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Series approximations for moments of order statistics using MAPLE

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Abstract

In this paper, we present a series of Maple procedures that will approximate the means, variances, and covariances of order statistics from any continuous population using series approximations in the form of David and Johnson's (Biometrika 41 (1954) 228–240) approximation. These procedures will allow one to improve upon the accuracy of David and Johnson's approximation by extending the series to include higher order terms. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $X_1, X_2, \dots, X_n$ be a random sample from a continuous population with cdf $F(x)$ and $U_1, U_2, \dots, U_n$ be a random sample from the standard uniform distribution. Further, let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ and $U_{1:n} \leqslant U_{2:n} \leqslant \cdots \leqslant U_{n:n}$ be the order statistics obtained from these samples. Moments of uniform order statistics may be obtained from the following formula (see, for example, Arnold et al., 1992):

$$E\left(\prod_{j=1}^{l} U_{i_j:n}^{m_j}\right) = \frac{n!}{\left(n + \sum_{j=1}^{l} m_j\right)!} \prod_{j=1}^{l} \frac{(i_j + m_1 + m_2 + \cdots + m_j - 1)!}{(i_j + m_1 + m_2 + \cdots + m_{j-1} - 1)!}. \quad (1.1)$$

It is well known that for any continuous cdf $F(x)$,

$$X_{i:n} \stackrel{d}{=} F^{-1}(U_{i:n}). \quad (1.2)$$

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This distributional relation was originally observed by Scheffé and Tukey (1945) (see also Arnold et al., 1992). By expanding $F^{-1}(U_{i:n})$ in a Taylor series around the point $E(U_{i:n})=i/(n+1)=p_i$, we obtain the following series expansion for $X_{i:n}$:

$$X_{i:n}=F^{-1}(p_i)+F^{-1(1)}(p_i)(U_{i:n}-p_i)+\tfrac{1}{2}F^{-1(2)}(p_i)(U_{i:n}-p_i)^2$$
$$+\tfrac{1}{6}F^{-1(3)}(p_i)(U_{i:n}-p_i)^3+\tfrac{1}{24}F^{-1(4)}(p_i)(U_{i:n}-p_i)^4+\cdots, \tag{1.3}$$

where $F^{-1(1)}(p_i)$, $F^{-1(2)}(p_i)$, $F^{-1(3)}(p_i)$, $F^{-1(4)}(p_i),\ldots$ are the successive derivatives of $F^{-1}(u)$ evaluated at $u=p_i$. David and Johnson (1954) used the series expansion in (1.3) along with the moments of uniform order statistics in (1.1) (written in inverse powers of $n+2$) to obtain the following approximations (where $q_i=1-p_i$):

$$\mu_{i:n}\approx F^{-1}(p_i)+\frac{p_iq_i}{2(n+2)}F^{-1(2)}(p_i)$$
$$+\frac{p_iq_i}{(n+2)^2}\left[\frac{1}{3}(q_i-p_i)F^{-1(3)}(p_i)+\frac{1}{8}p_iq_iF^{-1(4)}(p_i)\right]$$
$$+\frac{p_iq_i}{(n+2)^3}\left[-\frac{1}{3}(q_i-p_i)F^{-1(3)}(p_i)+\frac{1}{4}\{(q_i-p_i)^2-p_iq_i\}F^{-1(4)}(p_i)\right.$$
$$\left.+\frac{1}{6}p_iq_i(q_i-p_i)F^{-1(5)}(p_i)+\frac{1}{48}p_i^2q_i^2F^{-1(6)}(p_i)\right] \tag{1.4}$$

and

$$\sigma_{i,j:n}\approx\frac{p_iq_j}{n+2}F^{-1(1)}(p_i)F^{-1(1)}(p_j)+\frac{p_iq_j}{(n+2)^2}[(q_i-p_i)F^{-1(2)}(p_i)F^{-1(1)}(p_j)$$
$$+(q_j-p_j)F^{-1(1)}(p_i)F^{-1(2)}(p_j)+\frac{1}{2}p_iq_iF^{-1(3)}(p_i)F^{-1(1)}(p_j)$$
$$\left.+\frac{1}{2}p_jq_jF^{-1(1)}(p_i)F^{-1(3)}(p_j)+\frac{1}{2}p_iq_jF^{-1(2)}(p_i)F^{-1(2)}(p_j)\right]$$
$$+\frac{p_iq_j}{(n+2)^3}[-(q_i-p_i)F^{-1(2)}(p_i)F^{-1(1)}(p_j)$$
$$-(q_j-p_j)F^{-1(1)}(p_i)F^{-1(2)}(p_j)+\{(q_i-p_i)^2-p_iq_i\}F^{-1(3)}(p_i)F^{-1(1)}(p_j)$$
$$+\{(q_j-p_j)^2-p_jq_j\}F^{-1(1)}(p_i)F^{-1(3)}(p_j)$$
$$+\left\{\frac{3}{2}(q_i-p_i)(q_j-p_j)+\frac{1}{2}p_jq_i-2p_iq_j\right\}F^{-1(2)}(p_i)F^{-1(2)}(p_j)$$
$$+\frac{5}{6}p_iq_i(q_i-p_i)F^{-1(4)}(p_i)F^{-1(1)}(p_j)+\frac{5}{6}p_jq_j(q_j-p_j)F^{-1(1)}(p_i)F^{-1(4)}(p_j)$$
$$+\left\{p_iq_j(q_i-p_i)+\frac{1}{2}p_iq_i(q_j-p_j)\right\}F^{-1(3)}(p_i)F^{-1(2)}(p_j)$$

