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On the exact and numerical solution of the time-delayed Burgers equation

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The time-delayed Burgers equation is introduced and the improved tanh-function method is used to construct exact multiple soliton and triangular periodic solutions. For an understanding of the nature of the exact solutions that contained the time-delay parameter, we calculated the numerical solutions of this equation by using the Adomian decomposition method to the boundary value problem.

Keywords: travelling wave; tanh-function method; time-delayed Burgers equation; Adomian decomposition method

AMS Subject Classifications:

1. Introduction

A number of nonlinear phenomena in many branches of sciences such as physical [6], chemical, economical [20] and biological processes [15,14] are described by the interplay of reaction and diffusion or by the interaction between convection and diffusion. One of the well-known partial differential equations (PDEs) which governs a wide variety of them is Burgers equation, which provides the simplest nonlinear model of turbulence [5]. The existence of relaxation time or delay time is an important feature in reaction diffusion and convection diffusion systems [9,1,3].

The generalized time-delayed Burgers–Fisher equation can be derived by a similar manner to that presented in [9]. It takes the following form:

$$\tau u_{tt} + [1 - \tau f_u]u_t = u_{xx} - pu^s u_x + f(u), \quad f(u) = qu(1-u) \tag{1}$$

where τ, p, are any real numbers and $s \in \mathbb{N}$.

It is clear that when $q = \tau = 0$ and $p = s = 1$, Equation (1) assumes the form of the classical Burgers equation [5]. Recently, many methods have been used to find exact solutions of nonlinear partial differential equations, such as the tanh function method [16,13,10,19,8], Jacobi elliptic function method [11], simplist equation method [12], unified algebraic method [18]. Also, many

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methods have been used to find numerical solutions of nonlinear partial differential equations, such as the Adomian decomposition method [2,4,17,7]. In this paper, the improved tanh-function method is used to find exact travelling wave solutions for Equation (1) for $q = 0$ and different values of τ, p, s, *i.e.*,

$$\tau u_{tt} - u_t \Box u_{xx} - pu^s u_x. \tag{2}$$

The paper is organized as follows: In Section 2, the improved tanh function method is presented. In Section 3, we use the improved tanh-function method to obtain multiple soliton and triangular periodic solutions of time-delayed Burgers equation. The numerical solutions of the problem are obtained by using the Adomian decomposition method (ADM) [2] in Sections 4, 5. We compare the numerical and exact results in Section 6 and Section 7 is devoted for the conclusion.

2. The improved tanh-function method

It is useful to summarize the main steps for using the improved tanh-function method:

1. Consider a general form of nonlinear PDE

$$N(u, u_t, u_x, u_{xx}, \dots) = 0. \tag{3}$$

2. We introduce the wave variable $\zeta = k(x - \omega t)$ to find the travelling wave solution, then

$$u(x, t) = U(\zeta), \tag{4}$$

where k and ω are the wave number and the wave speed respectively. Thus, we have

$$u_t = -k\omega U'(\zeta), \quad u_{tt} = k^2\omega^2 U''(\zeta), \quad u_x = kU'(\zeta), \quad u_{xx} = k^2 U'(\zeta), \dots, \tag{5}$$

and the PDE (3) reduces to an ordinary differential equation (ODE) given by,

$$N(U, U', U'', \dots) = 0. \tag{6}$$

3. If all terms of (6) contain derivatives in ζ, then by integrating this equation and taking the constant of integration to be zero, we obtain a simplified ODE.
4. Introduce

$$U(\zeta) = \sum_{i=0}^{n} a_i F^i(\zeta), \tag{7}$$

where n is a positive integer that can be determined by balancing the linear term with the nonlinear term in Equation (6); a_i, $i = 1, 2, \dots, n$, are parameters to be determined and $F(\zeta)$ is a solution of the Riccati equation that tanh-function satisfies, *i.e.*,

$$F' = CF^2 + A \tag{8}$$

where A, C are constants. The relations between values of A, C and corresponding $F(\zeta)$ in (8) is given in the following table.
5. Introducing (8) into (7) and then substituting (7) into Equation (6) yields a set of algebraic equations involving a_i, $(i = 1, 2, \dots, n)$, k, ω because all coefficients of $F^i(\zeta)$ have to vanish. Having determined these parameters, we obtain an analytic solutions in closed form.

