software/gauss/examples.html

http://www.rpi.edu/~persap/P2F07_persans/software/gauss/hidden2.html

Area Vector

- Represent an area as a vector $\Delta\vec{A}$, of length equal to the area, and direction of the "outward normal" to the surface.

- The flux of light through a hole of area ΔA is proportional to the area, and the cosine of the angle between the light direction and this area vector.
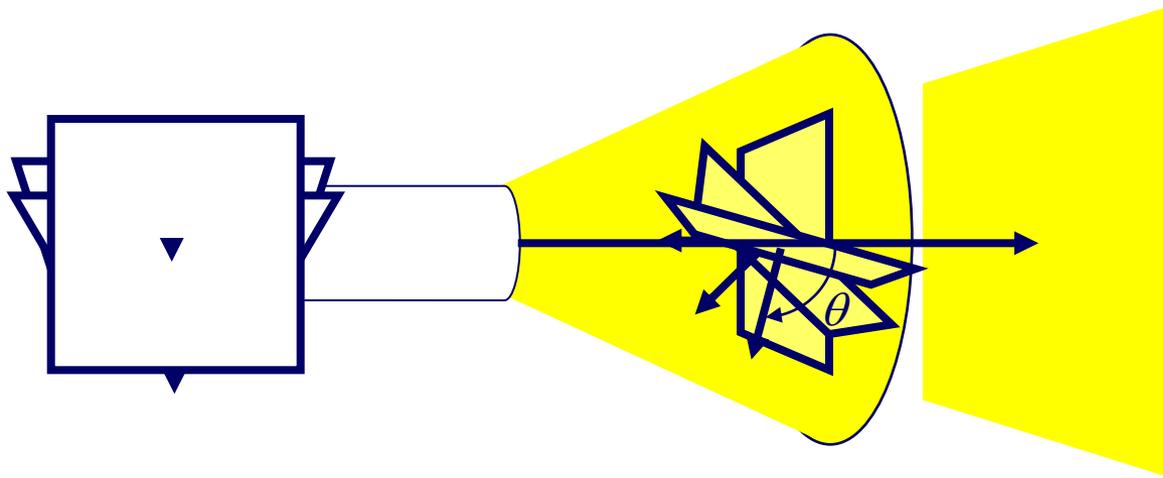
- If we use a vector $\vec{L}$ to represent the light energy per unit time, then the light out of the hole is $L\Delta A\cos\theta = \vec{L}\cdot\Delta\vec{A}$. In this case it is *negative* $(\theta > 90°)$ which means the light flux is into the hole.

Flux of Electric Field

- Like the flow of water, or light energy, we can think of the electric field as flowing through a surface (although in this case nothing is actually moving).

- We represent the flux of electric field as Φ (greek letter phi), so the flux of the electric field through an element of area ΔA is

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{A} = E\,\Delta A\cos\theta$$

- When $\theta < 90°$, the flux is positive (out of the surface), and when $\theta > 90°$, the flux is negative.

- When we have a complicated surface, we can divide it up into tiny elemental areas:

$$d\Phi = \vec{E}\cdot d\vec{A} = E\,dA\cos\theta$$
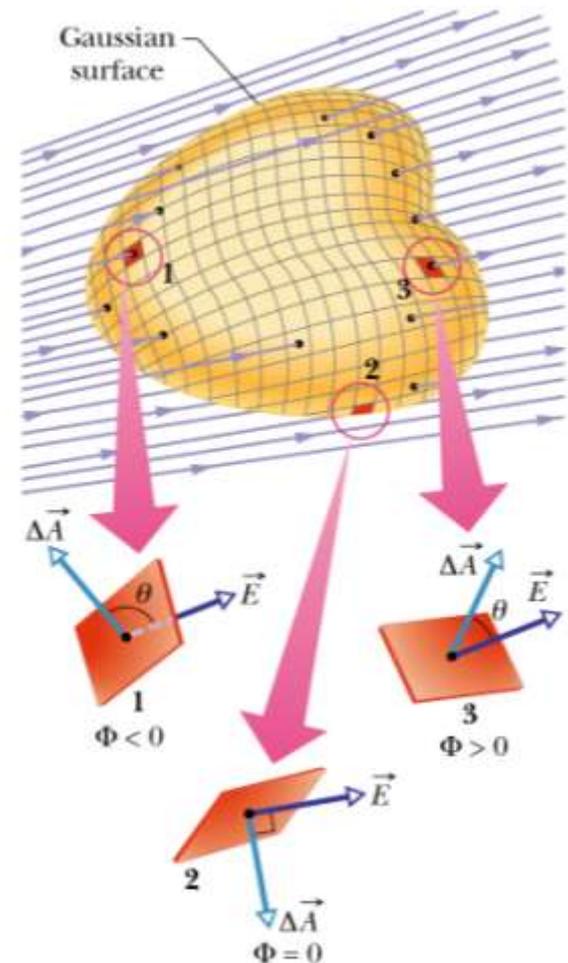
Gauss' Law

❑ We are going to be most interested in *closed* surfaces, in which case the outward direction becomes self-evident.

❑ We can ask, what is the electric flux out of such a closed surface? Just integrate over the closed surface:
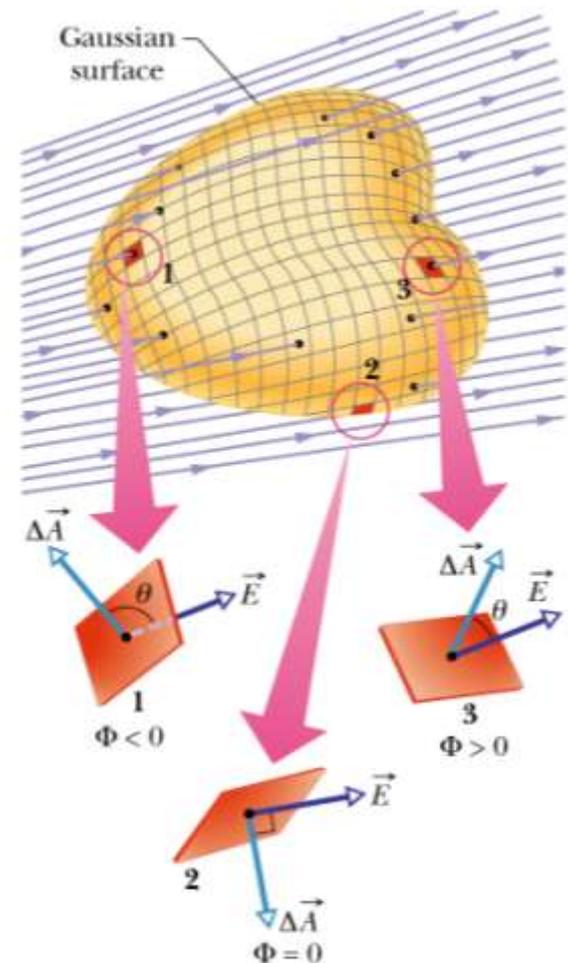
$$\Phi = \oint d\Phi = \oint \vec{E} \cdot d\vec{A}$$

Flux positive => out
Flux negative => in

❑ The $\oint$ symbol has a little circle to indicate that the integral is over a closed surface.