$$+\left\{p_iq_j(q_j-p_j)+\frac{1}{2}p_jq_j(q_i-p_i)\right\}F^{-1(2)}(p_i)F^{-1(3)}(p_j)$$
$$+\frac{1}{8}p_i^2q_i^2F^{-1(5)}(p_i)F^{-1(1)}(p_j)+\frac{1}{8}p_j^2q_j^2F^{-1(1)}(p_i)F^{-1(5)}(p_j)$$
$$+\frac{1}{4}p_i^2q_iq_jF^{-1(4)}(p_i)F^{-1(2)}(p_j)+\frac{1}{4}p_ip_jq_j^2F^{-1(2)}(p_i)F^{-1(4)}(p_j)$$
$$\left.+\frac{1}{12}\{2p_i^2q_j^2+3p_ip_jq_iq_j\}F^{-1(3)}(p_i)F^{-1(3)}(p_j)\right] \tag{1.5}$$

for the mean, variance ($i=j$ in (1.5)), and covariance of order statistics. David and Johnson (1954) have given similar series approximations for the first four cumulants and cross-cumulants of order statistics. Clark and Williams (1958) have developed similar series approximations by making use of the exact expressions of the central moments of uniform order statistics where the kth central moment is of order $\{(n+2)(n+3)\cdots(n+k)\}^{-1}$ instead of inverse powers of $(n+2)$. These developments, as well as other methods of approximation can be found in David (1981), Arnold and Balakrishnan (1989), and Arnold et al. (1992).

The approximations in (1.4) and (1.5), which are up to order three, work quite well for central order statistics in most cases, as illustrated by David and Johnson (1954). However, they are clearly very tedious to compute. Furthermore, the approximations do not usually provide satisfactory results for extreme order statistics, where the convergence of this approximation to the true value may be very slow in some cases. Improved approximations for extreme order statistics can only be obtained by using a higher order expansion. That is, by including terms of order $1/(n+2)^4$ and higher. However, the necessary expressions will then become exceedingly complicated. In this paper, we will provide several Maple procedures designed to enable one to extend the series expansions of David and Johnson (1954) to include higher order terms. With these Maple procedures, one will be able to compute series approximations for the means, variances, and covariances of order statistics up to any desired order.

In Section 2, we will show how our Maple procedures can be used to obtain and extend the expressions in (1.4) and (1.5), and we will describe how to numerically obtain approximate moments. In Section 3, we will give some specific examples to show how these procedures can be used to improve the approximations for extreme order statistics, and in Section 4 we will discuss applications. The procedures themselves are available at the following web site: http://icarus.math.mcmaster.ca/childsa/maple.html, and details of how they work will be given in Appendix A.

2. Using the procedures

Extensions of (1.4) and (1.5) can be obtained from the procedures "mean" and "covariance" given on the above mentioned web site. The arguments for each series are k and o. k specifies how many terms to use in the original Taylor series (1.3), and o specifies the desired order of the expansion. In (1.4) and (1.5) $k=6$ and

$o=3$. Eq. (1.4) can be produced by typing the command "mean(6, 3)" in Maple (after reading in the file containing the procedures), which results in the following output:

$$\begin{aligned}
&\mathrm{FI0}(p) - \frac{1}{2}\frac{\mathrm{FI2}(p)p(p-1)}{n2}\\
&\quad + \frac{\frac{1}{8}\mathrm{FI4}(p)p(p-1)(p^2-p) + \frac{1}{3}\mathrm{FI3}(p)p(2p-1)(p-1)}{n2^2}\\
&\quad + \left(\frac{1}{24}\mathrm{FI4}(p)(3p(p-1)(-7p^2+7p-2) - 9p(p-1)(p^2-p))\right.\\
&\quad - \frac{1}{3}\mathrm{FI3}(p)p(2p-1)(p-1) - \frac{1}{144}\mathrm{FI6}(p)p(p-1)(3p^4-6p^3+3p^2)\\
&\quad \left. - \frac{1}{30}\mathrm{FI5}(p)p(2p-1)(p-1)(5p^2-5p)\right)\Big/ n2^3. \qquad (2.1)
\end{aligned}$$

Here, $n2$ is an abbreviation for $n+2$, and FI4(p), for example, is an abbreviation for $F^{-1(4)}(p_i)$. (1.4) can be extended to include terms up to order 4, for example, with the command "mean(8, 4)":

$$\begin{aligned}
&\mathrm{FI0}(p) - \frac{1}{2}\frac{\mathrm{FI2}(p)p(p-1)}{n2}\\
&\quad + \frac{\frac{1}{8}\mathrm{FI4}(p)p(p-1)(p^2-p) + \frac{1}{3}\mathrm{FI3}(p)p(2p-1)(p-1)}{n2^2}\\
&\quad + \left(\frac{1}{24}\mathrm{FI4}(p)(3p(p-1)(-7p^2+7p-2) - 9p(p-1)(p^2-p))\right.\\
&\quad - \frac{1}{3}\mathrm{FI3}(p)p(2p-1)(p-1) - \frac{1}{144}\mathrm{FI6}(p)p(p-1)(3p^4-6p^3+3p^2)\\
&\quad \left. - \frac{1}{30}\mathrm{FI5}(p)p(2p-1)(p-1)(5p^2-5p)\right)\Big/ n2^3\\
&\quad + \left(\frac{1}{5760}\mathrm{FI8}(p)p(p-1)(15p^6-45p^5+45p^4-15p^3)\right.\\
&\quad + \frac{1}{24}\mathrm{FI4}(p)(-9p(p-1)(-7p^2+7p-2) + 21p(p-1)(p^2-p))\\
&\quad + \frac{1}{3}\mathrm{FI3}(p)p(2p-1)(p-1)\\
&\quad + \frac{1}{120}\mathrm{FI5}(p)(-4p(2p-1)(p-1)(-17p^2+17p-6)\\
&\quad + 24p(2p-1)(p-1)(5p^2-5p))\\
&\quad + \frac{1}{720}\mathrm{FI6}(p)(-5p(p-1)(-92p^4+184p^3-118p^2+26p)
\end{aligned}$$

