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ON TELEGRAPH REACTION DIFFUSION AND COUPLED MAP LATTICE IN SOME BIOLOGICAL SYSTEMS

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It is argued that telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion in biological, economic and social systems. Telegraph reaction diffusion (TRD) is studied in one and two spatial dimensions. Some exact and approximate results are obtained. A coupled map lattice (CML) corresponding to the spatial prisoner's dilemma game is constructed and studied in the weak diffusion limit. A formula is derived for Lyapunov exponents and it is shown that periodic solutions are dense in the weak coupling regime and that this system is structurally stable.

Keywords: Telegraph Reaction Diffusion Equation; Coupled Map Lattice; Prisoner's Dilemma Game; Sapatiotemporal Chaos; Lyapunov Exponent.

1. Introduction

Nonlinear dynamics[1] is an important topics in mathematics, physics, biology, economics, etc. In most realistic systems, spacial effects play an important role, hence spatiotemporal chaos (STC)[2–6] acquired its importance. A system is said to be a STC if the number of positive Lyapunov exponents diverges as its size increase. Some characteristic features of STC are:

- it may appear if the system size is large enough,
- it may appear if the system consists of different components.

Both of these properties exists in many biological and economic systems.

We begin by studying the telegraph equation,[7,8] which has been proposed by Ref. 9 to model diffusion in biological, social and economic systems. It will be shown that for large enough systems, the constant solution is unstable. This is a characteristic property of STC. Some exact and approximate solutions of the telegraph equation in one and two spatial dimensions will be derived. The corresponding reaction diffusion equation (TRD) will be studied spatially corresponding to the prisoner's dilemma game.[10] Piecewise linear approximation[11,12] is used to study the two-spatial dimension problem and it is shown that it admits nonzero cooperation.

To study STC for reaction diffusion systems, we use the relation[2] between one dimensional CML, with logistic equation evolution equation, such that the parameter depends on the local site is studied analytically in the weak diffusion limit. The results are generalized to arbitrary evolution equations and a formula for the corresponding Lyapunov exponents is derived. Finally, a results of Afraimovich[13] is used to prove that a CML corresponding to the prisoner dilemma game is structurally stable and that periodic solutions are dense in the weak coupling regime.

2. Basics of Telegraph Equation

The standard diffusion equation depends on the continuity equation plus Fick's first law, i.e.,

$$\frac{\partial c(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial t}\,, \quad J(x,t) = -D\frac{\partial c(x,t)}{\partial x}\,, \tag{1}$$

where $J(x,t)$ is the current of the diffusing object, e.g., technology, concepts etc., $c(x,t)$ is the distribution function of the diffusing quantity and D is the diffusion constant. The resulting standard diffusion equation is[12]:

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}\,. \tag{2}$$

A basic weakness of this equation is that the flux $J(x,t)$ reacts simultaneously to the gradient of $c(x,t)$, consequently an unbounded propagation speed is assumed. This manifests itself in the familiar solution to Eq. (2),

$$c(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}\,, \quad c(x,0) = \delta(x)\,,$$

i.e.

$$c(x,0) > 0\,\forall\, x \quad \text{and} \quad \forall\, t > 0\,.$$

This is unrealistic specially in biological and economical systems, where it is known that in many cases propagation speeds is typically small. To rectify this weakness, Fick's law, Eq. (1), is replaced by:

$$J + \tau\frac{\partial J}{\partial t} = -D\frac{\partial c(x,t)}{\partial x}\,. \tag{3}$$

Combining Eq. (3) with Eq. (1), one obtains the telegraph equation

$$\tau\frac{\partial^2 c(x,t)}{\partial t^2} + \frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}\,, \tag{4}$$

where τ is a time constant, which is a measure of the memory (delay) effect. This has been discussed in detail in Ref. 13. The corresponding telegraph reaction diffusion (TRD) equation is:

$$\tau \frac{\partial^2 c(x,t)}{\partial t^2} + \left(1 - \tau \frac{df}{dc}\right) \frac{\partial c(x,t)}{\partial t} = D \frac{\partial^2 c(x,t)}{\partial x^2} + f(c)\,, \tag{5}$$

where $f(c)$ represents the reaction term.