3. Analytic solutions of the time-delayed Burgers equation

The time-delayed Burgers equation is given by

$$\tau u_{tt} + u_t + pu^s u_x - u_{xx} = 0. \quad (9)$$

where τ is the time -delay.

In order to obtain travelling wave solutions of (9), we set

$$u(x,t) = U(\zeta), \quad \zeta = k(x - \omega t). \quad (10)$$

Substituting (10) into (9), we find that

$$(\tau\omega^2 - 1)k^2 U'' - k\omega U' + pkU^s U' = 0. \quad (11)$$

Balancing U'' with $U^s U'$ gives $n = 1/s$ which is not an integer as $s \neq 1$. But we need the balancing number to be a positive integer so as to apply the ansatz (7) and (8). We make a transformation

$$U = V^{\frac{1}{s}}. \quad (12)$$

Using this transformation, Equation (11) changes to the form:

$$(\tau\omega^2 - 1)k^2 \left[V''V + \left(\frac{1}{s} - 1\right) V'^2 \right] - k\omega V V' + pkV^2 V' = 0. \quad (13)$$

Balancing V'^2 with $V^2 V'$ gives $n = 1$. Therefore, we may choose the following ansatz:

$$V(\zeta) = a_0 + a_1 F(\zeta). \quad (14)$$

Then

$$U(\zeta) = [a_0 + a_1 F(\zeta)]^{1/s}. \quad (15)$$

Now, substituting (14) into (7) along with Equation (8) and using Mathematica., yields a system of equations with respect to F^i. Setting the coefficients of F^i in the resulting system of equations to zero, we can deduce the following set of algebraic polynomials with respect to the unknowns a_0, a_1, namely:

$$\begin{aligned}
-A\omega a_0 a_1 + Apa_0^2 a_1 + A^2 a_1^2 - \frac{A^2 a_1^2}{s} + A^2\tau\omega^2 a_1^2 + \frac{A^2\tau\omega^2 a_1^2}{s} &= 0,\\
-2ACa_0 a_1 + 2A\omega^2\tau C a_0 a_1 - A\omega a_1^2 + 2Apa_0 a_1^2 &= 0,\\
-2C^2 a_0 a_1 + 2C^2\tau\omega^2 a_0 a_1 - C\omega a_1^2 + 2Cpa_0 a_1^2 &= 0,\\
-C\omega a_0 a_1 + Cpa_0^2 a_1 - \frac{C^2\omega^2\tau a_1^2}{s} + \frac{2AC\omega^2 a_1^2}{s} + Apa_1^3 &= 0,\\
-C^2 a_1^2 - \frac{C^2 a_1^2}{s} + \frac{C^2\omega^2\tau a_1^2}{s} + C^2\omega^2\tau a_1^2 + Cpa_1^3 &= 0.
\end{aligned} \quad (16)$$

Solving the above system of equations by using Mathematica., we find the following set of solutions that correspond to some values of A and C, see Table 1.

Case 1 $A = \frac{1}{2}, C = -\frac{1}{2}$,

$$a_0 = \frac{(1+s)\omega}{2p}, \quad a_1 = \mp\frac{(1+s)\omega}{2p}, \quad k_1 = \mp\frac{s\omega}{(\tau\omega^2 - 1)}. \quad (17)$$

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Table 1.

Case	A	C	$F(\zeta)$
1	$\frac{1}{2}$	$-\frac{1}{2}$	$\coth\zeta \pm \cosh\zeta$, $\tanh\zeta \pm I\,\mathrm{sech}\zeta$, $I = \sqrt{-1}$
2	$\frac{1}{2}$	$\frac{1}{2}$	$\sec\zeta \pm \tan\zeta$
3	$-\frac{1}{2}$	$-\frac{1}{2}$	$(\csc\zeta \mp \cot\zeta)$
4	1	-1	$\tanh\zeta$, $\coth\zeta$
5	1	1	$\tan\zeta$
6	-1	-1	$\cot\zeta$

Case 2 $A = C = \frac{1}{2}$,

$$a_0 = \frac{(1+s)\omega}{2p}, \quad a_1 = \mp\frac{(1+s)\omega I}{2p}, \quad k_2 = \pm\frac{s\omega I}{(\tau\omega^2 - 1)}. \tag{18}$$

Case 3 $A = C = -\frac{1}{2}$,

$$a_0 = \frac{(1+s)\omega}{2p}, \quad a_1 = \mp\frac{(1+s)\omega I}{2p}, \quad k_3 = \mp\frac{s\omega I}{(\tau\omega^2 - 1)}. \tag{19}$$