$$+50p(p-1)(3p^4-6p^3+3p^2))$$

$$+\frac{1}{840}\mathrm{FI7}(p)p(2p-1)(p-1)(35p^4-70p^3+35p^2)\Bigg)\Bigg/n2^4.$$

Similarly, (1.5) can be obtained with the command "covariance(6, 3)", and extended with the command "covariance(8, 4)" which will produce the 3-page long output given in Appendix B.

Numerically, these series can be evaluated with the procedure "evaluate" after defining $F(x)$. The first five arguments for "evaluate" are k, o (as above), i, j, and n. The last argument '*flag*' indicates whether or not the inverse function $F^{-1}(x)$ can be solved for explicitly or not. *flag* = 1 is used if $F^{-1}(x)$ cannot be solved for explicitly, and *flag* = 2 is used otherwise. (Actually, *flag* = 1 can always be used, but *flag* = 2 is more efficient.) Whenever $k = 6$ and $o = 3$, this procedure is equivalent to David and Johnson's (1954) approximation. For example, to obtain David and Johnson's approximation for $\mu_{9:11}, \sigma_{9,9:11}$ and $\sigma_{9,10:11}$ using the exponential distribution $F(x) = 1 - e^{-x} (x > 0)$, first define the function in Maple

```
> F:=x - > 1 - exp(-x);
```

then type

```
> evaluate(6, 3, 9, 10, 11, 2);
```

The resulting output,

$$\ln(4) + \frac{1173}{8788}, \frac{672}{2197}, \frac{672}{2197}$$

gives approximate values for $\mu_{9:11}, \sigma_{9,9:11}$, and $\sigma_{9,10:11}$, respectively. To get a numerical answer, just type

```
> evalf(%);
```

to get

1.519771830, .3058716431, .3058716431

For the normal distribution, define $F(x)$ as follows:

```
> f: = x - > 1/sqrt(2*Pi)* exp(-x^2/2);

> F: = x - > int(f(t),    t = - infinity..x);
```

Then, for example, to approximate $\mu_{9:11}, \sigma_{9,9:11}$ and $\sigma_{9,10:11}$ using $k = 8$, $o = 4$, type

```
> evaluate(8, 4, 9, 10, 11, 1);
```

[followed by evalf(%)] to get

.7288425408, .1657292761, .1403153903

which gives approximate values for $\mu_{9:11}$, $\sigma_{9,9:11}$ and $\sigma_{9,10:11}$, respectively.

A second procedure for numerical evaluation called "evaluate2" allows the user to specify the desired accuracy of the approximate moments as measured by the relative error between successive approximations. The arguments for this procedure are

$i, j, n, flag$, and tol. This procedure continues to calculate successive approximations (using the procedure "evaluate") until the maximum relative error in the approximations for the mean, variance, and covariance is less than the specified tolerance (tol). The output for this procedure are the values of k and o in the final approximation, the maximum relative error, and the values of the mean, variance, and covariance. For example, for the exponential distribution, the following command:

$$> \text{evaluate2}(2, 3, 7, 2, 0.01)$$

results in the output

8, 4, .006969052536, .3095171241, .04812387452, .04812387452,

which indicates that the procedure terminated with the approximation $k = 8$, $o = 4$, and the maximum relative error between successive approximations was 0.00697 (< 0.01).

We should note here that the David–Johnson series, and its extensions provided by our programs, are not in general guaranteed to converge. Indeed, the desired moments might not even exist. Certain differentiability conditions on the cdf $F(x)$ will guarantee convergence for intermediate order statistics (i.e., as $n \to \infty$ with $i/n \to r$), but in that case convergence for extreme order statistics might be extremely slow, or even nonexistent. For details one may refer to van Zwet (1964). Therefore, it is not always possible to obtain an approximation up to any desired accuracy. In addition, due to the algebraic complexity of the series expansion, the procedures can become quite slow as k and o increase. And so we caution the reader that the procedure "evaluate2" could be extremely slow, or indeed might never terminate. However, we should add that such convergence problems are not unique to the David–Johnson series. Indeed similar problems can arise when evaluating moments directly using numerical integration; see, for example, Barnett (1966).

