Exact solution of the generalized time-delayed Burgers equation through the improved tanh–function method

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Abstract

The generalized time-delayed Burgers equation is introduced and the improved tanh-function method is used to construct exact multiple soliton and triangular periodic solutions. The obtained solutions reduces to the generalized and classical Burgers equations as a special cases.

Key words: *Travelling wave, tanh-function method, time-delayed Burgers equation,.*

1 Introduction

A number of nonlinear phenomena in many branches of sciences such as physical [1], chemical, economical [2] and biological processes [3,4] are described by the interplay of reaction and diffusion or by the interaction between convection and diffusion. One of the well known partial differential equations which governs a wide variety of them is Burgers equation which provides the simplest nonlinear model of turbulence [5]. The existence of relaxation time or delay time is an important feature in reaction diffusion and convection diffusion systems [6-8].

The generalized time-delayed Burgers equation can be derived by a similar manner to that presented in [6]. It takes the following form:

$$\tau u_{tt} + [1 - \tau f_u] u_t = u_{xx} - p u^s u_x, \tag{1}$$

where τ, p, are any real numbers and $s \in \mathbb{N}$.

It is clear that when $p = s = 1$, equation (1) reduces to the form of the classical Burgers equation [5]. Recently, many methods have been used to find exact solutions of nonlinear partial differential equations, such as the tanh function method [9-13], Jacobi elliptic function method [14], Simplest equation method [15], unified algebraic method [16], and the factorization method [17-19] .

The paper is organized as follows: In Section 2, the improved tanh function method is presented. In Section 3, we use the improved tanh-function method to obtain multiple soliton and triangular periodic solutions of the generalized time-delayed Burgers equation. Section4, is devoted for the conclusion.

2 The improved tanh-function method

1. Consider a general form of nonlinear partial differential equation (PDE)

$$N(u, u_t, u_x, u_{xx}, \cdots) = 0. \tag{2}$$

2. We introduce the wave variable $\zeta = k(x - \omega t)$ to find the travelling wave solution, then

$$u(x,t) = U(\zeta), \tag{3}$$

where k and ω are the wave number and the wave speed respectively. Thus, we have

$$u_t = -k\omega U'(\zeta), \quad u_{tt} = k^2\omega^2 U''(\zeta), \quad u_x = kU'(\zeta), \quad u_{xx} = k^2 U'(\zeta), \ \cdots, \tag{4}$$

and the PDE (2) reduces to an ordinary differential equation (ODE) given by,

$$N(U, U', U'', \cdots) = 0. \tag{5}$$

3. If all terms of (5) contain derivatives in ζ, then by integrating this equation and taking the constant of integration to be zero, we obtain a simplified ODE.
4. Introduce

$$U(\zeta) = \sum_{i=0}^{n} a_i F^i(\zeta), \tag{6}$$

where n is a positive integer that can be determined by balancing the linear term with the nonlinear term in equation (5); a_i , $i = 1, 2, \cdots, n$, are parameters to be determined and $F(\zeta)$ is a solution of the Riccati equation that tanh function satisfies, i.e.

$$F' = CF^2 + A, \tag{7}$$

where A, C are constants. The relations between values of A, C and corresponding $F(\zeta)$ in (7) is given in the following table.

Case	A	C	$F(\zeta)$
1	$\frac{1}{2}$	$-\frac{1}{2}$	$\coth\zeta \pm \cosh\zeta$, $\tanh\zeta \pm I\,\text{sech}\,\zeta$, $I=\sqrt{-1}$
2	$\frac{1}{2}$	$\frac{1}{2}$	$\sec\zeta \pm \tan\zeta$
3	$-\frac{1}{2}$	$-\frac{1}{2}$	$(\csc\zeta \mp \cot\zeta)$
4	1	-1	$\tanh\zeta$, $\coth\zeta$
5	1	1	$\tan\zeta$
6	-1	-1	$\cot\zeta$

5. Introducing (7) into (6) and then substituting (6) into equation (5) yields a set of algebraic equations involving a_i, $(i = 1, 2, \cdots .n)$, k, ω because all coefficients of $F^i(\zeta)$ have to vanish. Having determined these parameters, we obtain an analytic solutions in closed form.