Now, we can address the critical size problem in TRD system. One considers a population undergoing logistic growth and diffusion. It is required to find the minimum size of habitat required for the survival of the population corresponding to the boundary conditions $c(0,t) = c(L,t) = 0$. Here, $f(c) = \alpha c - \beta c^2$. We look for solutions in the form,

$$c(x,t) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right) e^{\lambda t}\,, \tag{6}$$

which by inspection satisfy the boundary conditions at $x = 0,\ L$. Substituting Eq. (6) into Eq. (5) and for small c, one gets after equating coefficients of $\sin(n\pi x/L)$,

$$\lambda = \frac{1}{2\tau}\left((-1+\alpha\tau) \pm \sqrt{(-1+\alpha\tau)^2 - 4\tau\left(\frac{Dn^2\pi^2}{L^2} - \alpha\right)}\right), \tag{7}$$

and so the solution is given by:

$$\begin{aligned} c = {} & \sum_{n=0} A_n \sin\left(\frac{n\pi x}{L}\right) \\ & \times \exp\left\{\frac{t}{2\tau}\left[(-1+\alpha\tau) + \sqrt{(-1+\alpha\tau)^2 + 4\tau\left(\frac{Dn^2\pi^2}{L^2} - \alpha\right)}\right]\right\} \\ & + \sum_{n=0} A_n \sin\left(\frac{n\pi x}{L}\right) \\ & \times \exp\left\{\frac{t}{2\tau}\left[(-1+\alpha\tau) - \sqrt{(-1+\alpha\tau)^2 + 4\tau\left(\frac{Dn^2\pi^2}{L^2} - \alpha\right)}\right]\right\}. \end{aligned} \tag{8}$$

It is noticed that, the dominant mode in Eq. (8) is that with the largest λ, namely that with $n = 1$, since

$$\left.\begin{aligned} & \frac{1}{2\tau}\left((-1+\alpha\tau) \pm \sqrt{(-1+\alpha\tau)^2 - 4\tau\left(\frac{Dn^2\pi^2}{L^2} - \alpha\right)}\right) < \\ & \frac{1}{2\tau}\left((-1+\alpha\tau) \pm \sqrt{(-1+\alpha\tau)^2 - 4\tau\left(\frac{D\pi^2}{L^2} - \alpha\right)}\right), \quad \forall n \geqslant 2 \end{aligned}\right\}. \tag{9}$$

So, for the population to survive, at least one of the exponents in Eq. (8) should be positive, hence

$$\alpha \geq D\left(\frac{\pi}{L}\right)^2 \quad \text{i.e. } L \geq L_c\,, \quad \text{where } L_c = \pi\sqrt{\frac{D}{f'(0)}}\,, \tag{10}$$

thus, one obtains the following proposition:

Proposition 1. *The critical size of a population undergoing logistic growth and satisfying the above boundary conditions is given by Eq.* (10).

Assuming that

$$f(c) = \alpha c + \beta c^2 + \gamma c^3\,,$$

and studying the stability of the homogeneous solution $c = 0$, one finds (for simplicity, we set the diffusion constant $D = 1$),

$$c(x,t) = \sum_{n=1} a_n(t) e^{inkx}\,,$$

$$a_1(t) = a_1^+(0)e^{r_+ t} + a_1^-(0)e^{r_- t}\,,$$

$$r_\pm = \frac{1}{2\tau}((-1+\alpha\tau) \pm \sqrt{(-1+\alpha\tau)^2 - 4\tau(\alpha - k^2)}\,)\,.$$

Now, if $\alpha > k^2$, i.e., the system size L satisfies $L > 2\pi/\sqrt{\alpha}$, then the solution $c = 0$ is unstable. This type of instability is frequent in spatiotemporal chaotic systems. This will be discussed further later on.

3. Application to Some Biological Problems

Prisoner's dilemma (PD) is a 2×2 symmetric game, where each of the two players has two possible strategies cooperate (C) or defect (D). The payoff matrix is

$$\begin{bmatrix} & C & D \\ C & R & S \\ D & T & U \end{bmatrix},$$

where $T > R > U > S$ and $2R > T + S$. Define $b = U - S$ and $a = R + U - S - T$ ($b > 0$), then the ordinary reaction diffusion equation for PD is:

$$\frac{\partial p}{\partial t} = \nabla^2 p + f(p)\,, \quad f(p) = -p(1-p)(b - ap)\,, \tag{11}$$

where p is the fraction of cooperators in the population. For the case $a = -b$, the following is an exact solution,

$$p = \frac{1}{\frac{1}{\sqrt{2}}\cosh(\sqrt{b}\,y) + C_1 e^{2bt + ix\sqrt{b}}}\,, \tag{12}$$

where C_1 is a constant of integration. This solution is not acceptable in our study since the fraction p is non-negative.

