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MINIMAX INEQUALITY AND EQUILIBRIA WITH A GENERALIZED COERCIVITY

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Abstract. In the first part of this paper, we prove a minimax inequality for maps satisfying a generalized coercivity type condition. As a consequence, we prove a result on the solvability of complementarity problems. In the second part, a result on the existence of maximal element in non-compact domains is obtained and as application, we prove the existence of equilibrium for an abstract economy (or generalized game) with non-compact choice sets.

1. Introduction

This paper is a study of minimax inequality and equilibrium for maps satisfying a "coercivity" type condition. We firstly recall the notion of coercing family for set-valued maps (also called correspondences) defined by Ben-El-Mechaiekh, Chebbi and Florenzano in [2]. As an example, we give the very general coercivity condition obtained by Ding and Tan in [5].

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In Section 2, we prove a minimax inequality for maps satisfying a generalized quasi-concavity condition and a coercivity type condition. Our result extends the minimax inequality obtained by Yen [11] to non-compact domains and generalizes also the minimax inequalities obtained in the non-compact case by Fan [6] and Ding and Tan [5]. As a consequence, we extend results on complementarity problems obtained by Karamardian [8] and Allen [1].

In Section 3, we prove the existence of maximal elements for preferences correspondences defined on non-compact subsets of a topological vector space and satisfying a coercivity type condition. As application, we prove an equilibrium existence result for generalized game (or abstract economy) with non-compact choice sets. The results of this section generalize corresponding results obtained in Borglin and Keiding [4], Toussaint [9], Tulcea [10] and Ding and Tan [5].

Throughout the paper, vector spaces are real and topological (vector) spaces are assumed to be Hausdorff. The convex hull of a subset A of a vector space is denoted by $\operatorname{co} A$, the closure of a subset A of a topological space is denoted by $\operatorname{cl} A$ and for any set X, $\langle X\rangle$ denotes the family of all non-empty finite subsets of X.

Let X be a subset of a topological vector space, Y a topological space and $F\colon X \to Y$ be a correspondence. In order to define the setting of this paper, we need the following definition given in [2]:

Definition 1. A family $\{(C_i, K_i)\}_{i\in I}$ of pair of sets is said to be *coercing* for F if and only if:

(i) For each $i \in I$, C_i is contained in a compact convex subset of X and K_i is a compact subset of Y.

(ii) For each $i, j \in I$, there exists $k \in I$ such that $C_i \bigcup C_j \subseteq C_k$.

(iii) For each $i \in I$, there exists $k \in I$ with $\displaystyle\bigcap_{x\in C_k} F(x) \subset K_i$.

For any correspondence $F\colon X \to Y$, let $F^*\colon Y \to X$ be the "dual" correspondence of F defined by $F^*(y) = X \setminus F^{-1}(y)$. Using the following equivalent formulation, we can easily see that (iii) is a coercivity type condition:

Remark 1. Let X be a subset of a topological vector space and $F\colon X \to Y$ be a correspondence. A family $\{(C_i, K_i)\}_{i\in I}$ of pair of sets is *coercing* for F if and only if it satisfies conditions (i), (ii) of Definition 1 and the following one:

$$\forall i \in I, \forall y \in X \setminus K_i, F^*(y) \cap C_k \neq \emptyset \quad \text{for some } k \in I. \tag{$\mathcal{C}$}$$

Definition 2. A family $\{(C_i, K_i)\}_{i\in I}$ of pair of sets is said to be $\mathcal{C}$*-coercing* for F if and only if it satisfies conditions (i), (ii) of Definition 1 and condition $(\mathcal{C})$ in Remark 1.

Note that in case where the family is reduced to one element, condition $(\mathcal{C})$ appeared first in this generality (with two sets K and C) in [3] and generalizes condition of Karamardian [8] and Allen [1]. Condition $(\mathcal{C})$ is also an extension of the coercivity condition given by Fan [6]. Fore more examples about correspondences admitting a coercing family (when I is a singleton), see [2]. By the following example, we can see that the notion of coercing family is very general:

Example 1. If $F\colon X \to X$ is a correspondence satisfying the following condition given in [5]: There exists X_0 contained in a compact convex subset of X and K a compact subset of X such that:

$$\forall y \in X \setminus K, F(y) \cap \mathrm{co}(X_0 \cup y) \neq \emptyset.$$

Then F admits a $\mathcal{C}$-coercing family.

