King Saud University

College of Sciences
Department of Mathematics

Semester II (1427/1428)

PH. D. COMPREHENSIVE EXAMINATION
(Analysis)

Answer All Questions

Time : 3 hours
Number of pages : 3

Section A

Question 1.

(a) Let $\mathcal{B}(X,Y)$ be the set of all continuous linear operators from X into Y, where X and Y are normed spaces.
(i) Prove that $\mathcal{B}(X,Y)$ is a normed space.
(ii) Prove that if Y is complete, then $\mathcal{B}(X,Y)$ is complete.

(b) Let φ be a given summable function. Prove that the functional defined on $\mathcal{C}[a,b]$ by :

$$f(x) = \int_b^a \varphi(t)x(t)dt$$

is linear and that :

$$\|f\| = \int_a^b |\varphi(t)|\, dt$$

Question 2.

(a) Let U be a bounded linear operator on a Hilbert space H. Consider the functional :

$$f(x) = (U(x), y),$$

where y is a fixed element of H.
(i) Show that f is linear and bounded.
(ii) Give an equation defining the adjoint U^* of U.

(b) In part (a), let $H = L^2(a,b)$ and $U(x) = \int_a^b k(s,t)x(t)dt$.
(i) Check that the adjoint operator U^* of U is also an integral operator.
(ii) When does the operator U is self-adjoint in view of the expression for U^*.

Section B

Question 1.

Let $(X, \mathcal{M}, \mu)$ be a measure space with μ is non-negative measure. Let $f : X \to \mathbb{C}$ be a measurable function such that $\mu(\{x \in X, f(x) \neq 0\}) > 0$. For $p \in [1, \infty)$, consider :

$$\varphi(p) = \int_X |f|^p \, d\mu \qquad \text{and} \qquad J = \{p \in [1, \infty), \varphi(p) < \infty\}.$$

(a) Show that $\varphi(p) > 0$ for all $p \in J$.

(b) Let $p_0, p_1 \in J$ with $p_0 \leq p_1$.
(i) For $\theta \in [0, 1]$, prove that $p_\theta = (1-\theta)p_0 + \theta p_1$ is an element of J.
(Hint. use Hölder inequality).
(ii) Deduce that $\ln \varphi$ is convex on J.

(c) Suppose that there exists $r_0 \in [1, \infty)$ such that $f \in L^{r_0}(\mu) \cap L^\infty(\mu)$.
(i) Show that $f \in L^p(\mu)$ for $p \in [r_0, \infty)$.
(ii) Prove that $\lim_{p\to\infty} \|f\|_{L^p(\mu)} = \|f\|_{L^\infty(\mu)}$.

Question 2.

(a) Consider the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. For all $A \subset \mathbb{N}$, define the function :

$$\nu_j(A) = Card(A \cap [j, \infty)).$$

(i) Show that ν_j is a non-negative measure defined on $\mathcal{P}(\mathbb{N})$.
(ii) Prove that for all $A \subset \mathbb{N}$, $\nu_j(A) \geq \nu_{j+1}(A)$.

(b) Let :

$$\nu(A) = \inf_{j\in\mathbb{N}} \nu_j(A).$$

(i) Prove that $\nu(\mathbb{N}) = \infty$ and for all $k \in \mathbb{N}$, $\nu(\{k\}) = 0$.
(ii) Deduce that ν is not a measure on $\mathbb{N}$.

Section C

Question 1.

(a) Let f be an analytic function inside and on a simple closed curve C except for a pole $z = \alpha$ of order p inside C. Suppose also that inside C, f has only one zero $z = \beta$ of order n and no zeros on C. Prove that :

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = n - p.$$

(b) Let $P_n(t) = (\frac{1}{2^n n!}) \frac{d^n}{dt^n}(t^2 - 1)^n, \quad n = 0, 1, 2, 3, ...$
Prove that if C is a circle with center at t and radius $\sqrt{|t^2 - 1|}$, then :

$$P_n(t) = \frac{1}{2\pi} \int_0^{2\pi} (t + \sqrt{t^2 - 1} \ \cos\theta)^n d\theta.$$

Question 2.

(a) Let $\{f_n(z)\}$, $n = 1, 2, 3, ...$ be a sequence of analytic functions in a region R. Suppose that :

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

is uniformly convergent in R.
Prove that F is analytic in R.

(b) Find a function which is harmonic in the upper half of the z-plane which takes the following values on the x-axis given by :

$$G(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$