Case 4 $A = 1, C = -1$,

$$a_0 = \frac{(1+s)\omega}{2p}, \quad a_1 = \pm\frac{(1+s)\omega}{2p}, \quad k_4 = \pm\frac{s\omega}{2(\tau\omega^2 - 1)}. \tag{20}$$

Case 5 $A = C = 1$,

$$a_0 = \frac{(1+s)\omega}{2p}, \quad a_1 = \mp\frac{(1+s)\omega}{2p}, \quad k_5 = \pm\frac{s\omega I}{2(\tau\omega^2 - 1)}. \tag{21}$$

Case 6 $A = C = -1$,

$$a_0 = \frac{(1+s)\omega}{2p}, \quad a_1 = \mp\frac{(1+s)\omega I}{2p}, \quad k_6 = \mp\frac{s\omega I}{2(\tau\omega^2 - 1)}. \tag{22}$$

Substituting (17)–(22) into (15) and using spacial solutions of Equation (8), according to the cases introduced in Table 1, we obtain the following multiple soliton and triangular periodic solutions of Equation (9),

$$u_1(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega}{2p}(\tanh\zeta \pm i\,\mathrm{sech}\zeta)\right]^{1/s}, \quad \zeta = k_1\,(x - \omega t)\,, \tag{23}$$

$$u_2(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}(\coth\zeta \pm \mathrm{coch}\zeta\right]^{1/s}, \quad \zeta = k_2\,(x - \omega t)\,, \tag{24}$$

$$u_3(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}\tanh\zeta\right]^{1/s}, \quad \zeta = k_3\,(x - \omega t)\,, \tag{25}$$

$$u_4(x,t) = \left[\frac{(1+s)\omega}{2p} \pm \frac{(1+s)\omega}{2p}\coth\zeta\right]^{1/s}, \quad \zeta = k_4\,(x - \omega t)\,, \tag{26}$$

$$u_5(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega}{2p}\tan\zeta\right]^{1/s}, \quad \zeta = k_5\,(x - \omega t)\,, \tag{27}$$

$$u_6(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}\cot\zeta\right]^{1/s}, \quad \zeta = k_6\,(x - \omega t)\,. \tag{28}$$

In Equations (23)–(28), taking the limit when $\tau \to 0$, we obtain:

$$u_1(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega}{2p}(\tanh\zeta \pm i\,\mathrm{sech}\zeta)\right]^{1/s}, \quad \zeta = \pm s\omega\,(x-\omega t)\,, \tag{29}$$

$$u_2(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}(\coth\zeta \pm \mathrm{coch}\zeta\right]^{1/s}, \quad \zeta = \mp s\omega I\,(x-\omega t)\,, \tag{30}$$

$$u_3(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}\tanh\zeta\right]^{1/s}, \quad \zeta = \pm s\omega I\,(x-\omega t)\,, \tag{31}$$

$$u_4(x,t) = \left[\frac{(1+s)\omega}{2p} \pm \frac{(1+s)\omega}{2p}\coth\zeta\right]^{1/s}, \quad \zeta = \mp s\omega\,(x-\omega t)\,, \tag{32}$$

$$u_5(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega}{2p}\tan\zeta\right]^{1/s}, \quad \zeta = \mp s\omega I\,(x-\omega t)\,, \tag{33}$$

$$u_6(x,t) = \left[\frac{(1+s)\omega}{2p} \mp \frac{(1+s)\omega I}{2p}\cot\zeta\right]^{1/s}, \quad \zeta = \pm s\omega I\,(x-\omega t)\,. \tag{34}$$

4. The analysis of the Adomian decomposition method

We consider the equation in the following form

$$\tau u_{tt} + u_t + pu^s u_x - u_{xx} = 0, \quad (x,t) \in (a,b) \times [0,T] \tag{35}$$

with the boundary conditions:

$$u(a,t) = f_1(t), \quad u(b,t) = f_2(t), \tag{36}$$

and the initial conditions:

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_1(x), \tag{37}$$

where $f_1(t)$, $f_2(t)$, $g_1(x)$, $g_1(x)$, are known functions.