Proof. Take the family:

$$\{(C_{A_y}), K)\}_{\{y\in\langle X\setminus K\rangle, A_y\in\langle X\rangle\}},$$

where for each $y \in \langle X \setminus K\rangle$ and for each $A_y \in \langle X\rangle$, $C_{A_y} = \mathrm{co}(X_0 \cup A_y)$. This family verifies conditions (i) and (ii) of Definition 1, by putting $A_y = \{y\}$ for every $y \in X \setminus K$, condition $\mathcal{C}$ is satisfied. □

2. Minimax inequalities

Let us recall that if X is a subset of a vector space Y, a correspondence $F\colon X \to Y$ is called *KKM* if for any $A \in \langle X\rangle$:

$$\mathrm{co}(A) \subset \bigcup_{x\in A} F(x).$$

A subset X of a topological space Y is compactly closed (open, respectively) if for every compact set C of Y, $X \cap C$ is closed (open, respectively) in C.

The following minimax inequality is an equivalent analytic formulation of Theorem 3.1 in [2]:

Theorem 1. *Let X be a non-empty convex subset of a topological vector space E and $f\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function such that:*

(a) *For each fixed $x \in X$, the function: $y \to f(x, y)$ is lower semi-continuous on each non-empty compact subset of X.*

(b) *For each* $A \in \langle X \rangle$, $\sup_{y \in \operatorname{co} A} \min_{x \in A} f(x,y) \leq 0$.

(c) *There exists a family* $\{(C_i, K_i)\}_{i \in I}$ *satisfying conditions* (i) *and* (ii) *of Definition* 1 *and the following one: For each* $i \in I$, *there exists* $k \in I$ *such that:*

$$\{y \in X,\ f(x,y) \leq 0,\ \forall x \in C_k\} \subset K_i.$$

Then there exists $y_0 \in X$ *such that* $f(x, y_0) \leq 0$ *for all* $x \in X$.

Proof. For each $x \in X$, let $F(x) = \{y \in X \colon f(x,y) \leq 0\}$. We have to show that F satisfies all conditions of Theorem 3.1 in [2]. By (a), $F(x)$ is compactly closed in X for each $x \in X$. If F is not KKM, there exist $A \in \langle X \rangle$ and $y \in \operatorname{co} A$ such that $f(x,y) > 0$ for all $x \in A$, which contradicts (b). Condition (c) implies that F admits a coercing family, it follows that $\bigcap_{x \in X} F(x) \neq \emptyset$. Let $y_0 \in \bigcap_{x \in X} F(x)$, then $f(x, y_0) \leq 0$, for all $x \in X$. □

Theorem 1 extends Theorem 6 of Fan [6]. Using Example 1, Theorem 1 is also a generalization of Theorem 1 in [5].

Corollary 1. *Let* X *be a non-empty convex subset of a topological vector space* E *and* $f, g \colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ *be such that:*

(i) *For each* $(x,y) \in X \times X$, $f(x,y) \leq g(x,y)$.

(ii) *For each* $x \in X$, $g(x,x) \leq 0$.

(iii) *For each fixed* $x \in X$, *the function:* $y \to f(x,y)$ *is lower semi-continuous on each non-empty compact subset of* X.

(iv) *For each fixed* $y \in X$, *the set* $\{x \in X \colon g(x,y) > 0\}$ *is convex.*

(v) *There exists a family* $\{(C_i, K_i)\}_{i \in I}$ *satisfying conditions* (i) *and* (ii) *of Definition* 1 *and the following one: For each* $i \in I$, *there exists* $k \in I$ *such that:*

$$\{y \in X \colon f(x,y) \leq 0,\ \forall x \in C_k\} \subset K_i.$$

Then there exists $y_0 \in X$ *such that* $f(x, y_0) \leq 0$ *for all* $x \in X$.

Proof. It is sufficient to show that (i), (ii) and (iv) imply condition (b) of Theorem 1. If not, there exist $A \in \langle X \rangle$ and $y \in \operatorname{co} A$ such that $\min_{x \in A} f(x,y) > 0$. Then by (i), $\min_{x \in A} g(x,y) > 0$. It follows by (iv) that $g(y,y) > 0$, which contradicts (ii). □

The following minimax inequality, which includes a generalization of Theorem 1 of Yen [11], can be deduced from Corollary 1:

Corollary 2. *Let X be a non-empty convex subset of a topological vector space E and $f, g\colon X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that:*

(a) *For each $(x, y) \in X \times X$, $f(x, y) \leq g(x, y)$.*

(b) *For each fixed $x \in X$, the function: $y \to f(x, y)$ is lower semi-continuous on each non-empty compact subset of X.*