❑ The closed surface is called a <u>gaussian surface</u>, because such surfaces are used by Gauss' Law, which states that:
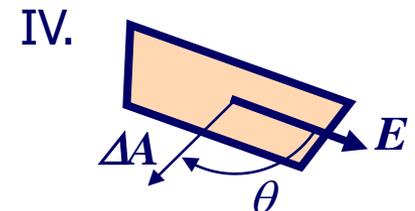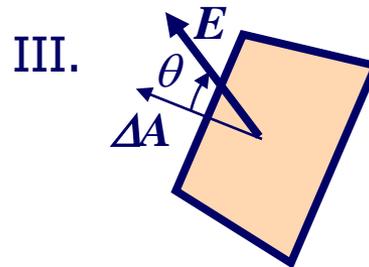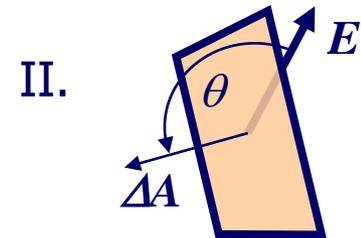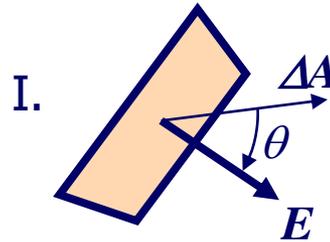
Gauss' Law

The flux of electric field through a closed surface is proportional to the charge enclosed.

Flux of Electric Field

1. Which of the following figures correctly shows a positive electric flux out of a surface element?

A. I.
B. II.
C. III.
D. IV.
E. I and III.

Mathematical Statement of Gauss' Law

❑ The constant of proportionality in Gauss' Law is our old friend ε_0.

$$\varepsilon_0 \Phi = q_{enc}$$

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc}$$

❑ Recall that I said that we would see later why Coulomb's constant is written

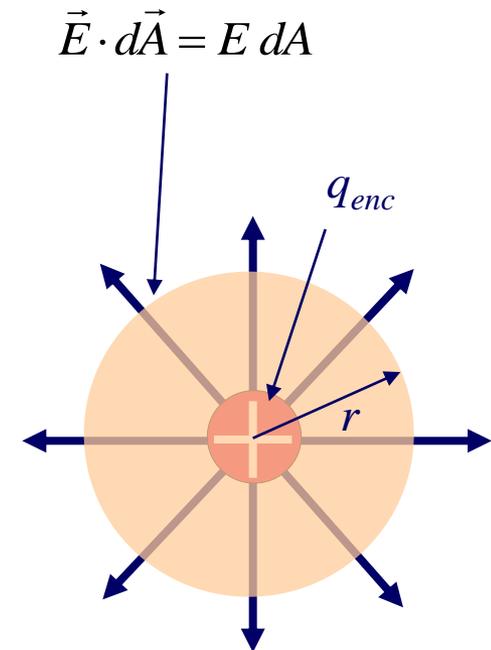$$k_E = \frac{1}{4\pi\varepsilon_0}$$?

❑ We can see it now by integrating the electric flux of a point charge over a spherical gaussian surface.

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E \oint dA = \varepsilon_0 E 4\pi r^2 = q_{enc}$$

❑ Solving for E gives Coulomb's Law.

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q_{enc}}{r^2}$$

$$\vec{E} \cdot d\vec{A} = E \, dA$$

q_{enc}

r

Example of Gauss' Law

- Consider a dipole with equal positive and negative charges.
- Imagine four surfaces S_1, S_2, S_3, S_4, as shown.
- S_1 encloses the positive charge. Note that the field is everywhere outward, so the flux is positive.
- S_2 encloses the negative charge. Note that the field is everywhere inward, so the flux through the surface is negative.
- S_3 encloses no charge. The flux through the surface is negative at the upper part, and positive at the lower part, but these cancel, and there is no net flux through the surface.
- S_4 encloses both charges. Again there is no net charge enclosed, so there is equal flux going out and coming in—no net flux through the surface.
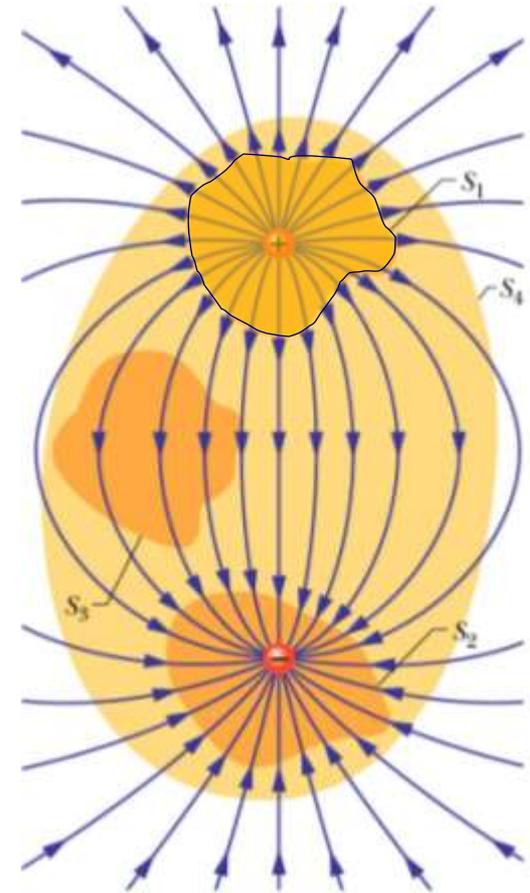
Interactive Examples

❑ Let's take a look at some interactive java applets that show the concepts.

❑ The first allows us to draw boxes or circles around both point and distributed charges, and uses the flux through those surfaces to determine the charge enclosed. Click on the link below.

   http://www.rpi.edu/~persap/P2F07_persans/software/gauss/examples.html

❑ The second applet allows us to do the same for an "unknown" charge distribution.

   *http://www.rpi.edu/~persap/P2F07_persans/software/gauss/hidden2.html*

• move mouse to see force on "test charge"
• right-click to draw field line through mouse point,
• shift right drag to draw gaussian box
• shift-ctrl right-drag to draw gaussian circle