3. Improved approximations

In Table 1, we present approximations for the means, variances, and covariances of order statistics from the normal distribution,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty;$$

the logistic distribution,

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}, \quad -\infty < x < \infty;$$

the exponential distribution,

$$f(x) = e^{-x}, \quad x > 0;$$

and the Type I extreme value distribution,

$$f(x) = e^{-e^{-x}} e^{-x}, \quad -\infty < x < \infty.$$

Table 1
Moments of order statistics using series approximations

k	o	i	j	n	Approximate			Exact		
					$\mu_{i:n}$	$\sigma_{i,i:n}$	$\sigma_{i,j:n}$	$\mu_{i:n}$	$\sigma_{i,i:n}$	$\sigma_{i,j:n}$
Normal										
6	3[a]	1	2	7	−1.35380	0.39419	0.19562	−1.35218	0.39192	0.19620
8	4				−1.35190	0.39343	0.19619			
10	5				−1.35160	0.39067	0.19621			
6	3[a]	2	3	7	−0.75740	0.25597	0.17392	−0.75737	0.25673	0.17448
8	4				−0.75739	0.25676	0.17444			
10	5				−0.75736	0.25672	0.17448			
6	3[a]	6	7	7	0.75740	0.25597	0.19562	0.75737	0.25673	0.19620
8	4				0.75739	0.25676	0.19619			
10	5				0.75736	0.25672	0.19621			
Logistic										
6	3[a]	1	2	7	−2.45273	1.75139	0.71291	−2.45000	1.79848	0.72366
8	4				−2.45044	1.80685	0.72250			
10	5				−2.44854	1.79941	0.72364			
6	3[a]	2	3	7	−1.28293	0.81505	0.51366	−1.28333	0.82626	0.51847
8	4				−1.28336	0.82499	0.51779			
10	5				−1.28332	0.82622	0.51839			
6	3[a]	6	7	7	1.28293	0.81505	0.71291	1.28333	0.82626	0.72366
8	4				1.28336	0.82499	0.72250			
10	5				1.28332	0.82622	0.72364			
Exponential										
6	3[a]	1	2	7	0.14283	0.02025	0.02025	0.14286	0.02041	0.02041
8	4				0.14285	0.02038	0.02038			
10	5				0.14286	0.02040	0.02040			
6	3[a]	2	3	7	0.30946	0.04779	0.04779	0.30952	0.04819	0.04819
8	4				0.30952	0.04812	0.04812			
10	5				0.30952	0.04818	0.04818			
6	3[a]	6	7	7	1.59240	0.50206	0.50206	1.59286	0.51180	0.51180
8	4				1.59287	0.51075	0.51075			
10	5				1.59284	0.51179	0.51179			
Extreme value										
6	3[a]	1	2	7	−0.84698	0.22123	0.13589	−0.84596	0.21964	0.13618
8	4				−0.84561	0.21966	0.13613			
10	5				−0.84564	0.21899	0.13619			
6	3[a]	2	3	7	−0.36542	0.20967	0.16434	−0.36531	0.21021	0.16497
8	4				−0.36532	0.21016	0.16489			
10	5				−0.36530	0.21019	0.16496			

Table 1 (continued.)

k	o	i	j	n	Approximate			Exact		
					$\mu_{i:n}$	$\sigma_{i,i:n}$	$\sigma_{i,j:n}$	$\mu_{i:n}$	$\sigma_{i,i:n}$	$\sigma_{i,j:n}$
6	3[a]	6	7	7	1.44363	0.63675	0.59668	1.44407	0.64691	0.60675
8	4				1.44409	0.64581	0.60567			
10	5				1.44406	0.64690	0.60674			
Normal										
6	3[a]	1	2	15	−1.73766	0.30526	0.14818	−1.73591	0.30104	0.14813
8	4				−1.73522	0.30214	0.14818			
10	5				−1.73533	0.29932	0.14813			
6	3[a]	4	5	15	−0.71488	0.12217	0.09968	−0.71488	0.12223	0.09973
8	4				−0.71488	0.12223	0.09973			
10	5				−0.71488	0.12223	0.09973			
6	3[a]	12	13	15	0.71488	0.12217	0.10816	0.71488	0.12223	0.10821
8	4				0.71488	0.12223	0.10822			
10	5				0.71488	0.12223	0.10821			
6	3[a]	14	15	15	1.24808	0.17927	0.14818	1.24794	0.17912	0.14813
8	4				1.24793	0.17921	0.14818			
10	5				1.24792	0.17910	0.14813			
Logistic										
6	3[a]	1	2	15	−3.25676	1.70266	0.67582	−3.25156	1.71387	0.67980
8	4				−3.25107	1.73201	0.68021			
10	5				−3.24960	1.71065	0.67988			
6	3[a]	4	5	15	−1.18651	0.36989	0.29082	−1.18654	0.37072	0.29133
8	4				−1.18655	0.37067	0.29130			
10	5				−1.18654	0.37072	0.29133			
6	3[a]	12	13	15	1.18651	0.36989	0.34344	1.18654	0.37072	0.34426
8	4				1.18655	0.37067	0.34421			
10	5				1.18654	0.37072	0.34426			
6	3[a]	14	15	15	2.18036	0.71501	0.67582	2.18013	0.71897	0.67980
8	4				2.18018	0.71937	0.68021			
10	5				2.18010	0.71906	0.67988			
Exponential										
6	3[a]	1	2	15	0.06667	0.00444	0.00444	0.06667	0.00444	0.00444
8	4				0.06667	0.00444	0.00444			
10	5				0.06667	0.00444	0.00444			
6	3[a]	4	5	15	0.29835	0.02238	0.02238	0.29835	0.02241	0.02241
8	4				0.29835	0.02241	0.02241			
10	5				0.29835	0.02241	0.02241			

Table 1 (continued.)

