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Cross-diffusional effect in a telegraph reaction diffusion Lotka–Volterra two competitive system

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Abstract

It is known now that, telegraph equation is more suitable than ordinary diffusion equation in modelling reaction diffusion in several branches of sciences. Telegraph reaction diffusion Lotka–Volterra two competitive system is considered. We observed that this system can give rise to diffusive instability only in the presence of cross-diffusion. Local and global stability analysis in the cross-diffusional effect are studied by considering suitable Lyapunov functional.

1. Introduction

The mathematical model proposed by Shigesada et al. [1] in their study of spatial segregation of interacting species is given by the following strongly coupled parabolic system,

$$\left.\begin{aligned}\frac{\partial u_1}{\partial t} &= \nabla^2[(D_{11}+\alpha_{11}u_1+\alpha_{12}u_2)u_1] + u_1(c_1 - a_{11}u_1 - a_{12}u_2) \quad \text{in } \Omega_T,\\ \frac{\partial u_2}{\partial t} &= \nabla^2[(D_{22}+\alpha_{21}u_1+\alpha_{22}u_2)u_1] + u_1(c_2 - a_{21}u_1 - a_{22}u_2) \quad \text{in } \Omega_T,\end{aligned}\right\} \tag{1}$$

$$\frac{\partial u_i}{\partial v} = 0, \quad i = 1,2 \quad \text{on } \partial\Omega_T. \tag{2}$$

where $\nabla^2 = \sum_{i=1}^{N}(\partial^2 x/\partial x_i^2)$, is the Laplace operator, Ω is a bounded smooth domain on $\mathbb{R}^N$ with $N \geqslant 1$, $\partial\Omega$ and $\overline{\Omega}$ are the boundary and the closure of Ω, respectively, $\Omega_T = \Omega \times [0,T[$ and $\partial\Omega_T = \partial\Omega \times [0,T[$ for some $T \in]0,\infty]$, v is the outward unit normal vector on $\partial\Omega$, D_{ii}, c_i, a_{ii} $(i = 1,2)$ are all positive constants while α_{ii} $(i = 1,2)$ denote non-negative constants. The initial values $u_i(x,0)$ are non-negative smooth functions which are not identically zero. $u_1(x,t)$ and $u_2(x,t)$ represents the densities of two competing species, D_{11} and D_{22} are their diffusion rates, c_1 and c_2 denote the intrinsic growth rates, a_{11} and a_{22} account for intra-specific competitions, a_{21} and a_{12} are the coefficients of intra-specific competitions, α_{11} and α_{22} are usually referred as *self-diffusion* pressures, and α_{12} and α_{21} are *cross-diffusion* pressures.

The term self diffusion implies passive transport and the population that behaves in this way displays movement of individuals from a higher to a lower concentration region. Cross-diffusion term expresses the population fluxes of one species due to the presence of other species. The value of cross-diffusion may be positive, negative or zero. Positive cross-diffusion term denotes one species tends to move in the direction of lower concentration of another species and negative cross-diffusion expresses the population fluxes of one species in the direction of higher concentration of another species. Many workers have observed the effect of cross-diffusion in biochemical systems [2–8]. The idea that

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cross-diffusion can induce pattern forming instability in an ecological situation have been examined in [9,10]. Some mathematical models for population dynamics with the inclusion of cross-diffusion as well as self diffusion are developed and showed that the effect of cross-diffusion may give rise to the segregation of two species [11].

The reaction diffusion Lotka–Volterra two species competition model can be written as [12]:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= D_{11}\frac{\partial^2 u_1}{\partial x^2} + u_1(c_1 - a_{11}u_1 - a_{12}u_2),\\ \frac{\partial u_2}{\partial t} &= D_{22}\frac{\partial^2 u_2}{\partial x^2} + u_2(c_2 - a_{21}u_1 - a_{22}u_2).\end{aligned} \tag{3}$$

System (3) has to be analyzed with the following initial and zero flux boundary conditions

$$u_i(x,t) > 0, \quad \left.\frac{\partial u_i}{\partial x}\right|_{x=0,R} = 0, \quad (0 \leqslant x \leqslant R) \quad i = 1,2. \tag{4}$$

Zero flux conditions imply no external input is imposed from outside [13].