Field At the Surface of a Conductor

$\vec{E}$

$\vec{E}_\perp$

$E_\parallel$

Copper surface

Gaussian surface
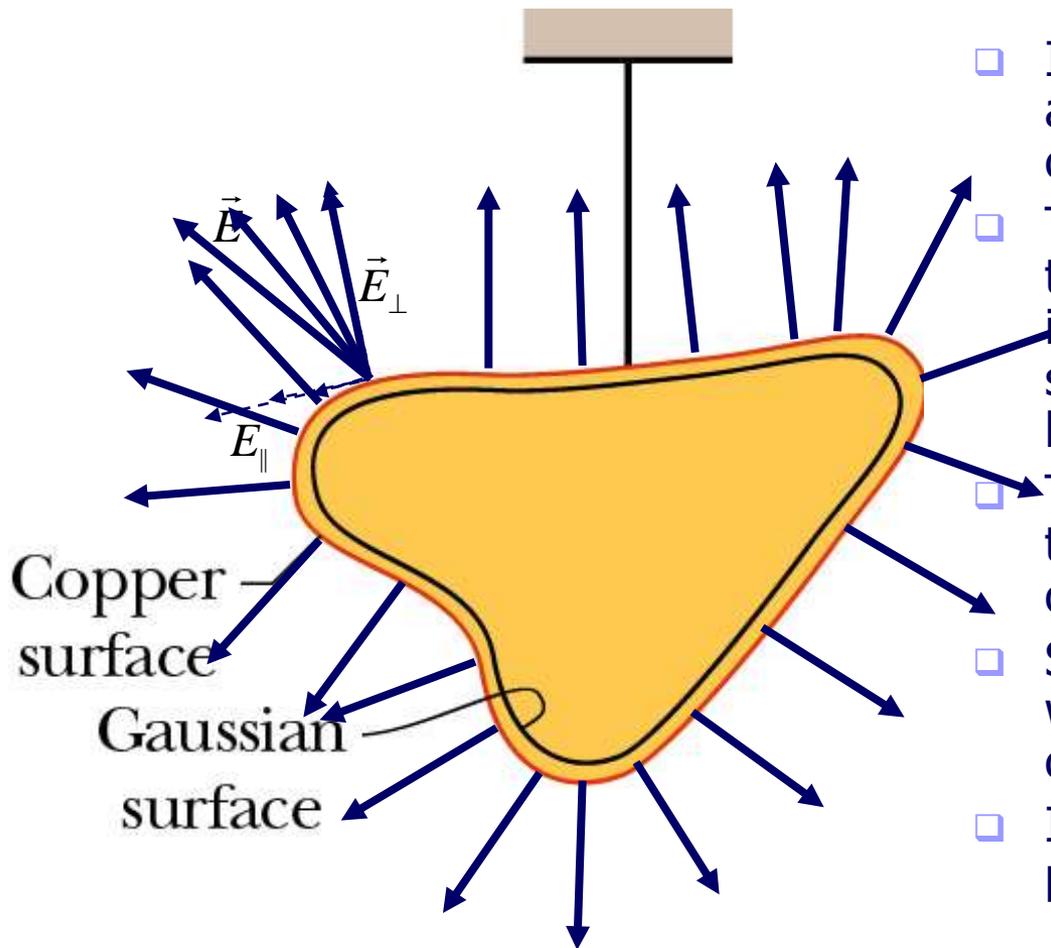
- ❑ Imagine an electric field at some arbitrary angle at the surface of a conductor.
- ❑ There is a component perpendicular to the surface, so charges will move in this direction until they reach the surface, and then, since they cannot leave the surface, they stop.
- ❑ There is also a component parallel to the surface, so there will be forces on charges in this direction.
- ❑ Since they are free to move, they will move to nullify any parallel component of E.
- ❑ In a very short time, only the perpendicular component is left.

Field Inside a Conductor

Copper
surface
Gaussian
surface

(a)

Gaussian
surface
Copper
surface

(b)
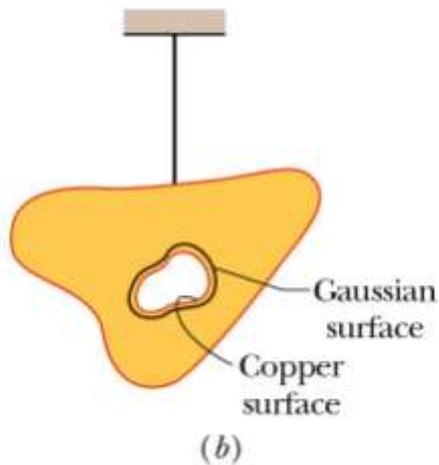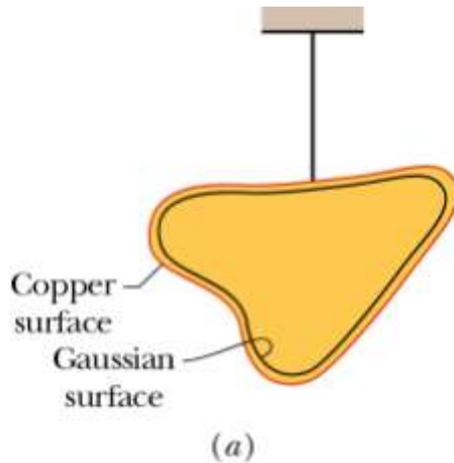
❑ We can use Gauss' Law to show that the inside of a conductor must have no net charge.

❑ Take an arbitrarily shaped conductor, and draw a gaussian surface just inside.

❑ Physically, we expect that there is no electric field inside, since otherwise the charges would move to nullify it.

❑ Since E = 0 everywhere inside, E must be zero also on the gaussian surface, hence there can be no net charge inside.

❑ Hence, all of the charge must be on the surface (as discussed in the previous slide).

❑ If we make a hole in the conductor, and surround the hole with a gaussian surface, by the same argument there is no E field through this new surface, hence there is no net charge in the hole.

Field Inside a Conductor

Gaussian surface

Copper surface
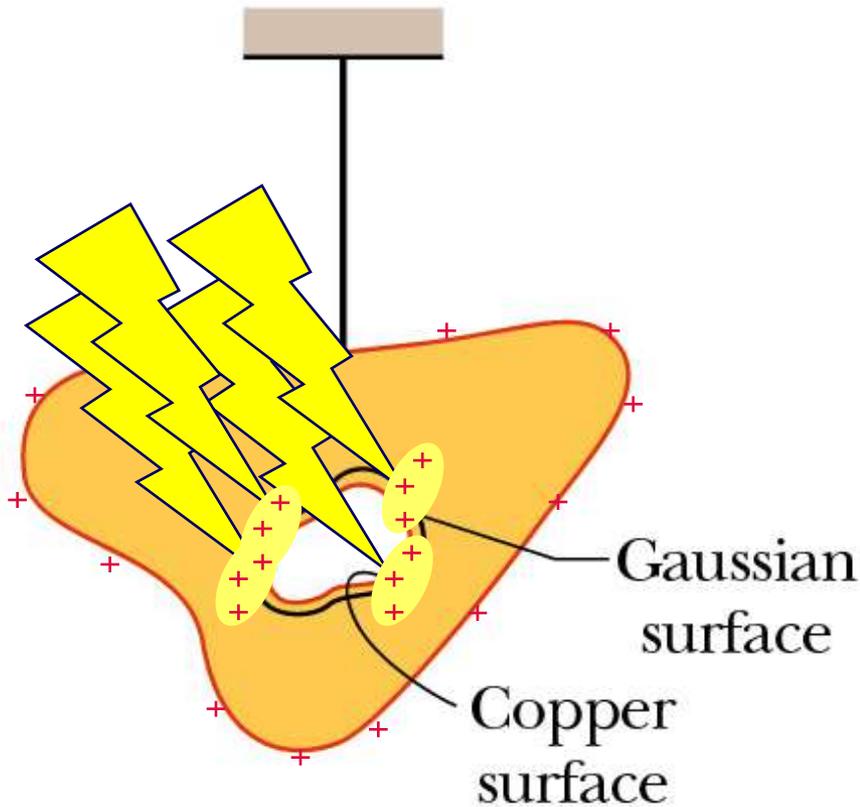
- ❑ We have the remarkable fact that if you try to deposit charge on the inside of the conductor…

- ❑ The charges all move to the outside and distribute themselves so that the electric field is everywhere normal to the surface.

- ❑ This is NOT obvious, but Gauss' Law allows us to show this!