Alternatively, we use piecewise linear approximation[11,12] in which the nonlinear $f(p)$ is approximated by a linear fraction (say $g(p)$), which preserve the essential features of $f(p)$.

For simplicity, set $a = 0$ (the results remain qualitatively correct even for $a < 0$). The piecewise linear approximation of $f(p)$ is[12]:

$$g(p) = \frac{p}{2} \quad \text{if} \;\; 0 \le p \le \frac{1}{2}\,, \quad \text{and} \quad g(p) = \frac{(1-p)}{2} \quad \text{if} \;\; \frac{1}{2} \le p \le 1\,.$$

The interesting region for us is $p > 1/2$, where one gets,

$$p = 1 - C_2 \exp\left[\frac{1}{2}\left(-v + \sqrt{v^2 - 4b + 4w^2}\,\right)\zeta\right] \cos\left(\sqrt{w^2 - \frac{b}{2}}\, y + C_3\right), \tag{13}$$

$\zeta = x - vt$, C_2, C_3 are constants and $w^2 \ge b/2$ and $v \ge \sqrt{2b}$ to guarantee convergence as $t \to \infty$, i.e., $\zeta \to -\infty$. This is an interesting solution since it allows cooperation ($p > 0$) as t increases. This is not the case in one dimension.

Assuming cylindrical symmetry, the standing wave for the system in Eq. (11) is:

$$\frac{\partial p}{\partial t} = 0\,, \quad p(r) = a_0 + a_1 r + \frac{1}{4}\, b a_0 (1 - a_0) r^2 + \frac{1}{9}\, b a_0 (1 - 2a_0) r^3 + O(r^4)\,,$$

where a_0, a_1 are constants to be determined from initial conditions. Since $0 \le p(r)$: then $0 \le a_0 \le 1$ and $a_1 > 0$. It is clear that for $a_0 > 0.5$, $p(r)$ initially increases then decreases to zero. This lends further support that the 2-spatial dimension PD reaction diffusion equation admits cooperation for large time.

Solving the same time dependent problem for the telegraph reaction diffusion, one gets $v = \sqrt{D/\tau}$, which is intrinsic for TRD (since in ordinary reaction diffusion, one has $\tau = 0$), then

$$p = 1 - C_3 \cos(wy + \epsilon) \exp\left[\zeta \left(w^2 - \frac{b}{2}\right)\left(v - \frac{vb\tau}{2}\right)\right], \quad p \ge \frac{1}{2}\,, \tag{14}$$

where C_3, ϵ are constants. Convergence requires $w > \sqrt{b/2}$ and $\tau b/2 < 1$, then $p \to 1$ as $t \to \infty$, i.e., cooperation is possible in two-spatial dimensions. This is not the case in one-dimension.

In many biological systems, one is studying interacting system, e.g., predator-prey, host-parasitoid, susceptible-infected, etc. Hence, the following generalization of TRD to systems is proposed:

$$\tau \frac{\partial^2 u_i}{\partial t^2} + \left(1 - \tau \frac{\partial f_i}{\partial u_i}\right) \frac{\partial u_i}{\partial t} = D_i \nabla^2 u_i + f_i(u_1, u_2, \ldots, u_n)\,, \tag{15}$$

where $i = 1, 2, \ldots, n$.

Applying Eq. (15) to a predator-prey system, then

$$\tau\frac{\partial^2 u_1}{\partial t^2} + (1-\tau a_1)\frac{\partial u_1}{\partial t} = D_1\nabla^2 u_1 + f_1(u_1,u_2)\,,\quad f_1 = a_1u_1 - b_1u_1u_2\,, \tag{16}$$

$$\tau\frac{\partial^2 u_2}{\partial t^2} + (1-\tau a_2)\frac{\partial u_2}{\partial t} = D_2\nabla^2 u_2 + f_2(u_1,u_2)\,,\quad f_2 = -a_2u_2 + b_2u_1u_2\,, \tag{17}$$

where u_1 and u_2 are the population densities of prey and predator, D_1 and D_2 are the diffusivities of the two populations, respectively. a_1, a_2 are the linear ration of birth and death for the individual species; b_1 and b_2 are the linear decay and growth factors due to interaction.

