INTERSECTION AND MINIMAX INEQUALITY WITH A GENERALIZED COERCIVITY IN H-SPACES[1]

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ABSTRACT. We define a generalized coercivity type condition for correspondence defined on a topological vector space endowed with a generalized convex structure. An extension of the Fan's matching theorem is obtained and used to prove results on coincidence, fixed point and minimax inequality with a weakned compactness condition.

Keywords. H-space, H-compact, H-coercing family, coincidence, fixed point, minimax inequality.

1. INTRODUCTION

This paper is a study of a coercivity type condition for correspondences defined on a topological space endowed with a generalized convex structure. We introduce the concept of coercing family in H-spaces and we propose the systematic development of the method based on the Fan's matching type theorem.

We firstly recall the structure of H-convexity defined by Horvath in [10] and H-KKM correspondence defined by Bardaro and Cepitelli in [4] and then we introduce the notion of H-coercing family for correspondences defined in H-spaces. In section 3, we prove a Fan's matching type theorem (see [7] and [8]) on intersection of correspondences defined in H-spaces and satisfying a weakened compactness condition. Theorem 1 and Theorem 2 of this section generalize recent results of Lassonde ([12], Theorem I), Horvath ([10], Theorem 1) and Bardaro and Cepitelli ([4], Theorem 1 and Theorem 2) as well as corresponding results obtained in Fan [8], Ben El-Mechaiekh, Deguire and Granas [3], Ding and Tan [6] and Ben El-Mechaiekh, Chebbi and Florenzano [2] when the H-convexity is replaced by the usual convexity of a topological vector space.

In this framework, we generalize the results on coincidence and fixed point obtained in Horvath [11], Bardaro and Cepitelli [5] and Lassonde [12] to coercive correspondences defined on noncompact H-spaces. The corresponding results in Park and Kim [13], Lassond [12] and Fan [7] are themselves generalizations of the well-known theorem on coincidence obtained by Ben El-Mechaiekh, Deguire and Granas in [3].

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In Section 4, we prove a minimax inequality for functions defined on an H-space and satisfying a generalized coercivity type condition. Our result generalizes minimax inequalities results obtained in Allen [1], Fan [7], Granas [9], Ben El-Mechaiekh, Deguire and Granas [3], Horvath [11] and Ding and Tan[?].

2. PRELIMINARIES

Let $\langle X \rangle$ denote the family of all nonempty finite subsets of X. In order to define the setting of this paper, we firstly recall some basic concepts:

Definition 2.1.

(a) (X, Γ) is said to be an *H-space* if X is a topological space and $\Gamma : \langle X \rangle \to E$ a set-valued map such that $\Gamma(A) \subset \Gamma(B)$ if $A \subset B$ and assumed to have nonempty C^∞ values[2]

(b) A subset $C \subset X$ is said to be *H-convex* if for every $A \in \langle C \rangle$, $\Gamma(A) \subset C$.

(c) A subset $K \subset X$ is said to be *H-compact* if for every $A \in \langle X \rangle$, there is a compact H-convex set D such that $A \cup K \subset D$.

Note that the class of H-spaces, which was firstly defined by Horvath in [11], contains topological vector spaces as well as a number of spaces with abstract topological convexity (the pseudo-convexity of Horvath [10] and the concept of convex space due to Lassonde [12] for example). For More details about generalized convexity, refer to [12], [10], [11], [4], [5], [13] and [2]. The notion of H-compactness generalizes the c-compactness of Lassonde in [12].

Definition 2.2. *Let (X, Γ) be an H-space. A set-valued map (simply called correspondence) $F : X \to X$ is called H-KKM if and only if:*

$$\forall A \in \langle X \rangle, \quad \Gamma(A) \subset \bigcup_{x \in A} F(x)$$

[2]A subset X of a topological space is said to be C^∞ (or ∞-*connected*) if for each integer n, any continuous function $f : \partial \Delta_n \to X$ can be continuously extended to a continuous function $g : \Delta_n \to X$.

Definition 2.3. *As was defined by Lassonde in [12], we say that a subet A of a topological space X is compactly closed (open, respectively) in X if for every compact set $C \subset X$, the set $A \cap C$ is closed (open respectively) in X.*

We now introduce the concept of generalized coercive correspondence:

Definition 2.4. *Let (X, Γ) be an H-space and Y a topological space. A family $\{(C_i, K_i)\}_{i \in I}$ is said to be H-coercing for a correspondence $F : X \to Y$ if and only if:*

(i) For each $i \in I$, C_i is an H-compact subset of X and K_i is a compact subset of Y;

(ii) For each $i, j \in I$, there exists $k \in I$ such that $C_i \bigcup C_j \subseteq C_k$;

(iii) For each $i \in I$, there exists $k \in I$ such that:

$$\bigcap_{x \in C_k} F(x) \subseteq K_i$$

Example 2.1. *If $F : X \to X$ is a correspondence satisfying the following condition given in [H ?]:*

For some $x_0 \in X$, $F(x_0)$ is compact.