There are two ideas here
- Electric field is zero inside conductors
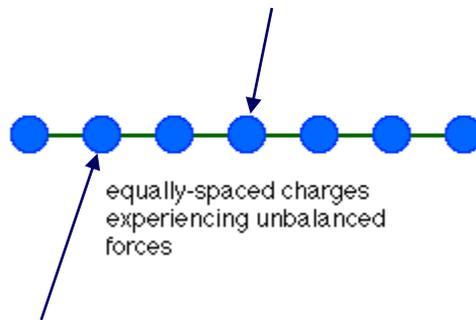- Because that is true, from Gauss' Law, cavities in conductors have E = 0

Charge Distribution on Conductors

charges collect near tip
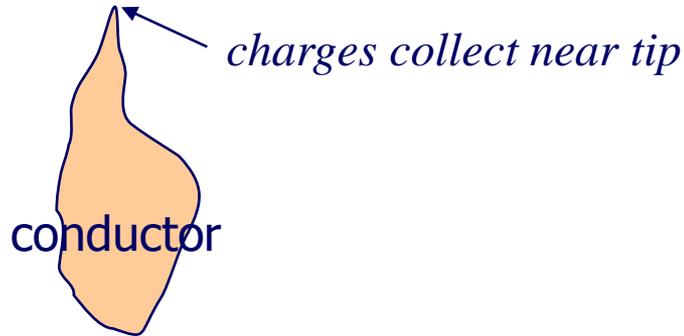
conductor

- For a conducting sphere, the charges spread themselves evenly around the surface.
- For other shapes, however, the charges tend to collect near sharp curvature.
- To see why, consider a line of charge.

equally-spaced charges experiencing balanced forces

balanced forces
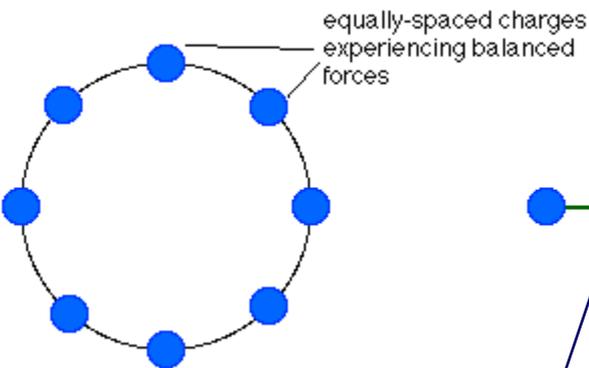
equally-spaced charges experiencing unbalanced forces

*unbalanced forces
(pushed on by one charge
from left, but by 5 charges
from right)*

unequally-spaced charges experiencing balanced forces
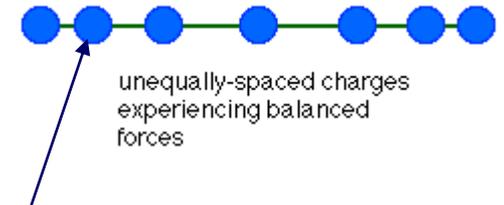
*redistributed charges
(pushed on by one nearby charge from left,
but by 5 more distant charges from right)*

A Charge Inside a Conductor

Spherical cavity

Conducting sphere

Positive point charge
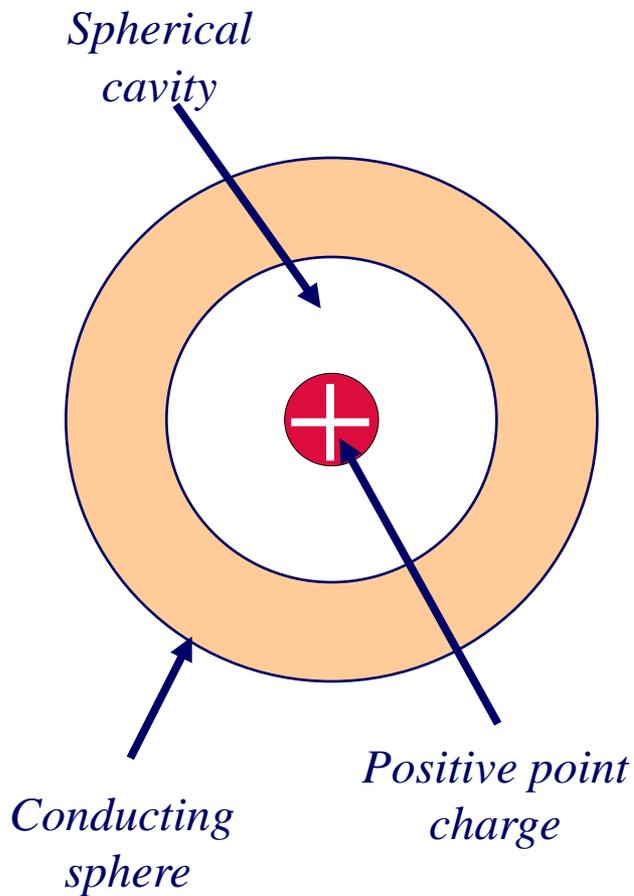
2. What will happen when we add a charge inside a conductor?

A. E field is still zero in the cavity.

B. E field is not zero in the cavity, but it is zero in the conductor.

C. E field is zero outside the conducting sphere.

D. E field is the same as if the conductor were not there (i.e. radial outward everywhere).

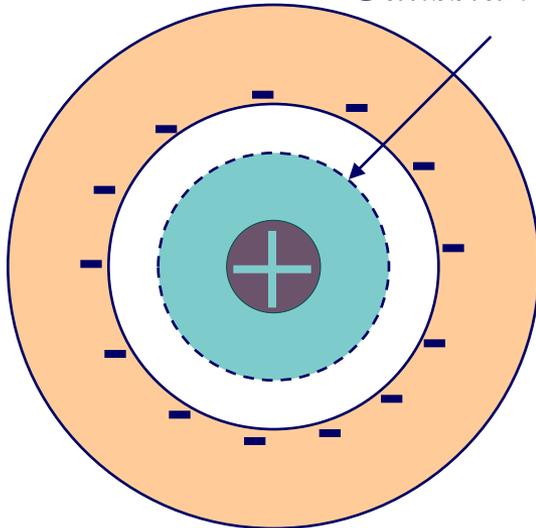E. E field is zero in the conductor, and negative (radially inward) outside the conducting sphere.

Use Gauss' Law to Find Out

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc}$$

Is $E = 0$ in the cavity?

No, because there is charge enclosed (Gauss' Law).

Gaussian Surface

Is $E = 0$ in the conductor?

Yes, because as before, if there were an electric field in the conductor, the charges would move in response (NOT Gauss' Law).
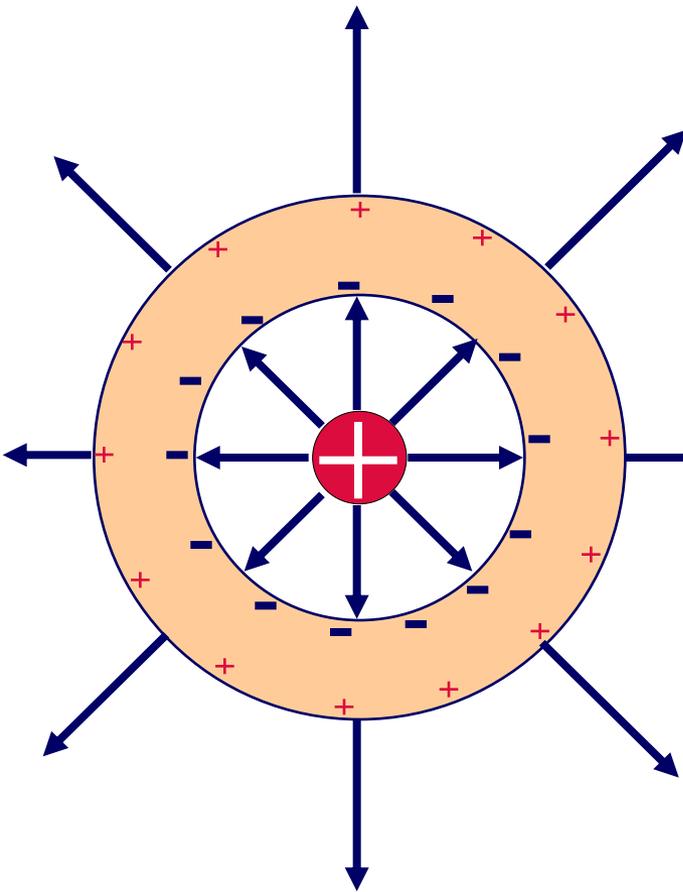
If we enlarge the gaussian surface so that it is inside the conductor, is there any net charge enclosed?