(c) *For each fixed $y \in X$, the function: $x \to g(x, y)$ is quasi-concave.*

(d) *For any $\alpha \in \mathbb{R}$, there exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying conditions (i), (ii) of Definition 1 and the following one: For each $i \in I$, there exists $k \in I$ such that:*

$$\{y \in X\colon f(x, y) \leq \alpha, \, \forall x \in C_k\} \subset K_i.$$

Then the following minimax inequality holds:

$$\inf_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} g(x, x).$$

Proof. We can assume that $\lambda = \sup_{x \in X} g(x, x)$ is finite, otherwise there is nothing to prove. The functions $f - \lambda$ and $g - \lambda$ satisfy conditions of Corollary 1, then there exists $y_0 \in X$ such that $f(x, y_0) \leq \lambda$, $\forall x \in X$. Hence the minimax inequality follows. □

The following extension of Theorem 3.1 of Karamardian [8] on the solvability of complementarity problems follows immediately from Theorem 1:

Corollary 3. *Let X be a non-empty convex subset of a topological vector space E and $f\colon X \to E^*$, where E^* denotes the topological dual of E, be such that:*

(i) *For each fixed $x \in X$, the function: $y \to \langle f(y), y - x\rangle$ is lower semi-continuous on each non-empty compact subset of X.*

(ii) *There exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying conditions (i), (ii) of Definition 1 and the following one: For each $i \in I$, there exists $k \in I$ such that:*

$$\{y \in X\colon \langle f(y), y - x\rangle \leq 0, \forall x \in C_k\} \subset K_i.$$

Then there exists $y^* \in X$ such that $\langle f(y^*), y^* - x\rangle \leq 0$ for all $x \in X$.

If moreover X is a cone and if X^0 denotes the polar cone of X, then $-f(y^*) \in X^0$ and $\langle f(y^*), y^*\rangle = 0$.

3. Equilibria in an abstract economy

Correspondences play a central role in the theory of economic equilibria. They usually represent preference relations (the value $P(x)$ of a correspondence P consists of all those commodities preferred to x). The issue there is to determine the existence of a so-called *maximal element* for a given preference P, i.e. an element $\bar{x}$ with $P(\bar{x}) = \emptyset$.

Definition 3. Using the terminology of Borglin and Keiding [4], given a map $P\colon X \to X$ of a non-empty subset X of a topological vector space, we say that:

(i) P is a KF *correspondence* if:
  (a) for all $y \in X$, $P^{-1}(y)$ is compactly open in X;
  (b) for all $x \in X$, $x \notin \operatorname{co} P(x)$.

(ii) A correspondence $\Psi_x\colon X \to X$ is a KF*-majorant of* P at $x \in X$ if Ψ_x is KF and $P(x') \subseteq \Psi_x(x')$, for all x' in some open neighborhood U_x of x in X.

(iii) P is KF*-majorized* if it admits a KF-majorant at each $x \in X$ with $P(x) \neq \emptyset$.

Remark 2. The concept of KF majoration is hereditary in the sense that it becomes global in the presence of paracompactness. More precisely, if a correspondence $P\colon X \to X$ is KF-majorized and if X is paracompact, then P is majorized by a KF correspondence Ψ, i.e., $P(x) \subseteq \Psi(x), \forall x \in X$ (see [4]).

Theorem 3.2 in [2] can be rephrased in terms of the existence of maximal elements as follows:

Proposition 1. *Let X be a non-empty convex and paracompact subset of a topological vector space. A correspondence $P\colon X \to X$ admits a maximal element provided that it is KF-majorized and has a $\mathcal{C}$-coercing family.*

Proof. Suppose that, for all $x \in X$, $P(x) \neq \emptyset$. Since P is KF-majorized and X is paracompact, it follows from Remark 2 that there exists a KF correspondence Ψ such that $P(x) \subseteq \Psi(x)$, $\forall x \in X$. By Theorem 3.2 in [2], the correspondence $\operatorname{co}\Psi$ admits a maximal element, which is also a maximal element for P. □

Theorem 1 in [5] follows from Example 1 and Proposition 1:

Corollary 4. *Let X be a non-empty convex and paracompact subset of a topological vector space and $P\colon X \to X$ a KF-majorized correspondence. If P satisfies the following coercivity condition: There exist X_0 contained in a compact convex subset of X and K a compact subset of X such that:*

$$\forall y \in X \setminus K, P(y) \cap \operatorname{co}(X_0 \cup y) \neq \emptyset.$$

Then P admits a maximal element.

Let now J be a (possibly infinite) set of agents. We consider the situation where each agent $j \in J$ has a non-empty choice set (or strategy set) X^j and a preference correspondence $P^j\colon X = \prod_{j\in J} X^j \to X^j$ such that $x^j \notin P^j(x)$, $x \in X$. Following Gale and Mas-Colell [7], we say that the collection $(X^j, P^j)_{j\in J}$ is a *qualitative game*.