6	3[a]	12	13	15	1.48486	0.21860	0.21860	1.48490	0.21933	0.21933
8	4				1.48490	0.21929	0.21929			
10	5				1.48490	0.21933	0.21933			
6	3[a]	14	15	15	2.31845	0.57657	0.57657	2.31823	0.58044	0.58044
8	4				2.31827	0.58084	0.58084			
10	5				2.31819	0.58052	0.58052			
Extreme value										
6	3[a]	1	2	15	−1.13349	0.13483	0.07775	−1.13268	0.13318	0.07774
8	4				−1.13213	0.13299	0.07773			
10	5				−1.13244	0.13255	0.07775			
6	3[a]	4	5	15	−0.34581	0.10135	0.08851	−0.34581	0.10139	0.08856
8	4				−0.34581	0.10139	0.08856			
10	5				−0.34581	0.10139	0.08856			
6	3[a]	12	13	15	1.34032	0.28445	0.27404	1.34035	0.28521	0.27480
8	4				1.34035	0.28517	0.27475			
10	5				1.34035	0.28521	0.27480			
6	3[a]	14	15	15	2.25059	0.64142	0.62384	2.25037	0.64533	0.62776
8	4				2.25042	0.64573	0.62816			
10	5				2.25034	0.64541	0.62783			
Normal										
6	3[a]	1	2	25	−1.96697	0.26296	0.12540	−1.96531	0.25850	0.12524
8	4				−1.96447	0.25938	0.12528			
10	5				−1.96476	0.25669	0.12524			
6	3[a]	7	8	25	−0.63690	0.07153	0.06345	−0.63690	0.07154	0.06346
8	4				−0.63690	0.07154	0.06346			
10	5				−0.63690	0.07154	0.06346			
6	3[a]	20	21	25	0.76405	0.07637	0.07083	0.76405	0.07638	0.07084
8	4				0.76405	0.07638	0.07084			
10	5				0.76405	0.07638	0.07084			
6	3[a]	24	25	25	1.52446	0.14705	0.12540	1.52430	0.14678	0.12524
8	4				1.52429	0.14685	0.12528			
10	5				1.52429	0.14675	0.12524			
Logistic										
6	3[a]	1	2	25	−3.78172	1.68757	0.66360	−3.77596	1.68574	0.66548
8	4				−3.77489	1.70607	0.66620			
10	5				−3.77385	1.67979	0.66550			
6	3[a]	7	8	25	−1.04510	0.20748	0.18048	−1.04511	0.20759	0.18030
8	4				−1.04511	0.20758	0.18056			
10	5				−1.04511	0.20759	0.18056			

Table 1 (continued.)

k	o	i	j	n	Approximate			Exact		
					$\mu_{i:n}$	$\sigma_{i,i:n}$	$\sigma_{i,j:n}$	$\mu_{i:n}$	$\sigma_{i,i:n}$	$\sigma_{i,j:n}$
6	3[a]	20	21	25	1.26440	0.23244	0.22195	1.26441	0.23259	0.22208
8	4				1.26441	0.23259	0.22210			
10	5				1.26441	0.23259	0.22210			
6	3[a]	24	25	25	2.73467	0.68562	0.66360	2.73429	0.68748	0.66548
8	4				2.73431	0.68819	0.66620			
10	5				2.73425	0.68751	0.66550			
Exponential										
6	3[a]	1	2	25	0.04000	0.00160	0.00160	0.04000	0.00160	0.00160
8	4				0.04000	0.00160	0.00160			
10	5				0.04000	0.00160	0.00160			
6	3[a]	7	8	25	0.32085	0.01483	0.01483	0.32085	0.01483	0.01483
8	4				0.32085	0.01483	0.01483			
10	5				0.32085	0.01483	0.01483			
6	3[a]	20	21	25	1.53262	0.14198	0.14198	1.53262	0.14211	0.14211
8	4				1.53263	0.14211	0.14211			
10	5				1.53262	0.14211	0.14211			
6	3[a]	24	25	25	2.81633	0.60387	0.60387	2.81596	0.60572	0.60572
8	4				2.81598	0.60643	0.60643			
10	5				2.81591	0.60575	0.60575			
Extreme value										
6	3[a]	1	2	25	−1.28894	0.10198	0.05669	−1.28826	0.10051	0.05666
8	4				−1.28765	0.10028	0.05665			
10	5				−1.28805	0.09992	0.05667			
6	3[a]	7	8	25	−0.28126	0.06211	0.05732	−0.28125	0.06212	0.05732
8	4				−0.28125	0.06212	0.05732			
10	5				−0.28125	0.06212	0.05732			
6	3[a]	20	21	25	1.40201	0.18201	0.17779	1.40201	0.18216	0.17793
8	4				1.40201	0.18215	0.17793			
10	5				1.40201	0.18216	0.17793			
6	3[a]	24	25	25	2.77592	0.64322	0.63289	2.77554	0.64507	0.63475
8	4				2.77556	0.64578	0.63547			
10	5				2.77550	0.64510	0.63478			

[a]The case $k=6$, $o=3$ gives the results of David and Johnson's (1954) approximation.

To illustrate the improvement in accuracy in the higher order approximations we calculated, using the procedure 'evaluate', $\mu_{i:n}$, $\sigma_{i,i:n}$, and $\sigma_{i,j:n}$ for $n=7,15$, and 25, $i=1,\frac{1}{4}(n+1),\frac{3}{4}(n+1)$, and $n-1$. We used $k=6$, $o=3$ (David and Johnson's (1954) approximation), $k=8$, $o=4$ and $k=10$, $o=5$. For the purposes of comparison, we also include the corresponding exact moments. Exact moments for the normal distribution were obtained from Tietjen et al. (1977). For the logistic and exponential distributions, exact moments were obtained using the recurrence relations given in Arnold et al. (1992). Exact moments for the extreme value distribution were obtained from Balakrishnan and Chan (1992).