Equation (35) can be written in the form:

$$L_t u(x,t) = \frac{1}{\tau}\left[u_{xx} - u_t - pN(u(x,t))\right], \tag{38}$$

where L_t is a linear operator defined as $L_t(.) = u_{tt}$, and the nonlinear term $N(u) = u^s u_x$. Since $L_t(.) = u_{tt}$ is always invertible for any t, the inverse operator L_t^{-1} exists and is defined as:

$$L_t^{-1}(.) = \int_0^t \int_0^t (.)\mathrm{d}t\,\mathrm{d}t. \tag{39}$$

Therefore, the solution of (35) can be written as:

$$u(x,t) = u(x,0) + tu_t(x,0) + L_t^{-1}\left[\frac{1}{\tau}\left(u_{xx} - u_t - pN(u(x,t))\right)\right]. \tag{40}$$

Following ADM [17], we can write the solution of (35) in series form as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{41}$$

Substituting the initial conditions identifying the zeroth component, $u(x,0) + tu_t(x,0)$, term arising from the initial conditions (37), we obtain the subsequence components by the following

The ADM is convergent if the following two hypotheses are satisfied:
H_1: $(\tau(L_t(u) - L_t(v)), u - v) \geqslant k\|u - v\|^2$; $k > 0, \forall\, u, v \in \mathcal{H}$,
H_2: $\exists\, M > 0, c(M)$ such that, $\forall\, u, v \in \mathcal{H}$ with $\|u\| \leq M$, $\|v\| \leq M$,
we have

$$\big(\tau(L_t(u) - L_t(v)), w\big) \leq c(M)\ \|u - v\|\|w\| \quad \forall w \in \mathcal{H}.$$

Theorem 1 (Sufficient condition for convergence) *For equation* (35) *without initial and boundary conditions, the ADM converges towards a particular solution.*

Proof We will verify the convergence hypotheses H_1 for the operator $L_t(u)$.

$$\tau(L_t(u) - L_t(v)) = -(u - v)_t + (u - v)_{xx} - p[u^s u_x - v^s v_x]. \tag{49}$$

Since ∂_t, ∂_{xx}, ∂_x are differential operators in $\mathcal{H}$, we have constants δ_1, δ_2 and δ_3 such that:

$$(-(u - v)_t, (u - v)) \geqslant \delta_1\|(u - v)\|^2, \tag{50}$$

$$(-(u - v)_{xx}, (u - v)) \geqslant \delta_2\|(u - v)\|^2, \tag{51}$$

$$\begin{aligned}\big(-(u^{s+1} - v^{s+1})_x, (u - v)\big) &\geqslant \delta_3\ \|(u^{s+1} - v^{s+1})\|^2\|(u - v)\| \\ &\geqslant (s + 1)\delta_3 M^s\|(u - v)\|^2.\end{aligned} \tag{52}$$

From (50)–(52), we get

$$\tau(L_t(u) - L_t(v)) \geqslant (\delta_1 - \delta_2 + \delta_3(s + 1)M^s)\|(u - v)\|^2 = k\|(u - v)\|^2, \tag{53}$$

where $k = (\delta_1 - \delta_2 + \delta_3(s + 1)M^s)$. Then, the hypothesis H_1 is fulfilled.
Now, we verify the convergence hypotheses H_2 for the operator $L_t(u)$.

$$\begin{aligned}(\tau(L_t(u) - L_t(v)), w) &= (-(u - v)_t, w) - (-(u - v)_{xx}, w) \\ &\quad + \frac{p}{s + 1}\big(-(u^{s+1} - v^{s+1})_x, w\big) \\ &\leq \|(u - v)\|\ \|w\| + pM^s\|(u - v)\|\ \|w\| - \|(u - v)\|\ \|w\| \\ &\leq c(M)\|(u - v)\|\ \|w\|,\end{aligned} \tag{54}$$

where $c(M) = pM^s$ and therefore H_2 holds ■

6. The test problem for the Adomian decomposition method

The principal purpose of the work reported in this Section is the testing of the ADM based on the method, which has been investigated in Section 4. We investigate how well the numerical scheme determines the solutions. The initial conditions can be obtained from the initial conditions (36) and (37). When we take $s = 1$, $p = 0.1$, $\tau = 0.5$, $k = -0.05$, $\omega = 0.1$, $t = 0.1$ and apply the recurrence relation (42), using Mathematica. program, we get $u_0(x,t), u_1(x,t), u_2(x,t), \ldots.$ as:

$$u_0(x,t) = 1 + 0.00502513t\,\mathrm{sech}^2(0.05025x) - \tanh(0.05025x) \tag{55}$$

