A Generalized KKMF Principle [1]

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We present in this paper a generalized version of the celebrated Knaster-Kuratowski-Mazurkiewicz-Fan's Principle on the intersection of a family of closed sets subject to a classical geometric condition and a weakened compactness condition. The fixed point formulation of this generalized principle extends the Browder-Fan fixed point theorem to set-valued maps of non-compact convex subsets of topological vector spaces.

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1. INTRODUCTION

Using the Sperner Lemma, Knaster, Kuratowski, and Mazurkiewicz established in 1929 the following result [17]: *let* X *consist of the set of vertices of a simplex in* $I\!R^n$ *and let* $F : X \to I\!R^n$ *be a set-valued map with non-empty compact values. Assume that* F *verifies the condition:*

$$\forall \{x_1, ..., x_k\} \subset X, conv(\{x_1, ..., x_k\}) \subset \bigcup_{i=1}^{k} F(x_i). \tag{1}$$

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

In 1961, Ky Fan [9] significantly extended this result to:

X an *arbitrary subset of a Hausdorff topological vector space,* and
F a map with *only one compact value*, the others being all closed, and satisfying the geometric condition (1).

The significance of this result was illustrated by numerous applications to the solvability of nonlinear problems (see e.g. Ky Fan [9], [10], [11], Dugundji-Granas [6], [7], Florenzano [12], Lassonde [18], Park [19], and references therein) and by a vigorous production of extensions and generalizations.

Using the terminology of Dugundji-Granas [6], we shall call *KKM maps* those set-valued maps with values in a vector space satisfying condition (1). We shall also refer to the Ky Fan's generalization of the KKM theorem as the *KKMF Principle.*

The depth of the KKMF Principle is perhaps better illustrated by the fact that it is equivalent to the more widely known Brouwer fixed point theorem as well as to its generalizations to infinite dimensions (fixed point theorem of Schauder-Tychonov) and to set-valued maps (fixed point theorem of Kakutani-Fan).

We are particularly interested in this paper with the (equivalent) fixed point formulation of the KKMF Principle, referred to as the Browder-Fan fixed point theorem:

every self set-valued map Φ *with non-empty convex values and open fibers of a convex compact subset* X *of a Hausdorff topological vector space admits a fixed point.*

A number of papers addressed the issue of weakening the compactness hypotheses in the KKMF Principle and the Browder-Fan fixed point theorem, replacing it by a "coercivity" type condition (see e.g. [11], [2], [3], [18], etc...)[2]. Noteworthy weaker compactness conditions are as follows:

- in the formulation of the KKMF Principle (see [11]):

[2] We draw the reader's attention to the fact that weakening the condition "X *compact*" to "$\Phi(X) \subset K$ *compact* $\subset X$" in the Browder-Fan theorem is still an open problem.

$$\exists X_0 \text{ contained in a compact convex } C \text{ of } X, \\ \text{such that } K = \bigcap_{x \in X_0} F(x) \text{ is compact.} \tag{2}$$

- in the formulation of the Browder-Fan Theorem (see [2]):

$$\exists K \text{ compact } \subset X, \exists C \text{ compact convex } \subset X, \\ \text{such that } \Phi(x) \cap C \neq \emptyset, \forall x \in X \setminus K. \tag{3}$$

The purpose of this note is to go a step further along the direction of conditions (2) and (3) with the "coercing" pair (C, K) replaced by an arbitrary family of pairs $\{(C_i, K_i)\}_{i \in I}$. The corpus of the paper consists of three parts. The definitions of coercing families in the contexts of the KKMF Principle and the Browder-Fan Theorem are given in Section 2 together with examples related to complementarity problems and to viability theory. The main existence results are described in Section 3. They include generalizations of the KKMF Principle and the Browder-Fan fixed point theorem.

Throughout the paper, vector spaces are real or complex and topological (vector) spaces are assumed to be Hausdorff. The convex hull of a subset A of a vector space is denoted by $conv(A)$. The closure of a subspace A of a topological space is denoted as usual by $cl(A)$.

Set-valued maps, simply called *maps,* are represented by capital letters, F, G, Φ, Ψ, ... The *fibers* of a map $\Phi : Y \to X$ are the inverse images $\Phi^{-1}(x) := \{y \in Y : x \in \Phi(y)\}, x \in X$.

2. COERCIVITY FOR MAPS

We are concerned in this section with relaxed compactness conditions used to prove the solvability of nonlinear problems on unbounded domains, particularly in the context of the KKMF Principle or its equivalent fixed point formulation.

