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TRAVELING WAVE SOLUTIONS FOR NONLINEAR WAVE EQUATION WITH DISSIPATION AND NONLINEAR TRANSPORT TERM THROUGH FACTORIZATIONS

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In this work, we use the factorization method to find explicit exact particular traveling wave solutions for the nonlinear wave equation with dissipation and nonlinear transport term. The two-parameter solution is obtained by using the particular solution and the known solutions for the Newell–Whitehead equation, Kolmogorov–Petrovsky–Piscounov equation, Fitzhugh–Nagumo equation, and the Burgers equation with cubic nonlinearity obtained as special cases from the solutions of the nonlinear wave equation with dissipation and nonlinear term.

Keywords: Factorization method; traveling wave; nonlinear wave equation.

1. Introduction

Solutions of nonlinear evolution equations appearing as traveling waves play a distinctive role in nonlinear phenomena [Murray (2003); Okubo (1980); Abdusalam (2006); Polyanin and Zaitsev (2004)]. These solutions are important in many applications of sciences. Many methods have been devoted to find the traveling wave solution for nonlinear evaluation equations [Clarkson and Mansfiet (1993)].

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Recently, the factorization of second-order linear differential equations and nonlinear differential equations became a well established technique to find solutions in an algebraic manner [Rosu and Cornejo-Pérez (2005); Cornejo-Pérez and Rosu (2005); Cornejo-Pérez *et al.* (2006); Smirnov (1999); Infeld and Hull (1951); Zwillinger (1992); Wang (1988); Reyes and Rosu (arXIV.math-ph/0510072 v2); Mielnik (1984); Fahmy and Bajunaid (2007); Abdusalam and Fahmy (2007)]. Rosu and Cornejo find one particular solution once the nonlinear equation is factorized with the use of two first order differential operators [Cornejo-Pérez and Rosu (2005a)]. They use the method for equations of types:

$$u'' + \gamma u' + f(u) = 0, \quad \gamma = \text{const.}, \tag{1}$$

and

$$u'' + g(u)u' + f(u) = 0, \tag{2}$$

where $'$ means the derivative $D = \frac{d}{dz}$, $g(u)$, $f(u)$ are polynomials in u. We applied the same method where the coefficient of u'' was also a polynomial function in u [Abdusalam and Fahmy (2007); Fahmy (2007)].

Now, Eq. (2) can be factorized as

$$[D - \varphi_2(u)]\,[D - \varphi_1(u)]\,u = 0, \tag{3}$$

which leads to the equation

$$u'' - \frac{d\varphi_1}{du}uu' - \varphi_1 u' - \varphi_2 u' + \varphi_1\varphi_2 u = 0, \tag{4}$$

or

$$u'' - \left(\varphi_1 + \varphi_2 + \frac{d\varphi_1}{du}u\right)u' + \varphi_1\varphi_2 u = 0. \tag{5}$$

Comparing (5) and (2) we find

$$g(u) = -\left(\varphi_1 + \varphi_2 + \frac{d\varphi_1}{du}u\right) \quad \text{and} \quad f(u) = \varphi_1\varphi_2 u. \tag{6}$$

If Eq. (2) can be factorized as in Eq. (3), then a first particular solution can be easily found by solving

$$[D - \varphi_1(u)]\,u = 0. \tag{7}$$

If $\varphi_1(u)$ is a linear function of the dependent variable u of the form $\varphi_1(u) = c_1u + c_2$, Eq. (7) turns out to be a Riccati equation for this variable, $u' - c_1u^2 - c_2u = 0$ and if the particular solution u_1 of this equation is known the general solution, the two-parameter solution, can be found as [Reyes and Rosu (arXIV.math-ph/0510072 v2)],

$$u_\lambda = u_1 + \frac{e^{I_1}}{\lambda - c_1 I_2},$$

where

$$I_1(z) \equiv \int_{z_0}^{z} [2c_1 u_1(z') + c_2]\, dz'$$

and

$$I_2(z) \equiv \int_{z_0}^{z} e^{I_1(z')} dz'.$$

2. The Nonlinear Wave Equation with Dissipation and Nonlinear Transport Term

The nonlinear wave equation with dissipation and nonlinear transport term is given as:

$$\alpha u u_x + \beta u_{tt} + \gamma u_t = u_{xx} - f(u), \qquad f(u) = (u - a_1)(u - a_2)(u - a_3), \tag{8}$$

where a_1, a_2, a_3 are distinct real numbers and α, β, γ are constants.