It looks like there is, but there cannot be, because Gauss' Law says $E = 0$ implies $q_{enc} = 0$!

How do we explain this?

There must be an equal and opposite charge induced on the inner surface.

E Field of Charge In Conductor

This negative charge acts with the inner charge to make the field radial inside the cavity.

This negative charge cannot appear out of nowhere. Where does it come from?

It comes from the outer surface (electrons drawn inward, attracted to the positive charge in the center). Therefore, it leaves positive charge behind.

The net positive charge that appears conductor is exactly the same as the original charge in the center, so what do the field lines look like?

By spherical symmetry, the positive shell of charge acts like a point charge in the center, so field is the same as the field of the original point charge.

E Field of Charge In Conductor

This negative charge acts with the inner charge to make the field radial inside the cavity.
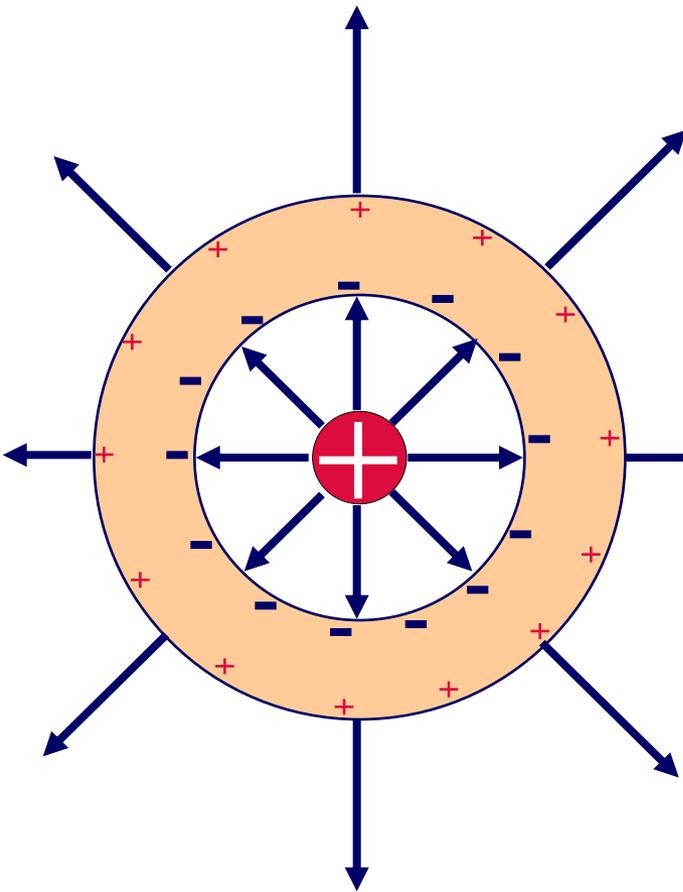
This negative charge cannot appear out of nowhere. Where does it come from?

It comes from the outer surface (electrons drawn inward, attracted to the positive charge in the center). Therefore, it leaves positive charge behind.

The net positive charge that appears conductor is exactly the same as the original charge in the center, so what do the field lines look like?

By spherical symmetry, the positive shell of charge acts like a point charge in the center, so field is the same as the field of the original point charge.

Back to the Previous Question

Spherical cavity

Conducting sphere

Positive point charge
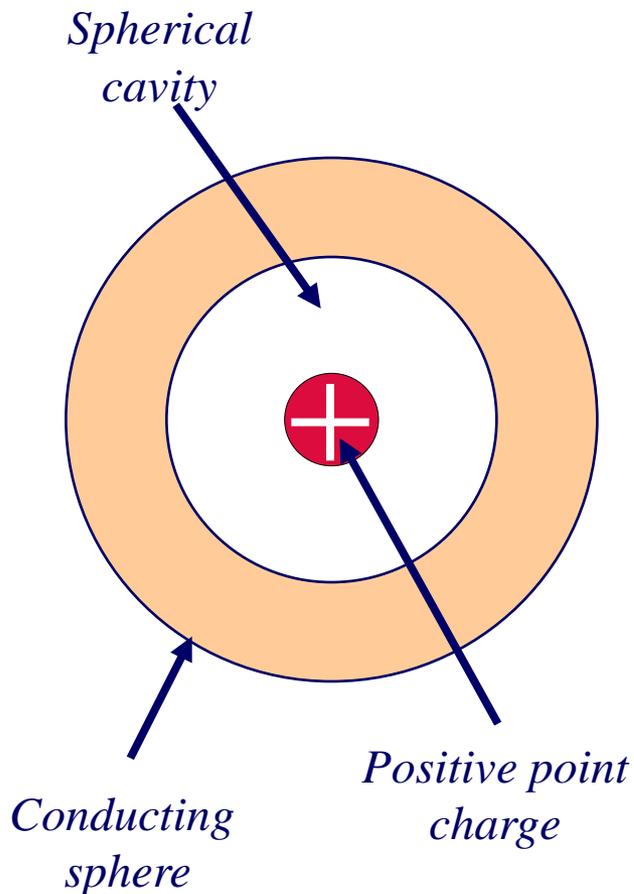
3. What will happen when we add a charge inside a conductor?