Definition 2.1. *Consider a subset X of a topological vector space and a topological space Y. A family $\{(C_i, K_i)\}_{i \in I}$ of pairs of sets is said to be coercing for a map $F : X \to Y$ if and only if:*

(i) For each $i \in I$, C_i is contained in a compact convex subset of X, and K_i is a compact subset of Y;

(ii) For each $i, j \in I$, there exists $k \in I$ such that $C_i \bigcup C_j \subseteq C_k$;

(iii) For each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in C_k} F(x) \subseteq K_i$.

Remark 2.2. *Our terminology is justified by the fact that (iii) in Definition 2.1 above is satisfied if and only if the "dual" map $\Phi : Y \to X$ of F, defined by $\Phi(y) = X \setminus F^{-1}(y), y \in Y$, verifies:*

$$\forall i \in I, \exists k \in I, \forall y \in Y \setminus K_i, \Phi(y) \cap C_k \neq \emptyset. \tag{4}$$

This is obviously a coercivity type condition that imposes suitable controls on the operator Φ outside of compact subsets of its domain. Whenever condition (4) is verified with a family $\{(C_i, K_i)\}_{i\in I}$ satisfying conditions (i) and (ii) of Definition 2.1, we shall also say that the family $\{(C_i, K_i)\}_{i\in I}$ is coercing for the map Φ.

REMARK 2.3. *In case I is a singleton, the notion of a coercing family was used by Ky Fan in [11]. Condition (4) again when I reduces to a singleton, appeared in this generality (with two sets K and C) first in [2] and generalizes conditions of Karamardian [16] and Allen [1].*

REMARK 2.4. *Obviously, if Y is compact, then Φ vacuously satisfies condition (4) with $K_i = Y$ for all i.*

Noteworthy instances of maps satisfying condition (4) arise naturally in the theory of minimax inequalities and complementarity problems. In this context and as the next examples suggest, condition (4) - in the simpler case where the index set I is a singleton - can be seen as a "boundary condition" that puts restrictions on unbounded sequences.

Consider a map $\Phi : Y \to X$ of the form:

$$\Phi(y) := \{x : \langle y - x, f(y)\rangle \geq 0\}$$

where $Y \subseteq X \subseteq (H, \langle ., .\rangle)$ is a Hilbert space, and $f : H \longrightarrow H$ is an operator. A subset $\{y_r\}_{r>0}$ of X is said to be an *exceptional family* for the operator f (see [16] and [14]) if:

(i) $\|y_r\| \to +\infty$ as $r \to +\infty$;

(ii) $\forall r > 0, \exists \mu_r > 0$, with $\langle y, f(y_r) + \mu_r y_r\rangle \geq 0, \forall y \in X$ and $\langle y_r, f(y_r) + \mu_r y_r\rangle = 0$.

EXAMPLE 2.5. *([15]) If Φ satisfies condition (4) with $I =$ singleton, then f does not admit an exceptional family in Y.*

Proof. Indeed, let $\{y_r\}_{r>0}$ be an exceptional family for f. We show that for every compact subset K and every compact convex subset C of Y, there exists y_r such that $y_r \in Y \setminus K$ and $\langle y_r - x, f(y_r)\rangle < 0$ for all $x \in C$. Let K and C be such subsets of Y. For each $r > 0$,

$$\begin{aligned}
\langle y_r - x, f(y_r)\rangle &= \langle y_r - x, f(y_r) + \mu_r y_r - \mu_r y_r\rangle \\
&= \langle y_r, f(y_r) + \mu_r y_r\rangle - \mu_r\langle y_r, yx_r\rangle - \langle x, f(y_r) + \mu_r y_r\rangle + \mu_r\langle x, y_r\rangle \\
&\leq \mu_r\|y_r\|(\|x\| - \|y_r\|).
\end{aligned}$$

Since both K and C are compact sets and $\|y_r\| \to +\infty$, one can choose $r_0 > 0$ large, and a positive constant M, in such a way that $y_{r_0} \in Y \setminus K, \|y_{r_0}\| > M$, and $\|x\| < M\ \forall x \in C$. It follows that $\langle y_{r_0} - x, f(y_{r_0})\rangle \leq \mu_{r_0}\|y_{r_0}\|(\|x\| - \|y_{r_0}\|) < 0\ \forall x \in C$. ∎

Consider now a metrisable subset $X = \bigcup_{n=1}^{\infty} C_n$ of a topological vector space where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex sets. A sequence $\{x_k\}$ is said to *escape from* X (relative to $\{C_n\}$) (cf [4]) if and only if

$$\forall n, \exists M \text{ such that } x_k \notin C_n, \forall k \geq M. \tag{5}$$