A Lotka–Volterra two species competition model without diffusion (uniform model) is given by

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= u_1(c_1 - a_{11}u_1 - a_{12}u_2),\\ \frac{\partial u_2}{\partial t} &= u_2(c_2 - a_{21}u_1 - a_{22}u_2).\end{aligned} \tag{5}$$

It is known that this system has a unique coexisting equilibrium namely

$$u_1^0 = \frac{c_1a_{22} - c_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \qquad u_2^0 = \frac{c_2a_{11} - c_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \tag{6}$$

if

$$\frac{a_{11}}{a_{21}} > \frac{c_1}{c_2} > \frac{a_{12}}{a_{22}}$$

or

$$\frac{a_{11}}{a_{21}} < \frac{c_1}{c_2} < \frac{a_{12}}{a_{22}}.$$

If $a_{11}a_{22} > a_{12}a_{21}$, then the uniform model (5) is said to describe a tolerant competition and if $a_{11}a_{22} < a_{12}a_{21}$ describe a severe competition. For a tolerant competition this model is locally and globally stable if it has positive equilibrium and in the severe case it is locally unstable if it admits a positive equilibrium and hence according to Turing theory [14], the severe competition of two species model cannot generate a real biological pattern.

The well known telegraph diffusion equation is given by [15]

$$\tau\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \tag{7}$$

where τ is a time constant. The corresponding telegraph reaction diffusion-equation (TRD) given by [16,17] is

$$\tau\frac{\partial^2 u}{\partial t^2} + \left(1 - \tau\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \tag{8}$$

where $f(u)$ represents the interacting term.

In many biological systems one studies interacting system, e.g., predator–prey, host–parasitoid, susceptible–infected, etc... Hence the following generalization of TRD systems as proposed by [17] is

$$\tau\frac{\partial^2 u_i}{\partial t^2} + \frac{\partial u_i}{\partial t} - \tau\sum_j \frac{\partial u_j}{\partial t}\frac{\partial f_i}{\partial u_j} = D_{il}\frac{\partial^2 u_i(x,t)}{\partial x^2} + f_i(u_1, u_2, \ldots, u_n), \tag{9}$$

where $i = 1,2,\ldots,n$, $l = 1,2,\ldots,n$ and D_{il} are the diffusion constants.

The case $n = 2$ to a predator (u_2)–prey (u_1) system was applied to study Turing instability by Ahmed et al. [18]. Here we suggest the following system:

$$\left.\begin{aligned} &\tau\frac{\partial^2 u_1}{\partial t^2}+\left(1-\tau\frac{\partial f_1}{\partial u_1}\right)\frac{\partial u_1}{\partial t}=D_{11}\frac{\partial^2 u_1}{\partial x^2}+f_1(u_1,u_2),\\ &\quad f_1=u_1(c_1-a_{11}u_1-a_{12}u_2),\\ &\tau\frac{\partial^2 u_2}{\partial t^2}+\left(1-\tau\frac{\partial f_2}{\partial u_2}\right)\frac{\partial u_2}{\partial t}=D_{22}\frac{\partial^2 u_2}{\partial x^2}+f_2(u_1,u_2),\\ &\quad f_2=u_2(c_2-a_{21}u_1-a_{22}u_2), \end{aligned}\right\} \tag{10}$$

where

$$u_i(x,t)>0,\quad \left.\frac{\partial u_i}{\partial x}\right|_{x=0,R}=0,\quad (0\leqslant x\leqslant R),\quad 0<t<\infty,\quad a_{ij}>0,\quad \forall\, i,j=1,2. \tag{11}$$

We call this system a telegraph reaction diffusion Lotka–Volterra two species competitive system.