$$\begin{aligned} u_1(x,t) = 2\{&-2.81986\times 10^{-8}t^3\mathrm{sech}^4(0.0502513x) \\ &\times[1.5 - 0.75\cosh(0.1005x) - 149.25\sinh(0.10050x)] \\ &+ 0.0000126259t^2\mathrm{sech}^2(0.05025x)\tanh(0.05025x) \\ &+ 2.114\times 10^{-8}t^4\mathrm{sech}^4(0.05025x)\tanh(0.05025x)\} \end{aligned} \tag{56}$$

$$\begin{aligned} u_2(x,t) = 2\{&-7.62931\times 10^{-14}t^7\mathrm{sech}^8(0.0502513x)[1.3333 - \cosh(0.100503x) \\ &+ 0.1\times\sinh(0.100503x)] + 1.33513\times 10^{-13}t^4\mathrm{sech}^5(0.0502513x) \\ &\times[2.79397\times 10^{-9}\cosh(0.0502513x) - 2.79397\times 10^{-9}\cosh(0.150754x) \\ &- 8.04119\times 10^6\sinh(0.0502513x) - 7.8804\times 10^6\sinh(0.150754x)] \\ &- 8.90086\times 10^{-14}t^6\mathrm{sech}^7(0.0502513x)[4.77.6\cosh(0.0502513x) \\ &- 238.8\cosh(0.15075x) + 648.4\sinh(0.0502513x) - 81.2\sinh(0.150754x)] \\ &- 1.0681\times 103t^5\mathrm{sech}^6(0.0502513x)[29692.5 + 19807\cosh(0.100503x) \\ &- 9900.5\cosh(0.201x) + 80197\sinh(0.100503x) - 99.5\sinh(0.201005x)] \\ &- 8.417\times 10^{-6}t^3\mathrm{sech}^2(0.0502x)\tanh(0.05025x)\}, \end{aligned} \tag{57}$$

$$\vdots$$

and so on. In this manner the other components of the decomposition series can be easily obtained of which $u(x,t)$ can be evaluated in a series form. To demonstrate the convergence of the ADM, the results of the numerical example are presented and few terms are required to obtain an accurate solution.

Table 2 represents the values of the exact $u_E(x,t)$, and numerical solution for $u_n(x,t)$ where $s = 1, p = 0.1, \tau = 0.5, k = -0.05, \omega = 0.1, t = 0.1$. It is clear from the Table 2 that the absolute error, $|u_n(x,t) - u_E(x,t)|$, is very small. For this reason we plot only the numerical solution of Equation (35), see Figure 1.

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Table 2. The.....

x_i	$u_n(x,t)$	$u_E(x,t)$	$\lvert u_n(x,t) - u_E(x,t)\rvert$
−100	1.99991	1.99991	9.09286×10^{-11}
−80	1.99936	1.99936	6.77907×10^{-10}
−60	1.99523	1.99523	5.01778×10^{-9}
−40	1.9649	1.9649	3.52062×10^{-8}
−20	1.76474	1.76474	1.67667×10^{-7}
0	1.00251	1.00251	2.65027×10^{-10}
20	0.237352	0.237353	1.67501×10^{-7}
40	0.0354437	0.0354437	3.51406×10^{-8}
60	0.00482283	0.00482284	5.0078×10^{-9}
80	0.000647525	0.000647525	6.76548×10^{-10}
100	0.000086781	0.000086781	9.0746×10^{-11}