Consider for a map $\Phi : X \to X$ the "boundary condition":

$$\forall \text{ escaping sequence } \{x_k\},\ \exists z \in X \text{ such that } z \in \Phi(x_k) \text{ for infinitely many } k's, \tag{6}$$

Example 2.6. *Assume that $\Phi : X \to X$ has open fibers in X. If Φ satisfies (6) then it satisfies (4).*

Proof. The set $K := \{y \in X : \Phi(y) = \emptyset\}$ is closed in X. Indeed, its complement is precisely the union $\bigcup \Phi^{-1}(x)$ of open subsets of X. We show that it is compact. Any sequence $\{x_k\}$ in K cannot escape X. By (5), there exists n_0 such that the set C_{n_0} contains a subsequence - again denoted - $\{x_k\}$. Since C_{n_0} is compact, $\{x_k\}$ converges to a limit in K. Hence K is compact. Obviously, $\Phi(y) \neq \emptyset$ for all $y \in Y \setminus K$, hence $\Phi(y) \cap C_k \neq \emptyset$ for some $k \in I$. Condition (4) is thus satisfied with $K_i \equiv K$ for all i. ■

Of particular interest is the situation where both X and Y are subsets with *filtrations* in a topological vector space E, that is $X := cl(\bigcup_{i\in I} X_i)$ and $Y := cl(\bigcup_{i\in I} Y_i)$ where $X_i = X \cap E_i \neq \emptyset, Y_i = Y \cap E_i \neq \emptyset$ and $\{E_i\}_{i\in I}$ is a filtering family of finite dimensional subspaces of E. The coercing family $\{(C_i, K_i)\}_{i\in I}$ of Definition 2.1 is such that $C_i \subseteq X_i$ and $K_i \subseteq Y_i$ for all i.

The coercivity condition (4) can be related to boundary conditions for maps which are necessary for the existence of trajectories of differential equations (see [13]). We consider below the case of non-compact viability domains. Indeed, consider the differential inclusion

$$\ell(x'(t)) \in \Gamma(x(t)), t \in [0, T], \tag{7}$$

where $\ell : I\!\!R^n \longrightarrow I\!\!R^m$ is a bounded linear operator, $\Gamma : I\!\!R^n \to I\!\!R^m$ is an upper hemicontinuous map with non-empty closed convex values, and X is a closed convex subset of $I\!\!R^n$. Let $\sigma(\Gamma(x), \psi) := \sup_{y\in\Gamma(x)}\langle\psi, y\rangle$ be the support functional of Γ ($\langle ., .\rangle$ being the inner product). The map $\Psi : X \to I\!\!R^n$ given by $\Psi(x) := \{\varphi \in I\!\!R^n : \varphi = \ell^*(\psi)$ and $\inf_{y\in\Gamma(x)}\langle\psi, y\rangle > 0\}$ has convex values and open fibers. Assuming it has non-empty values, an analogue of Michael's selection theorem [12, Proposition 8, chapitre 1] garantees the existence of a continuous single-valued selection s for Ψ. Consider now the map $\Phi : X \to X$, defined for all $x \in X$ by:

$$\Phi(x) := \{y : \langle s(x), y\rangle > \langle s(x), x\rangle\}$$

Example 2.7. *Assume that there exist a compact subset K of X and a convex compact subset C of X such that C linearly attracts trajectories starting outside K in the* *sense*

$$\begin{cases} \forall x_0 \in X \setminus K, \exists x(.) \textit{ solution of (7) starting at } x_0 \textit{ such that} \\ \quad \forall T' \in (0,T], \exists t \in (0,T'] \textit{ with } x(t) \in conv(\{x_0\} \cup C). \end{cases} \tag{8}$$

Then Φ *satisfies (4) with the pair* (C, K).

Proof. The first step consists in showing, for an arbitrary but fixed $x_0 \in X \setminus K$, that the map Γ satisfies the inwardness condition

$$cl(\ell(S_D(x_0))) \cap \Gamma(x_0) \neq \emptyset, \tag{9}$$

where $D := conv(\{x_0\} \cup C)$ is the drop with vertex at x_0 and base C, and $S_D(x_0)$ is the cone $\bigcup_{t>0} \frac{1}{t}(D - x_0)$ spanned by $D - x_0$. Indeed, hypothesis (8) implies the existence of a trajectory $x(.)$ of (7) and a positive sequence of real numbers $\{t_k\}_{k=1}^{\infty}$ converging to 0^+ such that $x(t_k) \in D$ for all k. The upper hemicontinuity of Γ amounts to the partial upper semicontinuiy,$\forall \psi \in I\!R^m$, of its support functional $x \longmapsto \sigma(\Gamma(x), \psi)$ (equivalently, the lower semicontinuity of the function $x \longmapsto \inf_{y \in \Gamma(x)} \langle \psi, y \rangle$). Thus, for a given ψ, $\forall \epsilon > 0, \exists \delta_\psi > 0$ such that:

$$\forall \tau \in [0, \delta_\psi], \langle \ell(x'(\tau)), \psi \rangle \leq \sigma(\Gamma(x(\tau)), \psi) < \sigma(\Gamma(x_0), \psi) + \epsilon \|\psi\|.$$

It follows that, for $\varphi = \ell^*(\psi)$,

$$\forall \tau \in [0, \delta_\psi], \langle \varphi, x'(\tau) \rangle < \sigma(\Gamma(x_0), \psi) + \epsilon \|\psi\|.$$

Consequently, $\forall k$,

$$\frac{1}{t_k} \int_0^{t_k} \langle \varphi, x'(\tau) \rangle d\tau < \frac{1}{t_k} \int_0^{t_k} [\sigma(\Gamma(x_0), \psi) + \epsilon \|\psi\|] d\tau = \sigma(\Gamma(x_0), \psi) + \epsilon \|\psi\|,$$

and, $\forall k$,

$$\langle \psi, y_k \rangle = \langle \varphi, v_k \rangle \leq \sigma(\Gamma(x_0), \psi) + \epsilon \|\psi\| \textit{ for } v_k := \frac{1}{t_k}(x(t_k) - x_0), y_k = \ell(v_k).$$

Being bounded by the Banach-Steinhauss boundedness principle, the sequence $\{y_k\}_{k=1}^{\infty}$ converges to some $y \in cl(\ell(S_D(x_0)))$ satisfying the inequality:

$$\langle \psi, y \rangle \leq \sigma(\Gamma(x_0), \psi) + \epsilon \|\psi\|.$$

Since ϵ and ψ are arbitrary and $\Gamma(x_0)$ is closed and convex, it follows (from the characterization of the closed convex sets in terms of their support function) that $y \in \Gamma(x_0)$, thus establishing (9).

The second step consists in observing that the inwardness condition (9) implies the inequality:

$$\inf_{y \in \Gamma(x_0)} \langle \psi, y \rangle \leq 0, \forall \psi \textit{ such that } \ell^*(\psi) \textit{ is normal to } D \textit{ at } x_0. \tag{10}$$

Indeed, let ψ be a vector such that $\ell^*(\psi)$ is normal to D at x_0, i.e. $\ell^*(\psi) \in N_D(x_0) := \{\varphi \in I\!R^n : \langle \varphi, x_0 \rangle \geq \sup_{x\in D}\langle \varphi, x\rangle\}$, and choose an element $y \in cl(\ell(S_D(x_0))) \cap \Gamma(x_0)$ which is the limit of a sequence $y_k = \ell(v_k), v_k := \frac{1}{t_k}(x_k - x_0) \in S_D(x_0), x_k \in D, t_k > 0, k = 1, 2, \ldots$. Obviously, $\forall k$, since $\langle \ell^*(\psi), x_0\rangle \geq \langle \ell^*(\psi), x_k\rangle$, then $\langle \psi, y_k\rangle = \langle \ell^*(\psi), v_k\rangle \leq 0$. It follows that $\inf_{y\in\Gamma(x_0)}\langle \psi, y\rangle \leq \langle \psi, y\rangle \leq 0$, thus establishing (10).

Finally, observe that the contrapositive of (10) precisely says that if $\ell^*(\psi) \in \Psi(x_0)$, then $\ell^*(\psi)(x_0) < \max_{x\in D} \ell^*(\psi)(x)$. D being compact, there exist $y_0 \in C$ and $\lambda \in [0, 1)$ such that $\max_{x\in D} \ell^*(\psi)(x) = \ell^*(\psi)(\lambda x_0 + (1-\lambda)y_0) = \lambda\ell^*(\psi)(x_0) + (1-\lambda)\ell^*(\psi)(y_0)$. This implies that $\ell^*(\psi)(x_0) < \ell^*(\psi)(y_0)$.

