King Saud University 1428–1429
Department of Mathematics

Mid Term Examination

M- 690 (2 hours)

Problem A

Let E be a normed space, E' be the topological dual space of E and C be a convex subset of E. For all $c \in C$, we consider the following two sets :

$$N_C(c) = \{f \in E' : f(c) \geq f(c')\} \quad ; \quad T_C(c) = \overline{\{t(c' - c) : t \geq 0, c' \in C\}}$$

1) Prove that for all $c \in C$. $N_C(c)$ and $T_C(c)$ are nonempty convex cones subsets of E.
2) Prove that $T_C(c) = (N_C(c))^0$ and $(T_C(c))^0 = N_C(c)$.
3) Verify that if c is in the interior of C, then $N_C(c) = \{0_{E'}\}$ and $T_C(c) = E$.
4) We suppose now that E is a Hilbert space. Prove that :

$$\forall x \in E, \quad x - proj_C(x) \in N_C(proj_C(x))$$

Problem B

Let C be a nonempty convex subset (not reduced to one element) of a finite dimensional space.

Part A. Let p be the dimension of the direction F of the affine sub-space generated by C (aff(C)). In this first part, we want to prove that C is homeomorphic to the closed unit ball of $\mathbb{R}^p$. We note that, by translation, C will be homeomorphic to a nonempty convex compact subset D of F and 0 is in the relatif interior of D. Hence we can define the jauge function j_D of D and then $D = \{x \in F : j_D(x) \leq 1\}$.

Let $B = (u_1, u_2, ..., u_p)$ be a basis of F and consider the following norm on F :

$$\forall x = (\xi_1, \xi_2, ..., \xi_p)_B \quad \|x\| = \sqrt{\sum_{j=1}^{p} {\xi_i}^2}$$

We denote by ψ the linear isomorphism between F and $\mathbb{R}^p$ and if $x \neq 0$, we define :

$$\varphi(x) = \frac{j_D(x)}{\|\psi(x)\|}\psi(x),$$

and if $x = 0$, $\varphi(x) = 0$.

1) Show that φ is continuous.

2) Prove that φ is one to one.

3) Deduce that D is homeomorphic to the closed unit ball of $\mathbb{R}^p$.

Part B. Prove that if $f : C \to C$ is a continuous function, then f has a fixed point.

Problem C

Let X ba metric space, Y be a normed space and $F : X \to Y$ is a correspondence.

1) Prove that if F has open inverse image, then the correspondence coF has also open inverse image.

2) We suppose that Y is finite dimensional normed space. Prove that if F is u.s.c. and $F(X)$ is contained in a fixed compact set K of Y, then the correspondence coF has closed graph.