Chattopadhyay and Chatterjee [12] showed that in the Lotka–Volterra diffusive model, cross-diffusional effect is necessary for forming spatial pattern in Turing sense and the pattern thus evolved is globally stable in a suitable parametric domain space. We will see in this paper that such results can be extended to our suggested system (10) and (11). At present, in Section 2, we study the boundedness of system (10) and (11). Section 3 is devoted to study the local and global stability analysis of our system. Section 4 is devoted for our conclusions.

2. Cross-diffusional effect in a telegraph reaction diffusion Lotka–Volterra two competitive system

Adding a cross-diffusion term in the first equation of (3), the system can be written as [12]

$$\begin{aligned} &\frac{\partial u_1}{\partial t}=D_{11}\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial}{\partial x}\left[D_{12}(u_1)\frac{\partial u_2}{\partial x}\right]+u_1(c_1-a_{11}u_1-a_{12}u_2),\\ &\frac{\partial u_2}{\partial t}=D_{22}\frac{\partial^2 u_2}{\partial x^2}+u_2(c_2-a_{21}u_1-a_{22}u_2), \end{aligned} \tag{12}$$

where $D_{12}(u_1)$ is the density dependent cross-diffusion coefficient of u_1 such that

$$D_{12}(u_1)\to 0,\quad \text{when } u_1\to 0.$$

System (12) has to be analyzed with the same initial and zero flux boundary conditions (4).

As a special case, Chattopadhyay and Chatterjee are assumed that the density dependent cross-diffusion coefficient is given by [12]

$$D_{12}(u_1)=D'_{12}\left(\frac{u_1}{\epsilon+u_1}\right)=D'_{12}\left(1-\frac{\epsilon}{u_1}+\cdots\right), \tag{13}$$

where $\epsilon>0$ is very small so that

$$D_{12}(u_1)\approx D'_{12}\quad \forall\ u_1\gg\epsilon.$$

Chattopadhyay and Chatterjee studied the local stability analysis of the system (12) and (4) and they concluded that for formation of biological pattern in Lotka–Volterra tolerant competitive system, cross-diffusion term in either species is essential, i.e., cross-diffusion has an important role for achieving spatial pattern in this particular case. Also, they prove the following important theorem:

Theorem 1. *The interior equilibrium point $E(u_1^0(x),u_2^0(x))$ of the system* (12) *and* (4) *in the domain W is globally asymptotically stable, if,*

(i) $4a_{11}a_{22}>(a_{12}+a_{21})^2$,
(ii) $D_{11}D_{22}/D'^2_{12}$ *have a lower and upper threshold value, where*

$$W=\{(u_1,u_2): u_1\gg\epsilon \text{ and } u_2>0\}. \tag{14}$$

The homogeneous (uniform) system of (10) is given by

$$\left.\begin{aligned} &\tau\frac{\partial^2 u_1}{\partial t^2}+\left(1-\tau\frac{\partial f_1}{\partial u_1}\right)\frac{\partial u_1}{\partial t}=f_1(u_1,u_2),\\ &\tau\frac{\partial^2 u_2}{\partial t^2}+\left(1-\tau\frac{\partial f_2}{\partial u_2}\right)\frac{\partial u_2}{\partial t}=f_2(u_1,u_2). \end{aligned}\right\} \tag{15}$$

At first we shall show that all the solutions of the system (15) are uniformly bounded. For this purpose, we have the following theorem:

Theorem 2. *All the solutions of the homogeneous system* (15) *which initiate in* R_2^+ *are uniformly bounded, where*

$$R_2^+ = \{(u_1, u_2) | u_1 > 0, \quad u_2 > 0\}. \tag{16}$$

Proof. Define the function

$$W = u_1 + u_2. \tag{17}$$

The time derivative of (16) along the solutions of the homogeneous system (15) is

$$\tau \frac{d^2 W}{dt^2} = (c_1 u_1 - a_{11} u_1^2 - a_{12} u_1 u_2) + (c_2 u_2 - a_{21} u_1 u_2 - a_{22} u_2^2) - \left(1 - \tau \frac{\partial f_1}{\partial u_1}\right) \frac{\partial u_1}{\partial t} - \left(1 - \tau \frac{\partial f_2}{\partial u_2}\right) \frac{\partial u_2}{\partial t}$$

and then

$$\tau \frac{d^2 W}{dt^2} - \tau (c_1 + c_2) \frac{dw}{dt} \leqslant (c_1 u_1 - a_{11} u_1^2) + (c_2 u_2 - a_{22} u_2^2) + c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t}$$

for $\lambda > 0$, we have

$$\tau \frac{d^2 W}{dt^2} - \tau (c_1 + c_2) \frac{dw}{dt} + \lambda W \leqslant (c_1 u_1 - a_{11} u_1^2 + \lambda u_1) + (c_2 u_2 - a_{22} u_2^2 + \lambda u_2) \leqslant \frac{(\lambda + c_1)^2}{4 a_{11}} + \frac{(\lambda + c_2)^2}{4 a_{22}}.$$

then, we can find a positive constant $K > 0$ such that

$$\tau \frac{d^2 W}{dt^2} - \tau (c_1 + c_2) \frac{dw}{dt} + \lambda W \leqslant K.$$

Applying a theorem on differential inequality [19], we have

$$0 \leqslant W(u_1, u_2) \leqslant A + B[e^{\sigma_1 t} + e^{\sigma_2 t}],$$

where $e^{\sigma_1 t}$, $e^{\sigma_2 t}$ are the solutions of the equation

$$\tau \frac{d^2 W}{dt^2} - \tau (c_1 + c_2) \frac{dw}{dt} + \lambda W = 0$$

and

$$\sigma_1 = \frac{(c_1 + c_2) + \sqrt{(c_1 + c_2)^2 - 4\lambda/\tau}}{2},$$

$$\sigma_2 = \frac{(c_1 + c_2) - \sqrt{(c_1 + c_2)^2 - 4\lambda/\tau}}{2}$$

and as $t \to \infty$, we have $0 \leqslant W \leqslant A$. Hence all the solutions of the homogeneous system (15) that initiate in R_2^+ are confined in the region $G = [(x, y) \in R_2^+ : W < A + h, \text{ for any } h > 0]$. □

3. Local and global stability analysis of the cross-diffusional effect in a telegraph reaction diffusion Lotka–Volterra two competitive system

To study the local stability analysis of telegraph reaction diffusion system (10) and (11), we must find the coexistence solution of (10) which is given by

$$(u_1^0, u_2^0) = \left(\frac{c_1 a_{22} - c_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}}, \quad \frac{c_2 a_{11} - c_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}} \right). \tag{18}$$

Linearizing the system (10) around (u_1^0, u_2^0), we get

$$|\tau\lambda^2\mathbf{I} - (\lambda\mathbf{B} - \mathbf{A}) + \mathbf{D}k^2| = 0,$$

where

$$\mathbf{D} = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \left(1 - \tau\frac{\partial f_1}{\partial u_1}\right) & 0 \\ 0 & \left(1 - \tau\frac{\partial f_2}{\partial u_2}\right) \end{pmatrix}_{(u_1^0, u_2^0)},$$

$$\mathbf{A} = \begin{pmatrix} c_1 - 2a_{11}u_1 - a_{12}u_2 & -a_{12}u_1 \\ -a_{21}u_2 & c_2 - 2a_{21}u_1 - 2a_{22}u_2 \end{pmatrix}_{(u_1^0, u_2^0)}.$$