From Table 1, we see that increasing the value of k and o increases the accuracy of the approximation, most notably for small samples. For example, when $n=7$, $k=6$, and $o=3$, the approximation of $\sigma_{2,3:7}$ is accurate to two decimal places—0.17392 compared to the exact value of 0.17448. However, with $k=10$ and $o=5$, the approximate value is the same as the exact value, to five decimal places. Similar improvements may be observed for the other distributions as well. For example, for the exponential distribution the approximation of $\sigma_{6,6:7}$ is accurate to one decimal place when $k=6$ and $o=3$—0.50206 compared to the exact value of 0.51180—but when $k=10$ and $o=5$, the approximation (0.51179) is accurate to four decimal places.

4. Applications

There are many applications that require knowledge of moments of order statistics. For example, coefficients of the best linear unbiased estimators of the location and scale parameters of a distribution require these moments. They are also used in a similar way in prediction problems. A more direct use of the expected value of order statistics is in correlation goodness-of-fit tests (discussed in the example below). However, all of these applications depend on the availability of tables of moments of order statistics for various distributions. Although such tables are available for the most commonly used distributions, they are somewhat limited in the sample sizes covered. For example, tables for the extreme value distribution are only available for sample sizes up to 30. Therefore, the main utility of the programs discussed in this paper is for situations when moments of order statistics are required for a given application, but are not available.

Example 1. Lawless (1982) considers the following data set which is a Type-II right censored sample with $n=13$, $r=10$, and is assumed to follow an extreme value distribution ($f(x)=\frac{1}{\sigma}\mathrm{e}^{(x-\mu)/\sigma}\mathrm{e}^{-\mathrm{e}(x-\mu)/\sigma}$, $-\infty < x < \infty$):

$$-1.541, -0.693, -0.128, 0, 0.278, 0.285, 0.432, 0.565, 0.916, 1.099.$$

The expected values of the first 10 order statistics in a sample of size 13 (taken from Balakrishnan and Chan, 1992) are as follows:

$$-3.14217, -2.10161, -1.55805, -1.17764, -0.87633, -0.61990,$$

$$-0.39038, -0.17642, 0.03079, 0.23992.$$

In a correlation goodness-of-fit test, the sample correlation coefficient r for the xy pairs $[X_{i:n}, E(X_{i:n})]$ is compared with the critical values of the distribution of the correlation coefficient ρ. Small values of r indicate that the distribution under consideration is not appropriate for the given data. For details, including tables of critical values of ρ for various distributions, one may refer to D'Agostino and Stephens (1986). For the above data, $r = 0.990824$, and the 10% critical value (obtained by simulating 10,000 samples) is 0.925305. Therefore, we conclude that the extreme value distribution is appropriate for the given data set.

In order to assess the accuracy of results obtained using the approximate moments from our procedures, we also calclulated the value of r and the 10% critical value using the approximate moments. These values are 0.990804 and 0.925297, respectively, which agree to four decimal places with the exact values. In the following example, however, exact moments are not available. But due to the larger sample size we can be confident that the results obtained are in even closer agreement with the exact results.

Example 2. The following data set, also from Lawless (1982), is a Type-II right censored sample with $n = 40$, $r = 28$: $-2.982, -2.849, -2.546, -2.350, -1.983, -1.492, -1.443, -1.394, -1.386, -1.269, -1.195, -1.174, -0.845, -0.620, -0.576, -0.548, -0.247, -0.195, -0.056, 0.013, 0.006, 0.033, 0.037, 0.046, 0.084, 0.221, 0.245, 0.296$. Is it reasonable to assume that the data follow an extreme value distribution? In this case, tables of expected values of the required order statistics are not available. Using our Maple program, we calculated the corresponding expected values of order statistics for the extreme value distribution: $-4.264, -3.253, -2.740, -2.394, -2.130, -1.916, -1.735, -1.578, -1.438, -1.311, -1.196, -1.089, -0.989, -0.894, -0.805, -0.720, -0.639, -0.560, -0.484, -0.411, -0.339, -0.268, -0.199, -0.130, -0.0621, 0.00560, 0.0734, 0.142$. In this case, the sample correlation coefficient is $r = 0.976$, and the 10% critical value (obtained by simulating 10,000 samples) is 0.956. Again, we conclude that the extreme value distribution is appropriate. Furthermore, the covariances obtained from our procedures (in addition to the expected values) allow us to calculate the variance–covariance matrix, and hence the best linear unbiased estimators of the location and scale parameters μ and σ. They turn out to be $\mu^* = 0.1797$, $\sigma^* = 0.9377$.

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Appendix A.