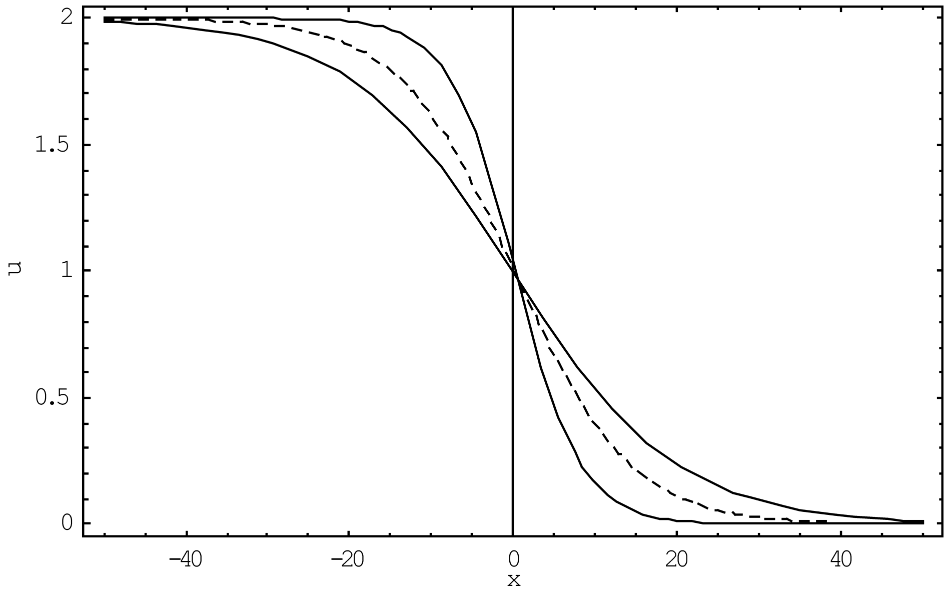

Figure 1. The solid graph represents the numerical solution of Equation (35) for $s = 1$, $p = 0.1$, $\omega = 0.1$, $k = -0.1002$, $\tau = 0.5$ and $t = 0.1$. The dashed and normal graphs represent the numerical solution for the same values as in the normal graph but for $t = 2$, $\tau = 30$ and $t = 3$, $\tau = 60$ respectively.

When we take $s = 2$, $p = 1$, $\tau = 0.2$, $k = -0.1002$, $\omega = 0.1$, $t = 1$ and apply the recurrence relation (42), we get

$$u_0(x,t) = \frac{0.000751503t\,\mathrm{sech}^2(0.1002x)}{\sqrt{0.15 - 0.15\tanh(0.1002x)}} + \sqrt{0.15 - 0.15\tanh(0.1002x)} \tag{58}$$

$$\begin{aligned}
u_1(x,t) = {} & 5.\{\mathrm{sech}^4(0.1002x)[-88.7111\cosh(0.1002x) + 620.978\sinh(0.1002x)] \\
& \times [1.77195 \times 10^{-10} t^4 \cosh(0.1002x) \\
& \times \sqrt{\mathrm{sech}(0.1002x)[0.15\cosh(0.1002x) - 0.15\sinh(0.1002x)]} \\
& + 1.77195 \times 10^{-10} t^4 \sqrt{\mathrm{sech}(0.1002x)[0.15\cosh(0.1002x) - 0.15\sinh(0.1002x)]} \\
& \times \sinh(0.1002x)] + \mathrm{sech}^5(0.1002x)[0.33333\cosh(0.1002x) - 1.\sinh(0.1002x)] \\
& \times [-1.41756 \times 10^{-10} t^5 \cosh(0.200401x)] \\
& \times \sqrt{\mathrm{sech}(0.1002x)[0.15\cosh(0.1002x) - 0.15\sinh(0.1002x)]} \\
& - 1 - 41756 \times 10^{-10} t^5 \sqrt{\mathrm{sech}(0.1002x)[0.15\cosh(0.1002x) - 0.15\sinh(0.1002x)]} \\
& \times \sinh(0.1002x)] + \mathrm{sech}^2(0.1002x)[1.77195 \times 10^{-10} t^2 \cosh(0.150301x) \\
& \times \sqrt{\mathrm{sech}(0.1002x)[0.15\cosh(0.1002x) - 0.15\sinh(0.1002x)]} \\
& + 1.77195 \times 10^{-10} t^2 \sqrt{\mathrm{sech}(0.1002x)[0.15\cosh(0.1002x) - 0.15\sinh(0.1002x)]} \\
& \times \sinh(0.150301x)][14165.4\cosh(0.0501002x) - 28330.8\cosh(0.150301x) \\
& - 3.11047 \times 10^{-10}\cosh(0.250501x) + 14165.4\sinh(0.0501002x) \\
& + 28330.8\sinh(0.150301x) - 3.1104 \times 10^{-10}\mathrm{Sinh}(0.250501x)]
\end{aligned}$$

$$+ \operatorname{sech}^3(0.1002x)[2.36259 \times 10^{-10} t^3 \cosh(0.x)$$
$$\times \sqrt{\operatorname{sech}(0.1002x)[0.15\cosh(0.1002x) - 0.15\sinh(0.1002x)]}\} \quad (59)$$

Table 3 represents the values of the exact $u_E(x,t)$, and numerical solution for $u_n(x,t)$ where $s = 2$, $p = 1$, $\tau = 0.2$, $k = -0.1002$, $\omega = 0.1$, $t = 1$. It is clear from theTable 3 that the absolute error, $|u_n(x,t) - u_E(x,t)|$, is very small. For this reason we plot only the numerical solution of Equation (42), see Figure 2.