Then F admits a coercing family.

*P*roof. Take, for all $i \in I$, $C_i = \{x_0\}$ and $K_i = F(x_0)$ ■

Example 2.2. *If $F : X \to X$ is a correspondence satisfying the following condition given in [DKT ?]: There exists a nonempty H-compact subset X_0 of X such that $\bigcap_{x \in X_0} F(x)$ is compact. Then F admits a H-coercing family.*

Proof. Take for each $i \in I$, $C_i = X_0$ and $K_i = \bigcap_{x \in X_0} F(x)$ ■

Note that when X is a subset of a topological vector space, the notion of coercing family in this generality was used by Ben El-Mechaiekh, Chebbi and Florenzano in [2] and generalized the concept of coercivity (with two sets K and C) used by Allen [1], Ben EL-Mechaiekh, Deguire and Granas [3] and Fan [8]. For more details about coercing family in topological vector space, see [2].

3. INTERSECTION, COINCIDENCE AND FIXED POINT THEOREM

The main result of this paper is the following extension of Theorem 4 in [8]:

Theorem 3.1. *Let (X,Γ) be a an H-space, Y any topological space and $F : X \to Y$ a correspondence such that:*

(1) For every $x \in X$, $F(x)$ is compactly closed in X.

(2) For some continuous map $s : X \to Y$ the correspondence $G : X \to X$ given by :

$$G(x) = s^{-1}(F(x))$$

is H-KKM.

(3) There exists an H-coercing family $\{(C_i, K_i)\}_{i\in I}$ for F.

Then $\bigcap\limits_{x\in X} F(x) \neq \emptyset$.

Proof. Let $\langle X\rangle$ be the family of all finite subsets of X. For $j = (i, a) \in J = I \times \langle X\rangle$, let $\hat{C}_j = C_i \cup a$ and $\hat{K}_j = K_i \cup a$. Since C_i is an H-compact subset, the family $\{(\hat{C}_j, \hat{K}_j)\}_{j\in J}$ is also H-coercing for F and, furthermore, $X = \bigcup\limits_{j\in J} \hat{C}_j$.

For every $j \in J$, let Z_j be the compact and H-convex set containing $\hat{C}_j$ and Let $Y_j = s(Z_j)$. We consider the correspondence $G_j : \hat{C}_j \to Z_j$ defined by :

$$G_j(x) = s_j^{-1}(F(x) \cap Y_j)$$

where s_j is the restriction of s to Z_j.

By (1), for each $x \in \hat{C}_j$, $G_j(x)$ is compact and it is easy to check that G_j is H-KKM since $G_j(x) = G(x) \cap Z_j$ and G is H-KKM. It follows from Corollary 1 of Horvath in [10] that $\bigcap\limits_{x\in\hat{C}_j} G_j(x)$ is not empty, so $\bigcap\limits_{x\in\hat{C}_j} F(x)$ is also not empty . Using condition (ii) of Definition 2.4, we can see that the family $\{ \bigcap\limits_{x\in\hat{C}_j} F(x)\}_{j\in J}$ has the finite intersection property. Since for some $j \in J$, $\bigcap\limits_{x\in\hat{C}_j} F(x)$ is contained in a compact set, we conclude that $\bigcap\limits_{j\in J} \bigcap\limits_{x\in\hat{C}_j} F(x)$ is not empty. Since $X = \bigcup\limits_{j\in J} \hat{C}_j$, we just have to notice that $\bigcap\limits_{j\in J} \bigcap\limits_{x\in\hat{C}_j} F(x) = \bigcap\limits_{x\in X} F(x)$, in order to complete the proof ■

Theorem 3.1 extends Theorem 1 in [4] which in tern generalizes Corollary 1 of Horvath in [10]. When I is a singleton and the H-convexity is replaced by the convexity of Lassonde, then Theorem 3.1 is reduced to Theorem I in [12].