The series in (1.4) is obtained by taking the expected value of the series in (1.3). The series in (1.5) is obtained by multiplying the expressions for $X_{i:n}$ and

$X_{j:n}$ obtained from (1.3), taking expectations, and then subtracting the approximation for $\mu_{i:n}\mu_{j:n}$ obtained from (1.3). It is therefore first necessary to be able to compute moments of the form

$$E[(U_{i:n} - p_i)^{m_1}(U_{j:n} - p_j)^{m_2}]. \tag{A.1}$$

This is accomplished by expanding each term above binomially, and then taking the expectation using the formula in (1.1) for the moments of uniform order statistics. When using (1.1), i is replaced by $(n+1)p_i$, and j by $(n+1)p_j$. For example, we have

$$\begin{aligned} &E[(U_{i:n} - p_i)^2(U_{j:n} - p_j)^2] \\ &\quad = \frac{p_i(p_j-1)(5p_j + 3p_ip_jn - p_jn - 15p_ip_j - 6 + 10p_i - 2p_in)}{(n+2)(n+3)(n+4)}. \end{aligned} \tag{A.2}$$

In order to write the above expression in powers of $(n+2)$, n's in the numerator are replaced by $(n+2)-2$, and $1/(n+3)(n+4)$ is replaced by its Taylor series expansion in inverse powers of $(n+2)$:

$$\frac{1}{(n+3)(n+4)} \approx \frac{1}{(n+2)^2} - \frac{3}{(n+2)^3} + \frac{7}{(n+2)^4} + \cdots \quad (\text{for large } n).$$

The resulting expression is what would be obtained by running the Maple procedure "central", by the command "central (3,2,2)". The first argument specifies the order of the desired expansion. The second and third arguments specify m_1 and m_2. The "central" procedure invokes the procedure "expected" which computes $E(U_{i:n}^{m_1}U_{j:n}^{m_2})$ using (1.1). The command "central (3,2,2)" produces the following output:

$$\begin{aligned} &\frac{p_i(p_j-1)(-p_j+3p_ip_j-2p_i)}{n2^2} \\ &\quad + \frac{p_i(p_j-1)(7p_j-21p_ip_j+14p_i-6)-3p_i(p_j-1)(-p_j+3p_ip_j-2p_i)}{n2^3}. \end{aligned}$$

Here again, $n2$ is an abbreviation for $n+2$.

The procedure "mean" with arguments k and o simply computes the expected value of (1.3) by invoking the procedure "central" when necessary.

The procedure "covariance" with arguments k and o first computes the expected value of the product of the series for X_i and X_j obtained from (1.3). That is, it first computes

$$\sum_{c_2=0}^{k}\sum_{c_1=0}^{k} F^{-1(c_1)}(p_i)F^{-1(c_2)}(p_j)E[(X_{i:n} - p_i)^{c_1}(X_{j:n} - p_j)^{c_2}] \tag{A.3}$$

by invoking the procedure "central" when necessary, then it subtracts the series for $\mu_{i:n}\mu_{j:n}$ obtained by invoking the procedure "mean".

Appendix B.

> covariance (8,4);