We have shown that $\forall x_0 \in X \setminus K, \exists y_0 \in C$, such that if $\varphi = \ell^*(\psi) \in \Psi(x_0)$, then $\varphi(x_0) < \varphi(y_0)$. Since $s(x_0) \in \Psi(x_0)$, it follows that $s(x_0)(y_0) > s(x_0)(x_0)$, i.e. $y_0 \in \Phi(x_0)$ thus establishing (4). ■

3. A GENERALIZATION OF THE KKMF PRINCIPLE

Let us say with Lassonde [18] that a subset A of a topological space X is compactly closed (open, respectively) if for every compact set C of X, $A \cap C$ is closed (open, respectively) in C. The fundamental result of the paper is the following generalization of the KKMF principle:

THEOREM 3.1. *Let E be a topological vector space, Y a convex subset of E, X a non-empty subset of Y, and $F : X \to Y$ a KKM map with compactly closed (in Y) values. If F admits a coercing family, then* $\bigcap_{x\in X} F(x) \neq \emptyset$.

Proof. Let $\{(C_i, K_i)\}_{i\in I}$ be a coercing family for F and let $\langle X\rangle$ be the family of all finite subsets of X. For $j = (i, a) \in J = I \times \langle X\rangle$, let $\hat{C}_j = C_i \cup a$ and $\hat{K}_j = K_i \cup a$. The family $\{(\hat{C}_j, \hat{K}_j)\}_{j\in J}$ is also a coercing family for F and, furthermore, $X = \bigcup_{j\in J} \hat{C}_j$. Define, for every $j \in J$, $F_j : \hat{C}_j \to Z_j$, Z_j being the convex compact subset of Y that contains C_j, by putting:

$$F_j(x) := F(x) \cap Z_j, x \in \hat{C}_j.$$

For each $j \in J$, F_j is a KKM map and for each $x \in X$, $F_j(x)$ is closed in Z_j. Since $F_j(x)$ is compact, it follows from the KKMF Principle [[9], Lemma 1] that $\bigcap_{x\in\hat{C}_j} F_j(x)$ is not empty. From Definition 1 (ii), it follows that the family $\{\bigcap_{x\in\hat{C}_j} F(x)\}_{j\in J}$ has the finite intersection property. Since it follows from Definition 2.1. (iii) that for some $j \in J$, $\bigcap_{x\in\hat{C}_j} F(x)$ is contained in a compact set, we conclude that $\bigcap_{j\in J}\bigcap_{x\in\hat{C}_j} F(x)$ is not empty. Since $X = \bigcup_{j\in J} \hat{C}_j$, we just have to notice that $\bigcap_{j\in J}\bigcap_{x\in\hat{C}_j} F(x) = \bigcap_{x\in X} F(x)$, in order to complete the proof. ■

If $C_i = C$ and $K_i = K$ for all $i \in I$, C is contained in a convex compact subset of X and K is a compact subset of Y, then Theorem 1 is reduced to Theorem 4 of Ky Fan [11] which in turn generalizes the KKMF principle.

The fixed point formulation of Theorem 3.1 is:

Theorem 3.2. *Let X be a non-empty convex subset of a topological vector space E and let $\Phi : X \to X$ be a map with compactly open fibers (in X) and non-empty values. If Φ admits a coercing family in the sense of Remark 2.2, then the map $conv(\Phi)$ has a fixed point.*

Proof. Assume for a contradiction that $conv(\Phi)$ is without fixed point, i.e. $x \notin conv(\Phi(x)), \forall x \in X$. Define $F : X \to X$ by :

$$F(x) := \{y \in X \mid x \notin \Phi(y)\}, x \in X.$$

Obviously, $\forall x \in X, F(x)$ is a compactly closed subset of X. We show that F is a KKM map. Suppose that for a finite subset $\{x_1, ..., x_n\}$of X there exists a convex combination $z = \sum_{i=1}^{n} \lambda_i x_{i,}$ with $z \notin \bigcup_{i=1}^{n} F(x_i)$. It follows that $x_i \in \Phi(z), i = 1, ..., n$, and $z \in conv(\Phi(z))$, which contradicts the assumption that $conv(\Phi)$ is without fixed point. To complete the proof, we remember that a coercing family for Φ is a coercing family for F (see Remark 2.2). Theorem 3.1 implies that $\bigcap_{x \in X} F(x) \neq \emptyset$ which contradicts the fact that Φ has non-empty values ■

This result generalizes the Browder-Fan fixed point theorem (see [5], [9], [11], [7], [2]). It extends Corollary of [3] (case where the coercing family consists of a single pair (C, K)) which contains results in e.g. [11], [8].

Theorem 3.1 and Theorem 3.2 can be used to extend existing results on the solvability of complementarity problems, existence of zero on non-compact domains and existence of equilibria for qualitative games and abstract economies.

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