To study the stability of the steady state (coexistence) solution, $f_1(u_1^*, u_2^*) = f_2(u_1^*, u_2^*) = 0$, i.e.,

$$u_1^* = \frac{a_2}{b_2}\,,\quad u_2^* = \frac{a_1}{b_1}\,,$$

assume

$$u_1 = u_1^* + \alpha\cos\left(\frac{n\pi x}{L}\right)e^{\lambda t}\,,\quad u_2 = u_2^* + \gamma\cos\left(\frac{n\pi x}{L}\right)e^{\lambda t}\,,$$

where λ is the eigenvalue, which determine the temporal growth, α and γ are arbitrary constants. So, λ are determined by the roots of the characteristic polynomial,

$$\left|\tau\lambda^2 I + (\lambda B)I - A + D\left(\frac{n\pi}{L}\right)^2\right| = 0\,,$$

where

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},\quad A = \begin{bmatrix} \dfrac{\partial f_1}{\partial u_1} & \dfrac{\partial f_1}{\partial u_2} \\[2ex] \dfrac{\partial f_2}{\partial u_1} & \dfrac{\partial f_2}{\partial u_2} \end{bmatrix}_{(u_1^*, u_2^*)}$$

and

$$B = \begin{bmatrix} \lambda(1-\tau a_1) & 0 \\ 0 & \lambda(1-\tau a_2) \end{bmatrix}.$$

Evaluating the above determinant, we get the eigenvalue λ as the roots of the following equation,

$$\left[\tau\lambda^2 + (1-\tau a_1)\lambda + D_1\left(\frac{n\pi}{L}\right)^2\right]\left[\tau\lambda^2 + (1-\tau a_2)\lambda + D_2\left(\frac{n\pi}{L}\right)^2\right] + a_1a_2 = 0\,.$$

Applying Routh–Hurwitz criteria,[16,17] one gets that for τ sufficiently small, the above coexistence if solution is asymptotically stable.

For example: if

$$n^2\tau \ll 1 \quad \text{and} \quad \tau < \frac{2(n\pi/L)^2}{(D_1 + D_2)},$$

then the solution $u_1^* = a_2/b_2$, $u_2^* = a_1/b_1$ is asymptotically stable.

Now, we look for wave solutions to Eqs. (16) and (17), so let,

$$\left.\begin{aligned} \frac{\ln|u_2|}{(-a_2 + b_2u_1)} - \tau \ln|u_2(-a_2 + b_2u_1)| = A_1 - v\zeta \\ \frac{\ln|u_1|}{(a_1 - b_1u_2)} - \tau \ln|u_1(a_1 - b_1u_2)| = A_2 - v\zeta \end{aligned}\right\}. \tag{18}$$

As $t \to \infty$, this wave tends to $u_1^* = a_2/b_2$, $u_2^* = a_1/b_1$, A_1, A_2 are constants.

A similar problem arises in epidemics, where infected and susceptible correspond to:

$$f_1 = -\mu u_1 + au_1u_2\,, \quad f_2 = \lambda u_2 - au_1u_2\,, \tag{19}$$

and the corresponding waves are:

$$\left.\begin{aligned} \frac{\ln|u_2|}{(\lambda - au_1)} - \tau \ln|u_2(au_1 - \lambda)| = A_1 - v\zeta \\ \frac{\ln|u_1|}{(au_2 - \mu)} - \tau \ln|u_1(au_2 - \mu)| = A_2 - v\zeta \end{aligned}\right\}. \tag{20}$$

As $t \to \infty$, this wave tends to $u_1^* = \lambda/a$, $u_2^* = \mu/a$, if $\lambda/a > 1$, $\mu/a > 1$.

Turing instability in TRD systems is richer than the one in ordinary diffusion. In this case,

$$f_1 = a_{11}u_1 + a_{12}u_2\,, \quad f_2 = a_{21}u_1 + a_{22}u_2\,. \tag{21}$$

Assume $u_j = u_j(0)\exp[\lambda t + ikx]$, then one gets the quartic equation,

$$[\tau\lambda + (1 - \tau a_{11})\lambda + D_1k^2 - a_{11}][\tau\lambda + (1 - \tau a_{22})\lambda + D_2k^2 - a_{22}] - a_{12}a_{21} = 0\,. \tag{22}$$

Turing instability corresponding to the case when the real part of, at least, one of the roots of Eq. (22) is positive, i.e., at least one of Routh–Hurwitz conditions is broken. One possibility is the familiar[18] Turing condition,

$$(D_1a_{22} + D_2a_{11})^2 > 4D_1D_2(a_{11}a_{22} - a_{12}a_{21})\,. \tag{23}$$

But, there are other possibilities, e.g.,

$$\left.\begin{aligned} k^2[\tau(2D_1a_{22} + D_1a_{11} + 2D_2a_{11} + D_2a_{22}) - 3(D_1 + D_2)] > \\ \frac{1}{\tau}\,[2 - 4\tau(a_{11} + a_{22}) + 2\tau^2(a_{11} + a_{22})^2 - \tau^3a_{11}a_{22}(a_{11} + a_{22})] \end{aligned}\right\}. \tag{24}$$