A. E field is still zero in the cavity.

B. E field is not zero in the cavity, but it is zero in the conductor.

C. E field is zero outside the conducting sphere.

D. E field is the same as if the conductor were not there (i.e. radial outward everywhere).

E. E field is zero in the conductor, and negative (radially inward) outside the conducting sphere.

E Field of Charge In Conductor
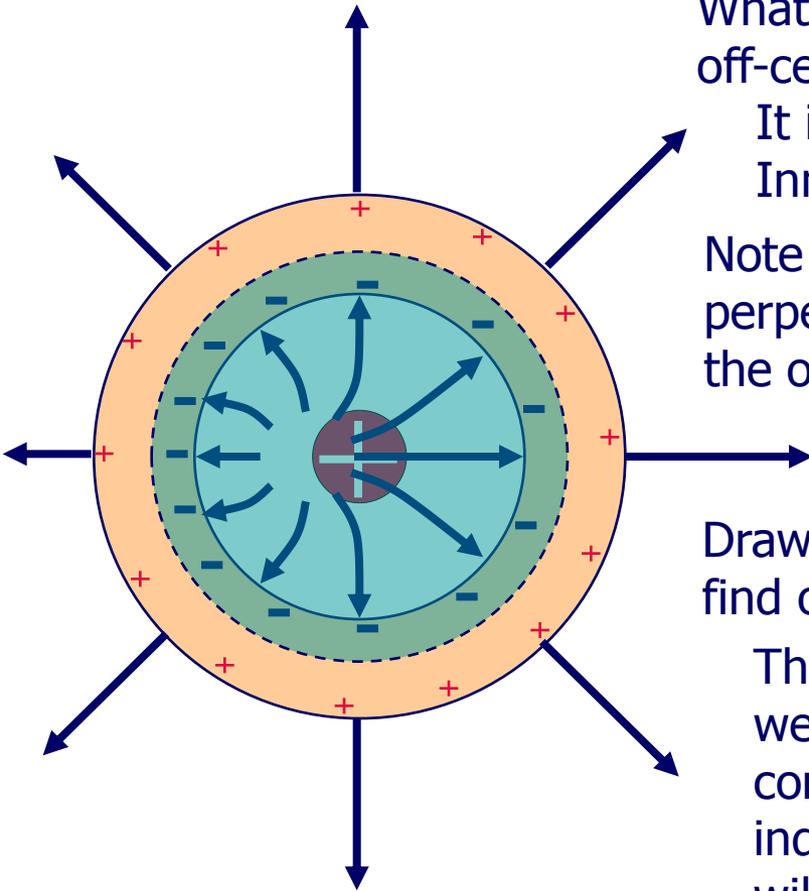
What happens when we move the inner charge off-center?

It induces an off-center charge distribution on the Inner wall.

Note that the field lines distorted, so they remain perpendicular to the inner wall. What happens to the outer positive charge distribution?

Draw a gaussian surface inside the conductor to find out.

The net charge enclosed is zero, so E = 0, which we already knew because it is inside the conductor. The inner charge is shielded by the induced charge distribution, so the outer charges will be evenly distributed.

Field Lines and Conductors

4. Why cannot the drawing of the conductor and its field lines be correct?

    A. The field lines are in both directions, coming to and leaving from the conductor.

    B. One of the field lines loops around, with both end points on the conductor.

    C. The field lines are not all perpendicular to the surface of the conductor.

   D. All of the above.

    E. B and C only.
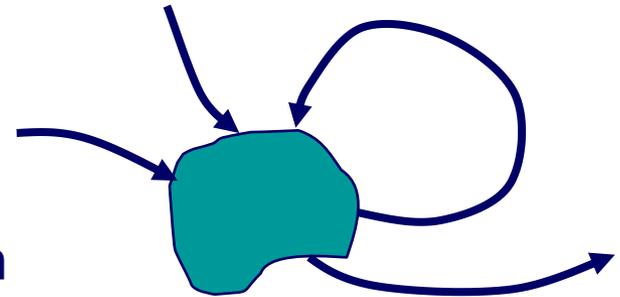
Other Geometries

❑ Always use the symmetry of the problem to determine what shape to make your gaussian surface.

❑ Here is a plate (plane) geometry, where the charges are evenly distributed on a flat surface. If the total charge on the plate is Q, and the plate has a total area A_{tot}, then the *surface charge density* is

$$\sigma = Q / A_{\text{tot}} \quad \text{C/m}^2$$

❑ The E field is everywhere perpendicular to the plate (again, if not, the charges will move until the part parallel to the surface is nullified). What is $\vec{E}$?

❑ Use a gaussian surface that is parallel to $\vec{E}$ on the sides (so no flux through side surfaces), and closes inside the conductor (no flux through that end).

❑ On the remaining side, the area vector $\vec{A}$ is parallel to the E field, so $\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 EA = \sigma A$ or $E = \dfrac{\sigma}{\varepsilon_0}$

(a)

$\vec{E} = 0$

Conducting Surface
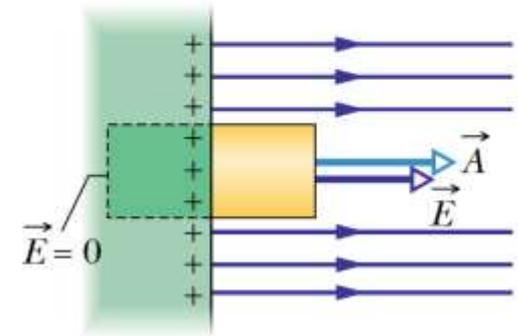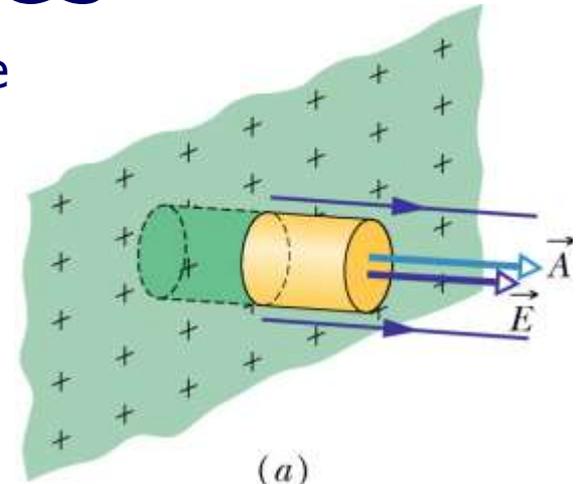
Line of Charge

- ❏ In the previous chapter, we calculated the E field on the axis of a line of charge, but with Gauss' Law we can now handle finding E off the line axis.

- ❏ Here is a line geometry, where the charges are evenly distributed on a long line. If the total charge on the line is Q, and the line has a total length L_{tot}, then the _linear charge density_ is

$$\lambda = Q / L_{tot} \quad C/m$$

- ❏ The E field is everywhere perpendicular to the line (again, if not, the charges will move until the part parallel to the line is nullified).

- ❏ Use a cylindrical gaussian surface that is parallel to $\vec{E}$ on the top and bottom (so no flux through those surfaces), and is perpendicular to $\vec{E}$ elsewhere.

- ❏ The area vector $d\vec{A}$ is parallel to $\vec{E}$, and the total area is $2\pi rh$ so

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E 2\pi rh = \lambda h \qquad \text{or} \qquad E = \frac{\lambda}{2\pi\varepsilon_0 r}$$
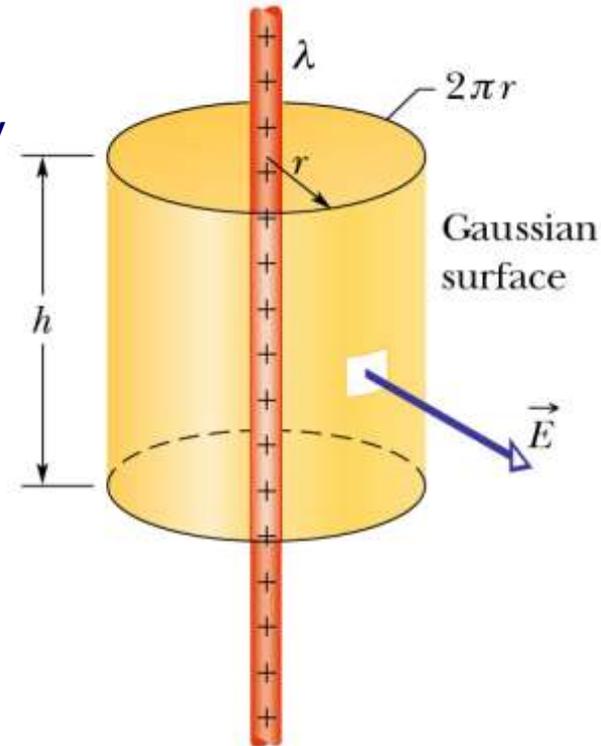
Line of Charge

Symmetry

5.   A charged sphere has an electric field just like that of a point charge of the same total charge, located at its center. What is the electric field of a long conducting cylinder like?