For any correspondence $F : X \to Y$, let $F^* : Y \to X$ be the correspondence defined by:

$$F^*(y) = X \setminus F^{-1}(y)$$

The following result is more specially adapted to the study of minimax inequality:

Theorem 3.2. *Let (X, Γ) be an H-space and $F, G : X \to X$ two correspondences such that:*

(a) For every $x \in X$, $G(x)$ is compactly closed and $F(x) \subset G(x)$.

(b) For every $x \in X$, $x \in F(x)$

(c) $F^*(x)$ is H-convex .

(d) There exists an H-coercing family $\{(C_i, K_i)\}_{i \in I}$ for G.

Then $\bigcap\limits_{x \in X} G(x) \neq \emptyset$.

Proof. By virtue of Theorem 3.1, it suffices to show that G is H-KKM. Suppose that for some finite subset $A \subset X$, there exists $y \in \Gamma(A)$ and $y \notin G(x)$ for every $x \in A$ and so $A \subset G^*(y)$. Since $G^*(y) \subset F^*(y)$ and by (c) $\Gamma(A) \subset F^*(y)$, hence $y \in F^*(y)$ which is equivalent to $y \notin F(y)$ and this contradicts (b) ■

Condition (d) of Theorem 3.2 extends the non-compactness condition of Theorem 2 in [10] and Theorem 1 in [11]. We now prove a generalization of Ky Fan's fixed point theorem as presented by Ben-El Mechaiekh, Deguire and Granas in [3]:

Proposition 3.1. *Let (X, Γ) be an H-space, Y a topological space and $S : X \to Y$ a correspondence such that:*

(i) For each $x \in X$, $S(x)$ is compactly open in Y.

(ii) For each $y \in X$, $S^{-1}(y)$ is nonempty and H-convex.

(iii) There exists an H-coercing family $\{(C_i, K_i)\}_{i \in I}$ for the correspondence $F : X \to Y$ defined by $F(x) = Y \backslash S(x), \ \forall x \in X$.

Then, for each continuous function s from X to Y, there exists an $x_0 \in X$ such that $s(x_0) \in S(x_0)$. In particular, S has a fixed point.

Proof. For each $x \in X$, $F(x)$ is compactly closed by (i). Let $s : X \to Y$ be any continuous map and $G : X \to X$ a correspondence defined, for all $x \in X$, by $G(x) = s^{-1}(F(x))$. G is not H-KKM, otherwise condition (ii) is not satisfied. Thus, there is a finite set $A \subset X$ and $x_0 \in \Gamma(A)$ such that $s(x_0) \in \cap_{x \in A} S(x)$ and so $s(x_0) \in S(x_0)$ ■

The following result is a geometrical formulation of Proposition 3.1:

Proposition 3.2. *Let (X, Γ) be an H-space, Y a topological space, Z an arbitrary set and $s : X \to Y$ a continuous function. Let $A \subset Z$ and $g : X \times Y \to Z$ be a function such that:*

(a) For every $x \in X$, the set $\{y \in Y : g(x, y) \in A\}$ is compactly open.

(b) For every $y \in Y$, the set $\{x \in X : g(x, y) \in A\}$ is nonempty and H-convex.

(c) There exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying conditions (i) and (ii) of Definition 1and the following one:

For all $i \in I$, there exist $k \in I$ such that $\{y \in Y : g(x, y) \notin A \ \ \forall x \in C_k\} \subset K_i$.

Then g satisfies at least one of the following properties:

(1) There exists $\hat{y} \in Y$ such that $g(x, \hat{y}) \notin A$ for all $x \in X$.

(2) There is $\hat{x} \in X$ such that $g(\hat{x}, s(\hat{x})) \in A$.

Proof. Proposition 3.3 $\Rightarrow$ Propsition 3.4: For each $x \in X$, consider the correspondence $F(x) = \{y \in Y \mid g(x, y) \in A\}$ and suppose property (1) does not holds, then F satisfies all conditions of Proposition 3.3. Consequently, there exists a point $\hat{x} \in X$ with $s(\hat{x}) \in S(\hat{x})$. Such a point satisfies property (2).

Proposition 3.4 $\Rightarrow$ Propsition 3.3: Take $Z = X \times Y$, $A = graph(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ and $g(x, y) = (x, y)$. All conditions of Proposition 3.4 are satisfied and property (1) does not hold since each $S^{-1}(y)$ is assumed to be nonempty. Hence property (2) holds ■

Observe that in case X or Y is compact, condition (iii) of Proposition 3.1 is automatically satisfied. When I is a singleton, Proposition 3.1 and Proposition 3.2 are respectively reduced to Theorem 1 and Theorem 1' in [5] which themselves generalize Theorem 1.1 and Theorem 1.1' in [12].

