King Saud University 1428–1429
Department of Mathematics

Mid Term Examination

M-690 (2 hours)

Problem 1

Let E be a normed space and $f : E \to \mathbb{R}$ be a convex l.s.c function.

Part A

1) For fixed $x_0 \in E$ and $\lambda_0 < f(x_0)$, prove that there exist $(g, k, \alpha) \in E' \times \mathbb{R} \times \mathbb{R}$, $(g, k) \neq (0, 0)$, such that :

$$g(x_0) + \lambda_0 k < \alpha < g(x) + \lambda k \qquad \forall (x, \lambda) \in Epi(f)$$

2) Prove that $k > 0$.
3) Deduce that f is minorized by an affine function.

Part B

Use results of Part A to prove that $f^{**} = f$, where f^* denotes the conjugate of f.

Problem 2

(1) Let E be a normed space, f be a sublinear functional on E.
Prove that if $Dom(f^*) \neq \emptyset$, then f^* is equal to :

$$f^*(g) = \chi_{Dom(f^*)} = \begin{cases} 0 & if \quad g \in Dom(f^*) \\ +\infty & if \quad g \notin Dom(f^*) \end{cases}$$

Deduce that $(\| \, . \, \|)^* = \chi_{\bar{B}_{E'(0,1)}}$.

(2) Prove that if C is a convex subset of a normed space E, then :

$$(\chi_C)^* = \sigma_C$$

where $\sigma_C(x) = \sup_{y \in C} \langle x, y \rangle$.

(3) Deduce that : $(\chi_{\bar{B}_E(0,1)})^* = \| \, . \, \|_{E'}$.

Problem 3

Solve the following optimization problem :

$$(\mathcal{P}) = \begin{cases} \min f(x_1, x_2) = 2x_1^2 + 2x_1 x_2 + x_2^2 - 10x_1 - 10x_2 \\ \\ \qquad x_1^2 + x_2^2 \leq 5 \\ \qquad 3x_1 + x_2 \leq 6 \end{cases}$$