Calculating the above determinant, we get

$$a_{21}a_{12}u_1^0u_2^0 = (\tau\lambda^2 + \lambda(1 + \tau a_{11}u_1^0) + a_{11}u_1^0 + D_{11}k^2)(\tau\lambda^2 + \lambda(1 + \tau a_{22}u_2^0) + a_{22}u_2^0 + D_{22}k^2)$$

which can be written as

$$\tau^2\lambda^4 + \beta_1\lambda^3 + \beta_2\lambda^2 + \beta_3\lambda + \beta_4 = 0, \tag{19}$$

where

$$\begin{aligned} \beta_1 &= \tau[(1 + \tau a_{11}u_1^0) + (1 + \tau a_{22}u_2^0)], \\ \beta_2 &= \tau[(a_{22}u_2^0 + D_{22}k^2) + (a_{11}u_1^0 + D_{11}k^2)] + (1 + \tau a_{11}u_1^0)(1 + \tau a_{22}u_2^0), \\ \beta_3 &= (a_{22}u_2^0 + D_{22}k^2)(1 + \tau a_{11}u_1^0) + (a_{11}u_1^0 + D_{11}k^2)(1 + \tau a_{22}u_2^0), \\ \beta_4 &= (a_{11}a_{22} - a_{21}a_{12})u_1^0u_2^0 + a_{12}a_{21}u_1^0k^2 + a_{11}D_{22}u_1^0k^2 + a_{22}D_{11}u_2^0k^2 + D_{11}D_{22}k^2. \end{aligned} \tag{20}$$

It is clear that $\beta_i > 0 \ \forall\ i = 1, 2, 3$ and for tolerant competition (weak competition) case $(a_{11}a_{22} - a_{21}a_{12}) > 0$, then $\beta_4 > 0$.

Hence, we may conclude that self-diffusion cannot alone generate spatial pattern in a two species telegraph reaction diffusion tolerant competition model.

Now, to generate spatial pattern in a two species Lotka–Volterra model (10), we add a cross-diffusion term in the first equation of that system. In this case the system take the new form

$$\left.\begin{aligned} \tau\frac{\partial^2 u_1}{\partial t^2} + \left(1 - \tau\frac{\partial f_1}{\partial u_1}\right)\frac{\partial u_1}{\partial t} &= D_1\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial}{\partial x}\left[D_{12}(u_1)\frac{\partial u_1}{\partial x}\right] + f_1(u_1, u_2), \\ \tau\frac{\partial^2 u_2}{\partial t^2} + \left(1 - \tau\frac{\partial f_2}{\partial u_2}\right)\frac{\partial u_2}{\partial t} &= D_2\frac{\partial^2 u_2}{\partial x^2} + f_2(u_1, u_2), \end{aligned}\right\} \tag{21}$$

as in the case of the Lotka–Volterra two competition cross-diffusion system, we assume that

$$D_{12}(u_1) = D'_{12}\left(\frac{u_1}{\epsilon + u_1}\right), \quad \epsilon > 0 \text{ and very small}$$

then $D_{12}(u_1) \simeq D'_{12} \ \forall\ u_1 > \epsilon$. The population domain is given by R_2^+.

The characteristic equation of the system (21) in the population domain R_2^+ is given by

$$\tau^2\lambda^4 + \beta_1\lambda^3 + \beta_2\lambda^2 + \beta_3\lambda + \left(\beta_4 - a_{21}D'_{12}u_2^0k^2\right) = 0, \tag{22}$$

where $\beta_i > 0 \ \forall\ i = 1, 2, 3, 4$, are given by (20) and k is non-zero constant wave length parameter and $2\pi k$ is the period of cosine.