In the following part, we need to study the Equation (35) when $\tau = 0$. So we use the same technique introduced in Section 4 but we take $L = \partial/\partial t$. Then for $s = 2$, $p = 0.1$, $\tau = 0$, $k = -0.1$, $\omega = 0.1$, $t = 1$ we have,

$$u_0(x,t) = \sqrt{1.5 - 1.5\tanh(0.1x)}, \quad (60)$$

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Table 3.

x_i	$u_n(x,t)$	$u_E(x,t)$	$\mid u_n(x,t) - u_E(x,t) \mid$
−100	0.547723	0.547723	1.81521×10^{-13}
−80	0.547723	0.547723	9.98857×10^{-12}
−60	0.547721	0.547721	5.49736×10^{-10}
−40	0.547634	0.547634	3.0227×10^{-8}
−20	0.542908	0.54291	1.57832×10^{-6}
0	0.389226	0.389234	8.13179×10^{-6}
20	0.0738971	0.0738915	5.69522×10^{-6}
40	0.0100512	0.0100503	8.31416×10^{-7}
60	0.00135506	0.00135495	1.12232×10^{-7}
80	0.000182654	0.000182639	1.51286×10^{-8}
100	0.0000246207	0.0000246186	2.03956×10^{-9}

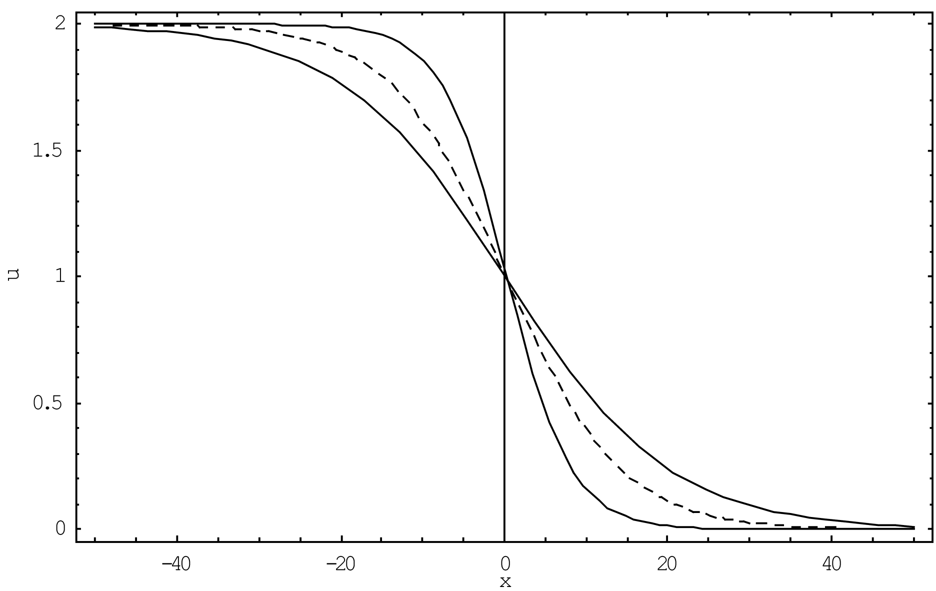

Figure 2. The solid graph represent the numerical solution of Equation (35) for $s = 2$, $p = 1$, $\omega = 0.1$, $k = -0.1002$, $\tau = 0.2$ and $t = 1$. The dashed and normal graphs represents the numerical solution for the same values as in the normal graph but for $t = 2$, $\tau = 30$ and $t = 3$, $\tau = 60$ respectively.