4. Coupled Map Lattices Corresponding to Reaction Diffusion Systems

In general, spatiotemporal chaos (STC) is not easy to study either analytically or numerically. Therefore, we use the procedure in Ref. 2 to derive the corresponding coupled map lattice (CML), which are easier to study and usually much richer in structure. The idea is to discretize the partial differential equation and to assume that the lattice spacing is unity in both space and time. Applying the above procedure to ordinary reaction diffusion equation, one gets the following CML,

$$u_i' = (1-2D)u_i + f_i(u) + D[u_{i+1}(x) + u_{i-1}(x)]\,, \quad u = (u_1, u_2, \ldots, u_n)\,, \tag{25}$$

where u' means u at the next time step. Studying Eq. (25) in the weak coupling domain, i.e., small D, we linearize in it and get the steady states,

$$\left.\begin{aligned} u_i^{ss} &= u_i^0 + Du_i^1\,, \quad \text{where} \;\; f_i(u_i^0) = 0 \\ u_i^1 &= \frac{2[-u_{i-1}^0 - u_{i+1}^0 + 2u_i^0]}{(\partial f_i(u^0)/\partial u_i)} \end{aligned}\right\}. \tag{26}$$

The eigenvalues of the system are given, in this approximation, by:

$$\lambda_i(t) = (1-2D) + \frac{\partial f_i(u^{ss}(t))}{\partial u_i}\,, \tag{27}$$

and the Lyapunov exponents are given by:

$$L_i = \lim \frac{1}{T}\sum_{t=1}^{T} \ln|\lambda_i(t)| \quad \text{for large } T\,. \tag{28}$$

For Prisoner's dilemma problem, Eq. (11), choosing $a = -b$ and using the results of Ref. 13, one gets the following proposition:

Proposition 2. *For sufficiently small diffusion, the symbolic dynamics of the CML, Eq.* (25), *with f given by $f(u) = -bu(1-u)(1+u)$, $b > 0$ is a direct product of the symbolic dynamics of the local map $u' = u[1+b(u^2-1)]$ on 3-symbols. Consequently, periodic orbits are dense in this CML.*

Now, the CML corresponding to the telegraph equation (25) is constructed. Due to the second order time derivative, a new variable is introduced since the local system is two-dimensional, hence one gets the telegraph CML (TCML),

$$\left.\begin{aligned} v_i' &= u_i\,, \\ u_i' &= v_i\left(-1 + \frac{1}{\tau} - f_i'\right) + u_i\left(2 - \frac{1}{\tau} - \frac{2D}{\tau} + f_i'\right) + \frac{f_i}{\tau} + \frac{D(u_{i+1}+u_{i-1})}{\tau} \end{aligned}\right\}, \tag{29}$$

where $f' = df/du$. Assuming $D/\tau \ll 1$, $D \ll 1$, then the steady state is given by Eq. (26) and Lyapunov exponents are:

$$L_i = \lim \frac{1}{2T} \sum_{t=1}^{T} \ln |\lambda_i^+ \lambda_i^-| , \tag{30}$$

$$\lambda_i^\pm = \frac{1}{2} [a_{ii} \pm \sqrt{a_{ii}^2 + 4b}] , \quad a_{ii} = \frac{\partial u_i'}{\partial u_i} , \quad b = \frac{\partial u_i'}{\partial v_i} . \tag{31}$$

5. Conclusions

The telegraph equation, which have been proposed by E. Ahmed and S. Z. Hassan[9] is studied, we have shown that for large enough systems, the constant solution is unstable, some exact and approximate solutions of the telegraph equation in one and two spatial dimensions are derived. The corresponding reaction diffusion (TRD) is studied in one and two spatial dimensions, spatially corresponding to the prisoner's dilemma game. Piecewise linear approximation is used to study the two-spatial dimensional problem and it is shown that it admits nonzero cooperation. Spatiotemporal chaos for the reaction diffusion systems is studied by using the relation between the one dimensional CML, with logistic equation, such that the parameters depends on the local site is studied analytically in the weak diffusion limit. These results are generalized to arbitrary evolution equations and a formula for the corresponding Lyapunov exponents is derived. Also, the results of Afraimovich is used to prove that a CML corresponding to the prisoner dilemma game is structurely stable and that periodic solutions are dense in the weak coupling regime.

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