For $\alpha = 0$, $\beta = 0$, $\gamma = 1$, Eq. (8) reduces to the nonlinear reaction-diffusion form

$$u_t = u_{xx} - (u - a_1)(u - a_2)(u - a_3), \tag{9}$$

for different choices of the parameters a_1, a_2, a_3. Equation (9) reduces to the well known nonlinear reaction diffusion equations appearing in many different branches of sciences: the Newell–Whitehead equation [Newell and Whitehead (1969)] or Kolmogorov–Petrovsky–Piscounov equation [Kolmogoroff *et al.* (1937)] ($a_1 = 0$, $a_2 = 1$, $a_3 = -1$), and Fitzhugh–Nagumo equation arising in population genetics [Aronson and Weinberger (1975)] ($a_1 = 0$, $a_2 = 1$, $a_3 = a$).

For $\beta = 0$, $\gamma = 1$, Eq. (8) reduces to the Burgers form with cubic nonlinear term [Wang *et al.* (1990)],

$$u_t + \alpha u u_x = u_{xx} - (u - a_1)(u - a_2)(u - a_3). \tag{10}$$

Using the coordinate transformation $z = x - ct$ (c is the propagation speed) in Eq. (8), we obtain the following nonlinear ordinary differential equation

$$(\beta c^2 - 1)u'' + (\alpha u - \gamma c)u' + f(u) = 0, \quad \beta c^2 > 1, \tag{11}$$

or

$$u'' + g(u)u' + F(u) = 0, \tag{12}$$

where

$$g(u) = \frac{-c\gamma}{(\beta c^2 - 1)} + \frac{\alpha}{(\beta c^2 - 1)} u, \quad \text{and} F(u) = \frac{1}{(\beta c^2 - 1)} f(u). \tag{13}$$

Using the transformation

$$w = (u - a_1), \tag{14}$$

Eq. (11) is reduced to

$$w'' + g(w)w' + F(w) = 0, \tag{15}$$

where

$$g(w) = \left(\frac{\alpha a_1 - c\gamma}{\beta c^2 - 1}\right) + \left(\frac{\alpha}{\beta c^2 - 1}\right) w, \quad F(w) = \frac{1}{\beta c^2 - 1} w(w - b_1)(w - b_2), \tag{16}$$

$b_1 = (a_2 - a_1)$ and $b_2 = (a_3 - a_1)$.

Using operator notation, Eq. (15) takes the form

$$\left[D^2 + g(w)D + \frac{F(w)}{w}\right] w = 0. \tag{17}$$

The factorization of (12) leads to

$$[D - \varphi_2(w)]\,[D - \varphi_1(w)]\, w = 0, \tag{18}$$

and then

$$w'' - \left[\varphi_2 + \varphi_1 + \frac{d\varphi_1}{dw} w\right] w' + \varphi_1 \varphi_2 w = 0. \tag{19}$$

Comparing (19) and (17), we obtain the conditions on φ_1 and φ_2 as:

$$-\left(\varphi_2 + \varphi_1 + \frac{d\varphi_1}{dw} w\right) = g(w), \quad \varphi_1 \varphi_2 = \frac{F(w)}{w}, \tag{20}$$

therefore

$$\varphi_1 \varphi_2 = (w - b_1)(w - b_2). \tag{21}$$

Now, we choose φ_1 and φ_2 such that

$$\varphi_1(w) = \frac{s}{(\beta c^2 - 1)} (w - b_1), \quad \varphi_2(u) = \frac{1}{s(\beta c^2 - 1)} (w - b_2), \quad s \neq 0. \tag{22}$$

From (22) and (20), we get

$$\left(s b_1 + \frac{1}{s} b_2\right) - \left(2s + \frac{1}{s}\right) w = \left(\frac{\alpha a_1 - c\gamma}{\sqrt{\beta c^2 - 1}}\right) + \left(\frac{\alpha}{\sqrt{\beta c^2 - 1}}\right) w \tag{23}$$

which implies that

$$s b_1 + \frac{1}{s} b_2 = \frac{\alpha a_1 - c\gamma}{\sqrt{\beta c^2 - 1}}, \tag{24}$$

$$-\left(2s + \frac{1}{s}\right) = \frac{\alpha}{\sqrt{\beta c^2 - 1}}, \tag{25}$$

then,

$$s^{\pm} = \frac{(\alpha a_1 - c\gamma) \pm \sqrt{(\alpha a_1 - c\gamma)^2 - 4 b_1 b_2 (\beta c^2 - 1)}}{2 b_1 \sqrt{\beta c^2 - 1}}, \tag{26}$$

then Eq. (18) reduces to

$$\left[D - \frac{1}{s^{\pm}} (w - b_2)\right] [D - s^{\pm}(w - b_1)] w = 0, \tag{27}$$

and the compatible first-order differential equation is

$$w' - s^{\pm}(w - b_1)w = 0. \tag{28}$$

By direct integration of (28), we get

$$u_1^{\pm}(z) = a_1 + (a_2 - a_1)\left(1 \pm \exp[s^{\pm}(a_2 - a_1)(z - z_0)]\right)^{-1}, \tag{29}$$

where z_0 is the integration constant.