For τ sufficiently small and neglecting term of order τ^2, Eq. (22) reduces to the following form

$$\lambda^3 + \frac{\beta_2}{\beta_1}\lambda^2 + \frac{\beta_3}{\beta_1}\lambda + \frac{\left(\beta_4 - a_{21}D'_{12}u_2^0k^2\right)}{\beta_1} = 0. \tag{23}$$

Thus, the condition for local *instability* is

$$\beta_2\beta_3 < \left(\beta_4 - a_{21}D'_{12}u_2^0k^2\right)\beta_1$$

and then

$$D'_{12} < \frac{(\beta_4\beta_1 - \beta_2\beta_3)}{k^2\beta_1 a_{21}u_2^0} \Rightarrow k^2 < \frac{(\beta_1\beta_4 - \beta_2\beta_3)}{D'_{12}a_{21}u_2^0\beta_1}. \tag{24}$$

Hence, we may conclude that for formation of biological pattern in telegraph Lotka–Volterra tolerant competitive system, cross-diffusion term in either species is essential.

Definition 1. A property is global if it cannot be analyzed in an arbitrary small neighborhoods of a single point.

Theorem 3 (Lyapunov first theorem). *Consider the system*

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n$$

for which x^0 is a fixed point. x^0 is stable fixed point if there exists a function $L(x)$ such that:

$$L(x^0) = 0 \quad \text{and} \quad L(x) > 0 \quad \text{if } x \neq x^0 \text{ in some } \Omega \text{ about } x^0;$$

the function $L(x)$ which satisfies these condition is called a Lyapunov function.

Now, to study the global stability analysis of the interior equilibrium point $E(u_1^0(x), u_2^0(x))$ for the system (21), we shall define the following Lyapunov function

$$\begin{aligned} V(u_1,u_2) = {} & (1+\tau(2a_{11}u_1^0 + a_{12}u_2^0)) \int_0^R \left[u_1 - u_1^0 - u_1^0 \ln\left(\frac{u_1}{u_1^0}\right)\right] dx \\ & - (1+\tau(a_{21}u_1^0 + 2a_{22}u_2^0)) \int_0^R \left[u_2 - u_2^0 - u_2^0 \ln\left(\frac{u_2}{u_2^0}\right)\right] dx, \end{aligned} \tag{25}$$

where $u_1 > 0$, $u_2 > 0$ and $u_1(x) = u_1^0$, $u_2(x) = u_2^0$.

It is easy to check that $V(u_1,u_2)$ is non-negative and $V(u_1,u_2) = 0$ iff $u_1(x) = u_1^0$, $u_2(x) = u_2^0$.

The following theorem is important to find out the criteria for global stability of the environmental pattern formed by the system (21).

Theorem 4. *The interior equilibrium point $E((u_1^0(x), u_2^0(x))$ of the system* (21) *in the domain R_2^+ is global asymptotically stable if the rate of change of the Lyapunov function* (25) *is negative i.e., if the following conditions hold*:

(i) $4k_1k_2a_{11}a_{22} > (k_1a_{12} + k_2a_{21})^2$,
(ii) $D_{11}D_{22}/(D'_{12})^2 > k_1u_2/4k_2u_1$ (*have a lower and upper threshold value*),
(iii) *the characteristic time constant τ is positive*,

where k_1 and k_2 are two arbitrary positive constants.

Proof. We can write system (21) as follows

$$\begin{aligned} (1+\tau(2a_{11}u_1^0 + a_{12}u_2^0))\frac{\partial u_1}{\partial t} &= D_1\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial}{\partial x}\left[D_{12}(u_1)\frac{\partial u_1}{\partial x}\right] - \tau\frac{\partial^2 u_1}{\partial t^2} + f_1(u_1,u_2), \\ (1+\tau(2a_{21}u_1^0 + a_{22}u_2^0))\frac{\partial u_2}{\partial t} &= D_2\frac{\partial^2 u_2}{\partial x^2} - \tau\frac{\partial^2 u_2}{\partial t^2} + f_2(u_1,u_2), \end{aligned} \tag{26}$$