A.   Also like that of a point charge at its center.
B.   Like a circular ring of charge at its center.
C.   Like a line charge along the cylinder axis.
D.   Cannot tell from the information given.

Uniform Sphere of Charge

❑ Here is a spherical geometry, where the charges are evenly distributed throughout the volume. If the total charge in the sphere is Q, and the sphere has a radius R, then the *volume charge density* is

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \ \ \text{C/m}^3$$

Enclosed charge is q — Gaussian surface
ρ
R
r
(a)

❑ By symmetry, the E field is everywhere radial from the center of the sphere.

❑ Use a spherical gaussian surface, which is perpendicular to $\vec{E}$ everywhere.

❑ The area vector $\vec{A}$ is parallel to $\vec{E}$, and the total area is $4\pi r^2$ so when the gaussian surface radius is $r < R$, then
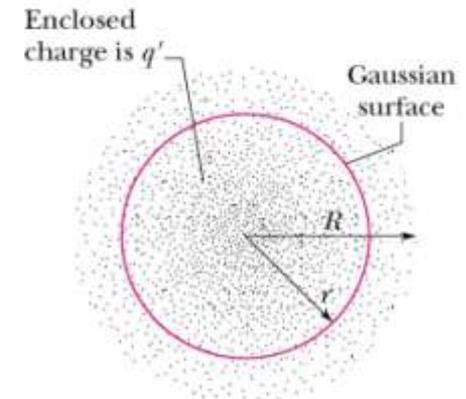
$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E 4\pi r^2 = \rho \frac{4}{3}\pi r^3 \quad \text{or} \quad E = \frac{\rho}{3\varepsilon_0} r$$

Enclosed charge is q' — Gaussian surface
R
r

❑ When $r > R$, then the charge enclosed is just Q, so

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E 4\pi r^2 = Q \quad \text{or} \quad E = \frac{Q}{4\pi\varepsilon_0 r^2}$$
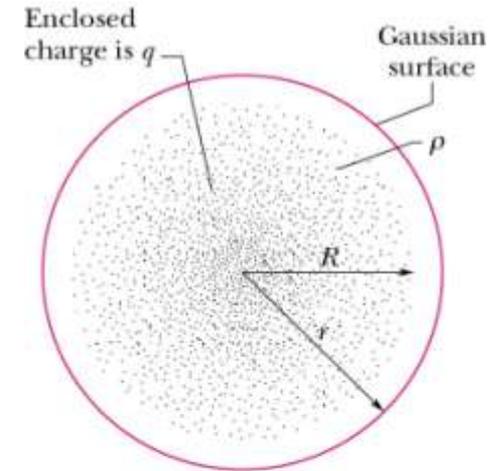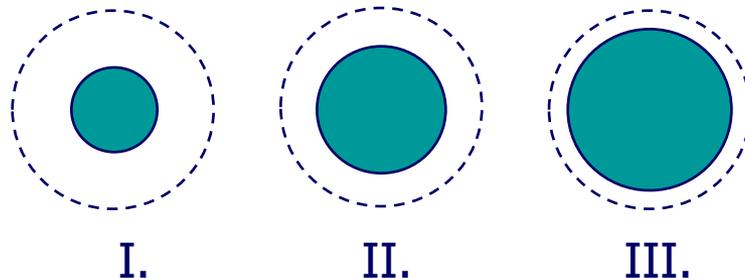
Coulomb's Law again

Field Lines and Conductors

6. The drawing shows three cylinders in cross-section, each with the same total charge. Each has the same size cylindrical gaussian surface (again shown in cross-section). Rank the three according to the electric field at the gaussian surface, greatest first.

   A. I, II, III
   B. III, II, I
   C. All tie.

I.   II.   III.

Nonconducting Sheet

❑ A nonconducting sheet with a uniform surface charge density has the same geometry as for the conducting plate, so use the same gaussian surface.

❑ The only difference is that now one end cannot close in a conductor, so there is electric flux out both ends.

❑ As you may expect, the resulting electric field is half of what we got before.

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 2EA = \sigma A$$

$$E = \frac{\sigma}{2\varepsilon_0}$$
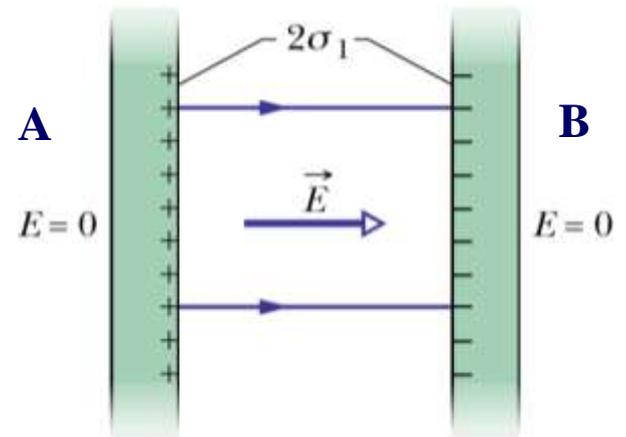
Sheet of Charge

(a)
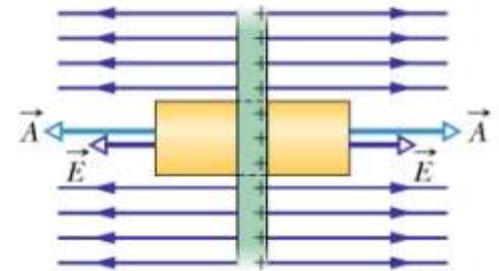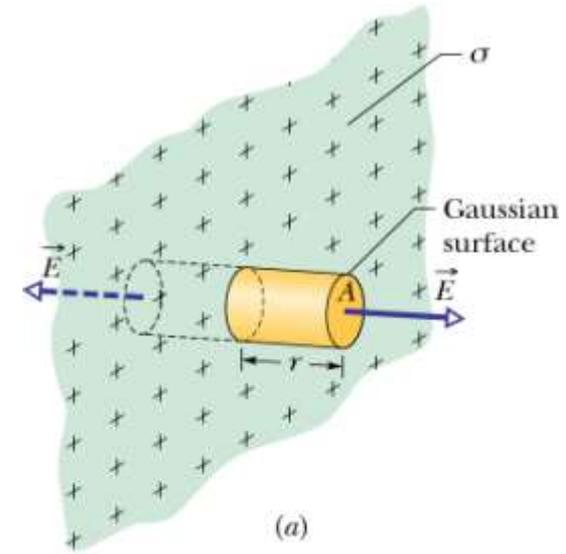
Two Parallel Conducting Plates

- When we have the situation shown in the left two panels (a positively charged plate and another negatively charged plate with the same magnitude of charge), both in isolation, they each have equal amounts of charge (surface charge density σ) on both faces.

- But when we bring them close together, the charges on the far sides move to the near sides, so on that inner surface the charge density is now 2σ.