4. MINIMAX INEQUALITY

Now by using the same argument as used by Bardaro and Ceppitelli in [4], we establish a generalized minimax inequality:

Theorem 4.1. *Let (X,Γ) be an H-space and let (E,C) be an order complete topological Riesz space, where C is the closed positive cone with a nonempty interior $int(C)$. Let $f,\ g : X \times X \to (E,C)$ be two functions satisfying the following conditions :*

(a) For every $(x,y) \in X \times X$, $g(x,y) \leq f(x,y)$.

(b) For every $y \in X$ and any $\lambda \in E$, the set $\{x \in X : f(x,y) \in \lambda + int(C)\}$ is H-convex.

(c) For every $x \in X$ and any $\lambda \in E$, the set $\{y \in X : g(x,y) \in \lambda + int(C)\}$ is compactly open.

(d) There exists a family $\{(C_i, K_i)\}_{i\in I}$ satisfying condition (i) and (ii) of Definition 2.4 and the following one:

$$\forall i \in I,\ \exists k \in I \text{ such that } \{\mathrm{y} \in \mathrm{Y} : \mathrm{g(x,y)} \notin \lambda + \mathrm{int(C)} \quad \forall \mathrm{x} \in \mathrm{C_k}\} \subset \mathrm{K_i}$$

Then, for every $\lambda \in E$, the following alternative holds:

(1) There exists $y_0 \in X$ such that for every $x \in X$, $g(x,y_0) \notin \lambda + int(C)$.

(2) There exists $x_0 \in X$, such that $f(x_0,x_0) \in \lambda + int(C)$.

Proof. For fixed $\lambda \in E$, we define $F(x) = \{y \in X : f(x,y) \notin \lambda + int(C)\}$ and $G(x) = \{y \in X : g(x,y) \notin \lambda + int(C)\}$. Condition (a) implies $F(x) \subset G(x)$ for every $x \in X$; indeed if $y \notin G(x)$, then $g(x,y) \in \lambda + int(C)$ and there is a neighborhood V of $0 \in E$ such that $g(x,y) + V \subset \lambda + int(C)$. But $g(x,y) \leq f(x,y)$ implies $\lambda < g(x,y) + v \leq f(x,y) + v$, for every $v \in V$ thus $f(x,y) + V \subset \lambda + int(C)$, that is $y \notin F(x)$.

If there exists $x_0 \in X$ with $x_0 \notin F(x_0)$, then $f(x_0,x_0) \in int(C)$ so we have (2). Otherwise $x \in F(x)$ for each $x \in X$. Hence all assumptions of Theorem 3.2 are satisfied, then $\bigcap_{x\in X} G(x) \neq \emptyset$ which implies property (1) of the alternative ■

Corollary 4.1. *Let (X,Γ) be an H-space, (E,C) a completely ordered topological Riesz space. Suppose that $f : X \times X \to (E,C)$ is a function satisfying the following properties:*

(a) f is upper bounded on the set $\triangle = \{(x,x) : x \in X\}$.

(b) For every $y \in X$ and any $\lambda \in E$, the set $\{x \in X : f(x,y) > \lambda\}$ is H-convex.

(c) For every $x \in X$ and any $\lambda \in E$, the set $\{y \in X : f(x,y) \leq \lambda\}$ is compactly closed.

(d) There exists a family $\{(C_i, K_i)\}_{i\in I}$ satisfying condition (i) and (ii) of Definition 4.2 and the following one:

$$\forall i \in I,\ \exists k \in I \text{ such that } \{\mathrm{y} \in \mathrm{Y} : \mathrm{f(x,y)} \leq \lambda \ \ \forall \mathrm{x} \in \mathrm{C_k}\} \subset \mathrm{K_i} \quad \forall \lambda \in \mathrm{E}$$

Then $\inf_{y\in X} \sup_{x\in X} f(x,y) \leq \sup f(x,x)$ (whenever the "inf" in the left-hand side exists).

Proof. Take $\lambda = \sup_{x\in X} f(x,x)$ which is well defined. By Proposition 4.1, there exists $y_0 \in X$ such that:

$$f(x,y_0) \leq \sup_{x\in X} f(x,x) \quad \forall x \in X$$

Since (E,C) is completely ordered, it follows that $\sup_{x\in X} f(x,y_0)$ exists and the result follows ∎

The results of this section generalize Theorem 3 in [4] and Proposition 5.1 in [10] by relaxing the compactness condition. In case $E = I\!R$, all those results generalize previously minimax inequality results obtained in [1] and [7].

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