Calculating the rate of change of the Lyapunov function (25), along the solution of (26), we have

$$\begin{aligned} \frac{dV}{dt} = {} & \left(1+\tau(2a_{11}u_1^0 + a_{12}u_2^0)\right) \\ & \times \int_0^R \left[(u_1 - u_1^0)\left\{ - a_{11}u_1 - a_{12}u_2 + \frac{D_{11}}{u_1}\frac{\partial^2 u_1}{\partial x^2} + \frac{D'_{12}}{u_1}\frac{\partial^2 u_2}{\partial x^2} + a_{11}u_1^0 + a_{12}u_2^0\right\} dx - \tau\frac{\partial^2 u_1}{\partial t^2}\right] dx \\ & - (1+\tau(a_{21}u_1^0 + 2a_{22}u_2^0)) \int_0^R \left[(u_2 - u_2^0)\left\{ - a_{21}u_1 - a_{22}u_2 + \frac{D_{22}}{u_2}\frac{\partial^2 u_2}{\partial x^2} + a_{21}u_2^0 + a_{22}u_2^0\right\} dx - \tau\frac{\partial^2 u_2}{\partial t^2}\right] dx. \end{aligned} \tag{27}$$

From Theorem 2, we have $(u_1^0 + u_2^0)$ is bounded, then there exist two positive constants k_1 and k_2 such that

$$(1+\tau(2a_{11}u_1^0 + a_{12}u_2^0)) \leqslant k_1, \quad (1+\tau(2a_{21}u_1^0 + a_{22}u_2^0)) \leqslant k_2 \tag{28}$$

then,

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -\int_0^R \left[\left\{ a_{11}k_1(u_1-u_1^0)^2 + (k_1a_{12}+k_2a_{21})(u_1-u_1^0)u_2-u_2^0) + a_{22}k_2(u_2-u_2^0)^2 \right\} + \left\{ -k_1\frac{D_{11}}{u_1}\frac{\partial u_1}{\partial x}\frac{\partial u_1^0}{\partial x} - k_1\frac{D'_{12}}{u_1}\frac{\partial u_1}{\partial x}\frac{\partial u_2^0}{\partial x} - k_2\frac{D_{22}}{u_2}\frac{\partial u_2}{\partial x}\frac{\partial u_2^0}{\partial x} + k_1D_{11}\frac{u_1^0}{u_1^2}\left(\frac{\partial u_1}{\partial x}\right)^2 + k_1D'_{12}\frac{u_1^0}{u_1^2}\frac{\partial u_1}{\partial x}\frac{\partial u_2^0}{\partial x} + k_2D_{22}\frac{u_2^0}{u_2^2}\left(\frac{\partial u_2}{\partial x}\right)^2 \right\} + \tau\left(k_1\frac{\partial^2 u_1}{\partial t^2} + k_2\frac{\partial^2 u_2}{\partial t^2}\right)\right]\mathrm{d}x. \tag{29}$$

We need all the solutions ended at the interior equilibrium, hence we must have $u_1(x,t) \geqslant u_1^0(x)$, $u_2(x,t) \geqslant u_2^0(x)$ then

$$\frac{u_1^0}{u_1^2} \leqslant \frac{1}{u_1}, \quad \frac{u_2^0}{u_2^2} \leqslant \frac{1}{u_2}. \tag{30}$$

Using (29) and (30) we get

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -\int_0^R \left[\left\{ a_{11}k_1(u_1-u_1^0)^2 + (k_1a_{12}+k_2a_{21})(u_1-u_1^0)(u_2-u_2^0) + a_{22}k_2(u_2-u_2^0)^2 \right\} + \left\{ -k_1\frac{D_{11}}{u_1}\frac{\partial u_1}{\partial x}\frac{\partial u_1^0}{\partial x} - k_1\frac{D'_{12}}{u_1}\frac{\partial u_1}{\partial x}\frac{\partial u_2^0}{\partial x} - k_2\frac{D_{22}}{u_2}\frac{\partial u_2}{\partial x}\frac{\partial u_2^0}{\partial x} + k_1D_{11}\frac{1}{u_1}\left(\frac{\partial u_1}{\partial x}\right)^2 + k_1D'_{12}\frac{1}{u_1}\frac{\partial u_1}{\partial x}\frac{\partial u_2^0}{\partial x} + k_2D_{22}\frac{1}{u_2}\left(\frac{\partial u_2}{\partial x}\right)^2 \right\} + \tau\left(k_1\frac{\partial^2 u_1}{\partial t^2} + k_2\frac{\partial^2 u_2}{\partial t^2}\right)\right]\mathrm{d}x. \tag{31}$$