- A gaussian surface shows that the net charge is zero (no flux through sides — dA perpendicular to E, or ends — $E = 0$). $E = 0$ outside, too, due to shielding, in just the same way we saw for the sphere.
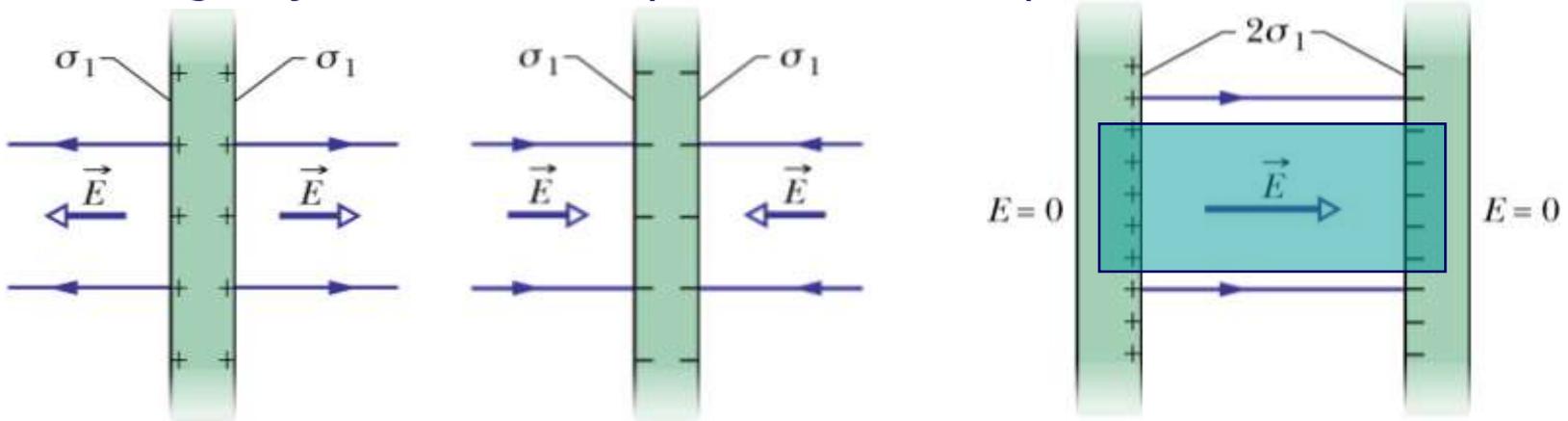
Parallel Conducting Plates

7. The sketch shows the case of two parallel conducting plates of equal and opposite charge. If the positively charged plate has a greater charge than the absolute value of the charge of the negatively charged plate, what would be the direction of E in the two regions A and B?

A. To the right in region A, and to the left in region B

B. To the left in region A, and to the right in region B

C. To the left in both regions.

D. To the right in both regions.

E. The field remains zero in both regions.

Two Parallel Nonconducting Sheets

❑ The situation is different if you bring two nonconducting sheets of charge close to each other.

❑ In this case, the charges cannot move, so there is no shielding, but now we can use the principle of superposition.

❑ In this case, the electric field on the left due to the positively charged sheet is canceled by the electric field on the left of the negatively charged sheet, so the field there is zero.

❑ Likewise, the electric field on the right due to the negatively charged sheet is canceled by the electric field on the right of the positively charged sheet.
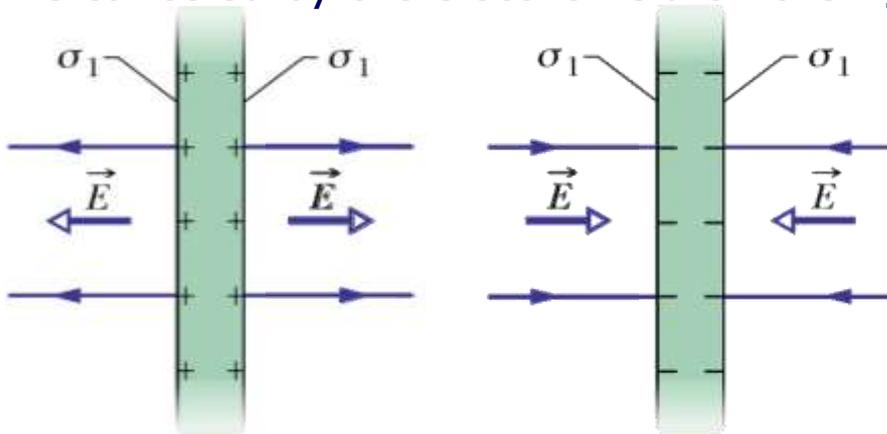
❑ The result is much the same as before, with the electric field in between being twice what it was previously.

Spherical Symmetry
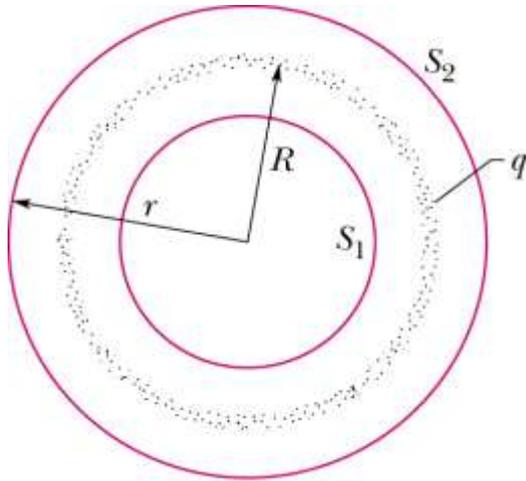
Spherical shell

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \quad (r \geq R)$$

$$E = 0 \quad (r < R)$$

We earlier said that a shell of uniform charge attracts or repels a charged particle that is outside the shell as if the shell's charge were concentrated at the center of the shell. We can now prove this using Gauss' Law.

We also said that a shell of uniform charge exerts no electrostatic force on a charged particle that is located inside the shell. Again, Gauss' Law can be used to prove this.

Summary

- Electric flux is the amount of electric field passing through a closed surface.
- Flux is positive when electric field is outward, and negative when electric field is inward through the closed surface.
- Gauss' Law states that the electric flux is proportional to the net charge enclosed by the surface, and the constant of proportionality is ε_0. In symbols, it is

$$\varepsilon_0 \Phi = q_{enc}$$

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc}$$

- There are three geometries we typically deal with:

Geometry	Charge Density	Gaussian surface	Electric field
Linear	$\lambda = q/L$	Cylindrical, with axis along line of charge	$E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$ — Line of Charge
Sheet or Plane	$\sigma = q/A$	Cylindrical, with axis along E.	$E = \dfrac{\sigma}{\varepsilon_0}$ — Conducting $\qquad E = \dfrac{\sigma}{2\varepsilon_0}$ — Nonconducting
Spherical	$\rho = q/V$	Spherical, with center on center of sphere	$E = \dfrac{q}{4\pi\varepsilon_0 r^2}$ — $r \geq R$ $\qquad E = \left(\dfrac{q}{4\pi\varepsilon_0 R^3}\right)r$ — $r < R$

Summary, cont'd

- The electric field is zero inside a conductor.
- The electric field is zero inside a cavity within a conductor, unless there is a charge inside that is not in contact with the walls.
- The electric field at the surface of a conductor is always perpendicular to that surface.
- Note, none of this is true for insulators.