Using Proposition 1, we obtain the following existence result for qualitative games:

Proposition 2. *Let $(X^j, P^j)_{j\in J}$ be a qualitative game such that the set $X = \prod_{j\in J} X^j$ is paracompact and satisfying the following conditions for each $j \in J$:*

(i) X^j is a non-empty convex subset of a topological vector space E^j.

(ii) P^j is KF-majorized.

(iii) $\{x \in X\colon P^j(x) \neq \emptyset\}$ is open in X.

(iv) P^j admits a $\mathcal{C}$-coercing family.

Then the game $(X^j, P^j)_{j\in J}$ has an equilibrium.

Proof. For each $x \in X$, let $J(x) = \{j \in J\colon P^j(x) \neq \emptyset\}$. Define $\Phi\colon X \to X$ by:

$$\Phi(x) = \begin{cases} \displaystyle\bigcap_{j\in J(x)} \operatorname{conv}(P'^j(x)) & \text{if } J(x) \neq \emptyset \\ \emptyset & \text{if } J(x) = \emptyset \end{cases}$$

where $P'^j\colon X \to X$ is defined by: $y \in P'^j(x) \iff y^j \in P^j(x)$. Using (ii), (iii), a standard argument (see [9]) shows that Φ is KF-majorized. Hypothesis (iv) implies that Φ admits a $\mathcal{C}$-coercing family. Hence, there exists an $\overline{x} \in X$ such that $\Phi(\overline{x}) = \emptyset$ i.e. $P^j(\overline{x}) = \emptyset$ for all $j \in J$. □

More generally, if each agent j is restricted in his choices to some non-empty subset of his strategy set due to the actions of the other players; this is formalized in terms of a *constraint correspondence* $B^j\colon X \to X^j$. The family $(X^j, B^j, P^j)_{j\in J}$ is called a *generalized qualitative game* or an *abstract economy*. We say that $\overline{x} \in X$ is an *equilibrium* of the game if for each $j \in J$:

$$\overline{x}^j \in \operatorname{cl}_{X^j} B^j(\overline{x}) \quad \text{and} \quad B^j(\overline{x}) \cap P^j(\overline{x}) = \emptyset.$$

Proposition 3. *Let $(X^j, B^j, P^j)_{j\in J}$ be a generalized qualitative game such that the set $X = \prod_{j\in J} X^j$ is paracompact and satisfying the following conditions for each $j \in J$:*

(*i*) *X^j is a non-empty convex subset of a topological vector space.*

(*ii*) *For each $x \in X$, $B^j(x)$ is non-empty and convex.*

(*iii*) *For each $y^j \in X^j$, $(B^j)^{-1}(y^j)$ is open in X.*

(*iv*) *$\mathrm{cl}_{X^j}(B^j)\colon X \to X^j$ is upper semi-continuous.*

(*v*) *$B^j \bigcap P^j$ is KF-majorized.*

(*vi*) *$\{x \in X\colon (B^j \cap P^j)(x) \neq \emptyset\}$ is open in X.*

(*vii*) *$P^j \cap B^j$ admits a $\mathcal{C}$-coercing family.*

Then the abstract economy $(X^j, B^j, P^j)_{j\in J}$ has an equilibrium.

Proof. For each $j \in J$, let $F^j = \{x \in X\colon x^j \notin \mathrm{cl}_{X^j} B^j(x)\}$. The set F^j is open in X by (iv). Define $Q^j\colon X \to X$ by:

$$Q^j(x) = \begin{cases} (B^j \cap P^j)(x) & \text{if } x \notin F^j \\ B^j(x) & \text{if } x \in F^j. \end{cases}$$

We can also show by a standard argument (see [9]) that the qualitative game $(X^j, Q^j)_{j\in J}$ satisfies the hypotheses (i)–(iii) of Proposition 2. By (vii) Q^j admits a $\mathcal{C}$-coercing family. We conclude that the qualitative game $(X^j, Q^j)_{j\in J}$ admits an equilibrium $\overline{x}$. Since $B^j(x)$ is non-empty for all $x \in X$, this implies that for each $j \in J$, $\overline{x}^j \in \mathrm{cl}_{X^j}(B^j(\overline{x}))$ and $B^j(\overline{x}) \cap P^j(\overline{x}) = \emptyset$. □

Proposition 3 generalizes Theorem 4 in [5]. If X_j is compact for each $j \in J$, then Proposition 2 reduces to Corollary 3 in [4], Theorem 2.5 in [9] and Proposition 3 in [10].

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