The first and second term of (31) in the curly brackets can be expressed in the form $(-\mathbf{X}^{\mathrm{T}}\mathbf{A}\mathbf{X})$ and $(-\mathbf{Y}^{\mathrm{T}}\mathbf{B}\mathbf{Y})$ respectively, where

$$\mathbf{Y} = \left(\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, \frac{\partial u_1^0}{\partial x}, \frac{\partial u_2^0}{\partial x}\right), \quad \mathbf{A} = \begin{pmatrix} k_1a_{11} & \frac{(k_1a_{12}+k_2a_{21})}{2} \\ \frac{(k_1a_{12}+k_2a_{21})}{2} & k_2a_{22} \end{pmatrix},$$

$$\mathbf{X} = (u_1-u_1^0, u_2-u_2^0), \quad \mathbf{B} = \begin{pmatrix} \frac{k_1D_{11}}{u_1} & \frac{k_1D'_{12}}{2u_1} & -\frac{k_1D_{11}}{2u_1} & 0 \\ \frac{k_1D'_{12}}{2u_1} & \frac{k_2D_{22}}{u_2} & -\frac{k_1D'_{12}}{2u_1} & -\frac{k_2D_{22}}{2u_2} \\ -\frac{k_1D_{11}}{2u_1} & -\frac{k_1D'_{12}}{2u_1} & 0 & 0 \\ 0 & -\frac{k_2D_{22}}{2u_2} & 0 & 0 \end{pmatrix}$$

and then

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -\int_0^R \left[\left[(-\mathbf{X}^{\mathrm{T}}\mathbf{A}\mathbf{X})(-\mathbf{Y}^{\mathrm{T}}\mathbf{B}\mathbf{Y}) \right] + \tau\left(k_1\frac{\partial^2 u_1}{\partial t^2} + k_2\frac{\partial^2 u_2}{\partial t^2}\right)\right]\mathrm{d}x. \tag{32}$$

The quantity $\mathrm{d}V/\mathrm{d}t$ is negative definite if the symmetric matrices **A** and **B** are positive definite, and the time constant $\tau > 0$. **A** is positive definite if $4k_1k_2a_{11}a_{22} > (k_1a_{12}+k_2a_{21})^2$ and **B** is positive definite if the second principle minor determinant of **B** is positive definite if

$$\frac{k_2D_{11}D_{22}}{k_1(D'_{12})^2} > \frac{u_2}{4u_1}.$$

Since both u_1, u_2 are bounded, then the above relation is true if $D_{11}D_{22}/(D'_{12})^2 > k_1u_2/4k_2u_1$, i.e., $D_{11}D_{22}/(D'_{12})^2$ have a lower and upper bound, hence **B** is positive definite if condition (ii) hold it is clear that, the 1st, 2nd and the 4th principles of the determinant of **B** are always positive. □

4. Conclusions

The boundedness of the homogeneous system of our suggested model are studied and we found that, it is uniformly bounded in the domain of solutions. Also, local stability analysis of the telegraph reaction diffusion system was studied. We conclude that, self-diffusion cannot alone generate spatial pattern in two species telegraph reaction diffusion tolerant competition model. After adding cross-diffusion term in either species, we found that, biological patterns are formed. An important criteria for global stability of the environmental pattern formed by the system are found.

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