3 Analytic solutions of the generalized time-delayed Burgers equation

The time-delayed Burgers equation is given by

$$\tau u_{tt} + u_t + pu^s u_x - u_{xx} = 0. \tag{8}$$

In order to obtain travelling wave solutions of (8), we set

$$u(x,t) = U(\zeta), \qquad \zeta = k(x - \omega t). \tag{9}$$

Substituting (9) into (8), we find that

$$(\tau\omega^2 - 1)k^2 U'' - k\omega U' + pkU^s U' = 0, \qquad \tau\omega^2 > 1. \tag{10}$$

Balancing U'' with $U^s U'$ gives $n = 1/s$ which is not an integer as $s \neq 1$. But we need the balancing number to be a positive integer so as to apply the ansatz (6) and (7). We make a transformation

$$U = V^{\frac{1}{s}}. \tag{11}$$

Using this transformation, equation (10) changes to the form:

$$(\tau\omega^2 - 1)k^2[V''V + (\frac{1}{s} - 1)V'^2] - k\omega VV' + pkV^2V' = 0. \tag{12}$$

Balancing V'^2 with V^2V' gives $n = 1$. Therefore, we may choose the following ansatz:

$$V(\zeta) = a_0 + a_1 F(\zeta). \tag{13}$$

Then

$$U(\zeta) = [a_0 + a_1 F(\zeta)]^{1/s}. \tag{14}$$

Now, substituting (13) into (6) along with equation (7) and using Mathematica, yields a system of equations with respect to F^i. Setting the coefficients of F^i in the resulting system of equations to zero, we can deduce the following set of algebraic polynomials with respect to the unknowns a_0, a_1, namely:

$$\begin{aligned}
-A\omega a_0a_1 + Apa_0^2a_1 + A^2a_1^2 - \frac{A^2a_1^2}{s} + A^2\tau\omega^2a_1^2 + \frac{A^2\tau\omega^2a_1^2}{s} &= 0,\\
-2ACa_0a_1 + 2A\omega^2\tau Ca_0a_1 - A\omega a_1^2 + 2Apa_0a_1^2 &= 0,\\
-2C^2a_0a_1 + 2C^2\tau\omega^2a_0a_1 - C\omega a_1^2 + 2Cpa_0a_1^2 &= 0, \quad (15)\\
-C\omega a_0a_1 + Cpa_0^2a_1 - \frac{C^2\omega^2\tau a_1^2}{s} + \frac{2AC\omega^2a_1^2}{s} + Apa_1^3 &= 0,\\
-C^2a_1^2 - \frac{C^2a_1^2}{s} + \frac{C^2\omega^2\tau a_1^2}{s} + C^2\omega^2\tau a_1^2 + Cpa_1^3 &= 0.
\end{aligned}$$

Solving the above system of equations by using Mathematica, we find the following set of solutions that correspond to some values of A and C, see Table 1.

Case 1: $A = \frac{1}{2}$, $C = -\frac{1}{2}$,

$$a_0 = \frac{(1+s)\omega}{2p}, \qquad a_1 = \mp\frac{(1+s)\omega}{2p}, \quad k_1 = \mp\frac{s\omega}{(\tau\omega^2-1)}. \tag{16}$$

Case 2: $A = C = \frac{1}{2}$,

$$a_0 = \frac{(1+s)\omega}{2p}, \qquad a_1 = \mp\frac{(1+s)\omega I}{2p}, \quad k_2 = \pm\frac{s\omega I}{(\tau\omega^2-1)}. \tag{17}$$

Case 3: $A = C = -\frac{1}{2}$,

$$a_0 = \frac{(1+s)\omega}{2p}, \qquad a_1 = \mp\frac{(1+s)\omega I}{2p}, \quad k_3 = \mp\frac{s\omega I}{(\tau\omega^2-1)}. \tag{18}$$

Case 4: $A = 1, C = -1$,

$$a_0 = \frac{(1+s)\omega}{2p}, \qquad a_1 = \pm\frac{(1+s)\omega}{2p}, \quad k_4 = \pm\frac{s\omega}{2(\tau\omega^2-1)}. \tag{19}$$

Case 5: $A = C = 1$,

$$a_0 = \frac{(1+s)\omega}{2p}, \qquad a_1 = \mp\frac{(1+s)\omega}{2p}, \quad k_5 = \pm\frac{s\omega I}{2(\tau\omega^2-1)}. \tag{20}$$