The solution (29) in hyperbolic form is given as:

$$u_1^{+}(z) = a_1 + (a_2 - a_1)\left[\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{1}{2}s^{+}(a_2 - a_1)(z - z_0)\right]\right], \tag{30}$$

$$u_1^{-}(z) = a_1 + (a_2 - a_1)\left[\frac{1}{2} - \frac{1}{2}\coth\left[\frac{1}{2}s^{-}(a_2 - a_1)(z - z_0)\right]\right]. \tag{31}$$

Now, if we choose the factorization terms as

$$\varphi_1(w) = \frac{(w - b_2)r}{\sqrt{(\beta c^2 - 1)}}, \quad \varphi_2(w) = \frac{1}{r\sqrt{(\beta c^2 - 1)}}(w - b_1), \quad r \neq 0, \tag{32}$$

and use (20) and (32) we obtain $c = \sqrt{\frac{1}{\beta}\left(1 - \frac{r\alpha}{(2r^2+1)}\right)}$ and

$$r^{\pm} = \frac{(\alpha a_1 - c\gamma) \pm \sqrt{(\alpha a_1 - c\gamma)^2 - 4b_1 b_2(\beta c^2 - 1)}}{2b_2\sqrt{\beta c^2 - 1}}, \tag{33}$$

and Eq. (15) is then factorized in the following different form

$$\left[D - \frac{1}{r^{\pm}}(w - b_1)\right]\left[D - r^{\pm}(w - b_2)\right]w = 0. \tag{34}$$

The corresponding compatible first-order equation is

$$w' - r^{\pm}(w - b_2)w = 0. \tag{35}$$

Direct integration of Eq. (35) gives a different first-order particular solution of Eq. (15),

$$u_2^{\pm}(z) = a_1 + (a_3 - a_1)\left(1 \pm \exp[r^{\pm}(a_3 - a_1)(z - z_0)]\right)^{-1}. \tag{36}$$

Eqs. (29) and (36) differ only through a constant value, i.e. if a_2 is interchanged with a_3, one gets the same solution.

The solution (36) in hyperbolic form is given as:

$$u_2^{+}(z) = a_1 + (a_3 - a_1)\left[\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{1}{2}r^{+}(a_3 - a_1)(z - z_0)\right]\right], \tag{37}$$

$$u_2^{-}(z) = a_1 + (a_3 - a_1)\left[\frac{1}{2} - \frac{1}{2}\coth\left[\frac{1}{2}r^{-}(a_3 - a_1)(z - z_0)\right]\right]. \tag{38}$$

Now, using the special values of α, β, γ, a_1, a_2, a_3, and the solutions (30), (31), (37), and (38), the explicit exact traveling wave solutions for the Newell–Whitehead equation or Kolmogorov–Petrovsky–Piscounov equation, Fitzhugh-Nagumo equation, and the Burgers equation with cubic nonlinearity are obtained.

Consider the solutions (29) and (36) and Riccati equations (28) and (35), the two-parameter solutions are obtained as:

$$u_\lambda^\pm = u_1^\pm(z) + a_1 + \left[\lambda e^{s^\pm(a_2-a_1)\zeta}\left(1+e^{s^\mp(a_2-a_1)\zeta}\right)^2 - \frac{(1+e^{s^\pm(a_2-a_1)\zeta})}{(a_2-a_1)}\right]^{-1}, \quad (39)$$

$$u_\mu^\pm = u_2^\pm(z) + a_1 + \left[\mu e^{r^\pm(a_3-a_1)\zeta}\left(1+e^{r^\mp(a_3-a_1)\zeta}\right)^2 - \frac{(1+e^{r^\pm(a_3-a_1)\zeta})}{(a_3-a_1)}\right]^{-1}, \quad (40)$$

where $\zeta = (z - z_0)$.

3. Conclusions

In conclusion, the efficient factorization method has been applied to find explicit exact particular traveling wave solutions for the nonlinear wave equation with dissipation and nonlinear transport term. The two-parameter solution is obtained by using the particular solution. The solutions of some famous nonlinear reaction diffusion equations are obtained by using the special values of α, β, γ, a_1, a_2, a_3 in the nonlinear wave equation with dissipation and nonlinear transport term.

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