$$u_1(x,t) = t\left\{ -\frac{0.005625\text{sech}^4 x}{(1.5-1.5)\tanh(0.1x)^{(3/2)}} + 0.0075\text{sech}^2(0.1x)\sqrt{1.5-1.5\tanh(0.1x)} + \frac{0.015\text{sech}^2(0.1x)\tanh(0.1x)}{\sqrt{1.5-1.5\tanh(0.1x)}} \right\}, \tag{61}$$

$$\begin{aligned} u_2(x,t) = \text{sech}^4(0.1x)\{&-3.90625\times 10^{-7}t^2\cosh(0.4x) \\ &\times\sqrt{\text{sech}(0.1x)(1.5\cosh(0.1x)-1.5\sinh(0.1x))}\times 3.90625\times 10^{-7}t^2 \\ &\times\sqrt{\text{sech}(0.1x)(1.5\cosh(0.1x)-1.5\sinh(0.1x))}\sinh(0.4x\} \\ &\{-8.0+1.42109\times 10^{-14}\cosh(0.2x)+24.0\cosh(0.4x)+16\cosh(0.6x) \\ &-9.9476\times 10^{-14}\times\sinh(0.2x)-24.0\sinh(0.4x)-16\sinh(0.6x)\}, \end{aligned} \tag{62}$$

$$\vdots$$

Table 4. **Q2**

x_i	$u_n(x,t)$	$u_E(x,t)$	$\mid u_n(x,t)-u_E(x,t)\mid$
−50	1.73201	1.73201	7.81028×10^{-9}
−40	1.73177	1.73177	5.80464×10^{-10}
−30	1.72995	1.72995	2.80599×10^{-9}
−20	1.71671	1.71671	1.88563×10^{-8}
−10	1.62747	1.76474	6.67383×10^{-8}
0	1.23085	1.23085	1.27032×10^{-7}
10	0.603288	0.603288	1.94961×10^{-8}
20	0.234582	0.234582	3.00297×10^{-8}
30	0.0869905	0.0869905	1.39277×10^{-8}
40	0.032037	0.032037	5.24568×10^{-9}
50	0.0117875	0.0117875	2.6091×10^{-9}

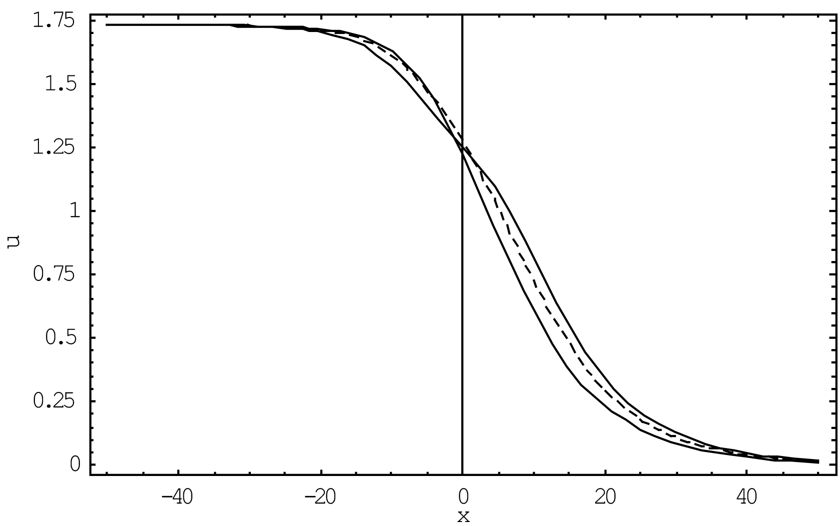

Figure 3. The normal graph represent the numerical solution of Equation (35) for $s=2$, $p=0.1$, $\omega=0.1$, $k=-0.1$, $\tau=0$ and $t=1$. The dashed and solid graphs represents the numerical solution for the same values as in the normal graph but for $t=15$ and $t=20$ respectively.

Table 4 represents the values of the exact $u_E(x,t)$, and numerical solution for $u_n(x,t)$ where $s=2$, $p=0.1$, $\tau=0$, $k=-0.1$, $\omega=0.1$, $t=1$. Also, it is clear from the Table 4 that, the absolute error, $|u_n(x,t)-u_E(x,t)|$, is very small. For this reason we plot only the numerical solution of Equation (42), see Figure 3.

7. Conclusions

We have presented the time-delayed Burgers differential equation and we have shown that this equation can be solved exactly for finite arbitrary time-delay by using the improved tanh-function method. Also, we obtained multiple soliton and triangular periodic solutions of the equation. The equation was solved numerically by the Adomian decomposition method using only the initial condition (27) and we obtained a continuous series of solutions which are convergent to the exact solution. The effect of the time-delay was shown in the three Figures 1–3 and we found that the time-delay as being effective in smoothing out the shock-wave nature of the travelling wave. Moreover, as for this equation we cannot find its exact solution, so we can then use the ADM which is excellent for solving nonlinear partial differential equations.

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