Case 6: $A = C = -1$,

$$a_0 = \frac{(1+s)\omega}{2p}, \qquad a_1 = \mp\frac{(1+s)\omega I}{2p}, \quad k_6 = \mp\frac{s\omega I}{2(\tau\omega^2-1)}. \tag{21}$$

Substituting (16) - (21) into (14) and using special solutions of equation (7), according to the cases introduced in Table 1, we obtain the following multiple soliton and triangular periodic solutions of equation (8),

$$u_1(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega}{2p}(\tanh\zeta \pm i\,\mathrm{sech}\,\zeta)\right]^{1/s}, \qquad \zeta = k_1\,(x-\omega t)\,, \tag{22}$$

$$u_2(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}(\coth\zeta \pm \mathrm{coch}\,\zeta\right]^{1/s}, \qquad \zeta = k_2\,(x-\omega t)\,, \tag{23}$$

$$u_3(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}\tanh\zeta\right]^{1/s}, \qquad \zeta = k_3\,(x-\omega t)\,, \tag{24}$$

$$u_4(x,t) = \left[\frac{(1+s)\omega}{2p} \pm \frac{(1+s)\omega}{2p}\coth\zeta\right]^{1/s}, \qquad \zeta = k_4\,(x-\omega t)\,, \tag{25}$$

$$u_5(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega}{2p}\tan\zeta\right]^{1/s}, \qquad \zeta = k_5\,(x-\omega t)\,, \tag{26}$$

$$u_6(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}\cot\zeta\right]^{1/s}, \qquad \zeta = k_6\,(x-\omega t)\,. \tag{27}$$

When $\tau \to 0$, equations (22)-(27), is reduced to the solutions of the generalized Burgers equation:

$$u_1(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega}{2p}(\tanh\zeta \pm i\,\mathrm{sech}\,\zeta)\right]^{1/s}, \qquad \zeta = \pm s\omega\,(x-\omega t)\,, \tag{28}$$

$$u_2(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}(\coth\zeta \pm \mathrm{coch}\,\zeta\right]^{1/s}, \qquad \zeta = \mp s\omega I\,(x-\omega t)\,, \tag{29}$$

$$u_3(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}\tanh\zeta\right]^{1/s}, \qquad \zeta = \pm s\omega I\,(x-\omega t)\,, \tag{30}$$

$$u_4(x,t) = \left[\frac{(1+s)\omega}{2p} \pm \frac{(1+s)\omega}{2p}\coth\zeta\right]^{1/s}, \qquad \zeta = \mp s\omega\,(x-\omega t)\,, \tag{31}$$

$$u_5(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega}{2p}\tan\zeta\right]^{1/s}, \qquad \zeta = \mp s\omega I\,(x-\omega t)\,, \tag{32}$$

$$u_6(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}\cot\zeta\right]^{1/s}, \qquad \zeta = \pm s\omega I\,(x-\omega t)\,. \tag{33}$$

In equations (28)-(33), putting $s = p = \omega = 1$, we obtain the solutions of the classical Burgers equation [5]:

$$u_1(x,t) = 1 \mp (\tanh\zeta \pm i\,\mathrm{sech}\,\zeta), \qquad \zeta = \pm\,(x-t)\,, \tag{34}$$

$$u_2(x,t) = 1 \mp I(\coth\zeta \pm \mathrm{coch}\,\zeta), \qquad \zeta = \mp I\,(x-t)\,, \tag{35}$$

$$u_3(x,t) = 1 \mp I\tanh\zeta, \qquad \zeta = \pm I\,(x-t)\,, \tag{36}$$

$$u_4(x,t) = 1 \pm \coth\zeta, \qquad \zeta = \mp\,(x-t)\,, \tag{37}$$

$$u_5(x,t) = 1 \mp \tan\zeta, \qquad \zeta = \mp I\,(x-t)\,, \tag{38}$$

$$u_6(x,t) = 1 \mp I\cot\zeta, \qquad \zeta = \pm I\,(x-t)\,. \tag{39}$$

4 Conclusions

We have presented the time-delayed Burgers differential equation and we have shown that this equation can be solved exactly for finite arbitrary time-delay by using the improved tanh-function method. Also, we obtained multiple soliton and triangular periodic solutions of the equation. More solutions of the generalized and classical Burgers equation are obtained as a special cases.

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