$$
\begin{aligned}
&-\frac{\mathrm{FI1}(p_i)\mathrm{FI1}(p_j)p_i(p_j-1)}{n+2}+\tfrac{1}{2}p_i(p_j-1)(\mathrm{FI1}(p_i)\mathrm{FI3}(p_j)p_j^2\\
&-\mathrm{FI1}(p_i)\mathrm{FI3}(p_j)p_j+4\mathrm{FI1}(p_i)\mathrm{FI2}(p_j)p_j+\mathrm{FI2}(p_i)p_i\mathrm{FI2}(p_j)p_j\\
&-\mathrm{FI3}(p_i)\mathrm{FI1}(p_j)p_i-2\mathrm{FI2}(p_i)\mathrm{FI1}(p_j)+4\mathrm{FI2}(p_i)\mathrm{FI1}(p_j)p_i\\
&-2\mathrm{FI1}(p_i)\mathrm{FI2}(p_j)-\mathrm{FI2}(p_i)\mathrm{FI2}(p_j)p_i+\mathrm{FI3}(p_i)\mathrm{FI1}(p_j)p_i^2)/(n+2)^2\\
&-\tfrac{1}{24}p_i(p_j-1)(180\mathrm{FI2}(p_i)p_i\mathrm{FI2}(p_j)p_j-14\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_i^2p_j\\
&+10\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_i^2p_j^2+72\mathrm{FI3}(p_i)\mathrm{FI2}(p_j)p_i^2p_j+6\mathrm{FI4}(p_i)\mathrm{FI2}(p_j)p_i^3p_j\\
&-96\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_ip_j+72\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_ip_j^2-6\mathrm{FI4}(p_i)\mathrm{FI2}(p_j)p_i^2p_j\\
&-48\mathrm{FI3}(p_i)\mathrm{FI2}(p_j)p_ip_j+6p_i\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_j+6p_i\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_j\\
&-6p_i\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_j^2+6\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_ip_j^3-12\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_ip_j^2\\
&-24\mathrm{FI2}(p_i)\mathrm{FI1}(p_j)+120\mathrm{FI1}(p_i)\mathrm{FI3}(p_j)p_j^2-120\mathrm{FI1}(p_i)\mathrm{FI3}(p_j)p_j\\
&-120\mathrm{FI3}(p_i)\mathrm{FI1}(p_j)p_i-120\mathrm{FI2}(p_i)\mathrm{FI2}(p_j)p_i+3p_i^2\mathrm{FI5}(p_i)\mathrm{FI1}(p_j)\\
&+12\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_j+20\mathrm{FI4}(p_i)\mathrm{FI1}(p_j)p_i-12\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_j^2\\
&+24\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_i+36\mathrm{FI3}(p_i)\mathrm{FI2}(p_j)p_i+40\mathrm{FI4}(p_i)\mathrm{FI1}(p_j)p_i^3\\
&-60\mathrm{FI4}(p_i)\mathrm{FI1}(p_j)p_i^2-6\mathrm{FI4}(p_i)\mathrm{FI2}(p_j)p_i^3+3\mathrm{FI5}(p_i)\mathrm{FI1}(p_j)p_i^4\\
&-6\mathrm{FI5}(p_i)\mathrm{FI1}(p_j)p_i^3-60\mathrm{FI1}(p_i)\mathrm{FI4}(p_j)p_j^2+6p_i^2\mathrm{FI4}(p_i)\mathrm{FI2}(p_j)\\
&+4p_i^2\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)+20\mathrm{FI1}(p_i)\mathrm{FI4}(p_j)p_j-60\mathrm{FI3}(p_i)\mathrm{FI2}(p_j)p_i^2\\
&+48\mathrm{FI1}(p_i)\mathrm{FI2}(p_j)p_j+48\mathrm{FI2}(p_i)\mathrm{FI1}(p_j)p_i-24\mathrm{FI1}(p_i)\mathrm{FI2}(p_j)\\
&+36\mathrm{FI2}(p_i)\mathrm{FI2}(p_j)+24\mathrm{FI1}(p_i)\mathrm{FI3}(p_j)+120\mathrm{FI3}(p_i)\mathrm{FI1}(p_j)p_i^2\\
&+3\mathrm{FI1}(p_i)\mathrm{FI5}(p_j)p_j^2+40\mathrm{FI1}(p_i)\mathrm{FI4}(p_j)p_j^3+3\mathrm{FI1}(p_i)\mathrm{FI5}(p_j)p_j^4\\
&-6\mathrm{FI1}(p_i)\mathrm{FI5}(p_j)p_j^3-60\mathrm{FI2}(p_i)\mathrm{FI2}(p_j)p_j+24\mathrm{FI3}(p_i)\mathrm{FI1}(p_j))/(n+2)^3\\
&+\tfrac{1}{48}(p_j-1)p_i(-70\mathrm{FI1}(p_i)\mathrm{FI6}(p_j)p_j^4+56\mathrm{FI1}(p_i)\mathrm{FI6}(p_j)p_j^3\\
&-14\mathrm{FI1}(p_i)\mathrm{FI6}(p_j)p_j^2-3\mathrm{FI1}(p_i)\mathrm{FI7}(p_j)p_j^5+3\mathrm{FI1}(p_i)\mathrm{FI7}(p_j)p_j^4\\
&-4p_i^3\mathrm{FI5}(p_i)\mathrm{FI3}(p_j)-2p_i^3\mathrm{FI4}(p_i)\mathrm{FI4}(p_j)-3p_i^3\mathrm{FI6}(p_i)\mathrm{FI2}(p_j)\\
&+144\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_j^2+144\mathrm{FI2}(p_j)p_j\mathrm{FI3}(p_i)-14p_i^2\mathrm{FI6}(p_i)\mathrm{FI1}(p_j)
\end{aligned}
$$

$$-48p_i^2\mathrm{FI4}(p_i)\mathrm{FI3}(p_j) - 38p_i^2\mathrm{FI5}(p_i)\mathrm{FI2}(p_j) - 24p_i^2\mathrm{FI3}(p_i)\mathrm{FI4}(p_j)$$
$$-52\mathrm{FI1}(p_i)\mathrm{FI5}(p_j)p_j - 6\mathrm{FI2}(p_i)\mathrm{FI5}(p_j)p_j^4 - 6\mathrm{FI2}(p_i)\mathrm{FI5}(p_j)p_j^2$$
$$+12\mathrm{FI2}(p_i)\mathrm{FI5}(p_j)p_j^3 + 28\mathrm{FI1}(p_i)\mathrm{FI6}(p_j)p_j^5 + 28\mathrm{FI6}(p_i)\mathrm{FI1}(p_j)p_j^5$$
$$+56\mathrm{FI6}(p_i)\mathrm{FI1}(p_j)p_i^3 - 72\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_i - 144\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_i$$
$$-124\mathrm{FI4}(p_i)\mathrm{FI2}(p_j)p_i - 52\mathrm{FI5}(p_i)\mathrm{FI1}(p_j)p_i - 52\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_j$$
$$-24\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_j + 24\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_j^2 - 92\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_j^3$$
$$-70\mathrm{FI6}(p_i)\mathrm{FI1}(p_j)p_i^4 + 3\mathrm{FI7}(p_i)\mathrm{FI1}(p_j)p_i^4 + 4\mathrm{FI5}(p_i)\mathrm{FI3}(p_j)p_i^4$$
$$+72\mathrm{FI4}(p_i)\mathrm{FI3}(p_j)p_i^3 - 70\mathrm{FI5}(p_i)\mathrm{FI2}(p_j)p_i^4 + 108\mathrm{FI5}(p_i)\mathrm{FI2}(p_j)p_i^3$$
$$+6\mathrm{FI6}(p_i)\mathrm{FI2}(p_j)p_i^4 - 3\mathrm{FI6}(p_i)\mathrm{FI2}(p_j)p_i^5 - 3\mathrm{FI7}(p_i)\mathrm{FI1}(p_j)p_i^5$$
$$+1008\mathrm{FI2}(p_i)p_i\mathrm{FI2}(p_j)p_j - 1128\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_i^2p_j$$
$$+792\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_i^2p_j^2 + 1488\mathrm{FI3}(p_i)\mathrm{FI2}(p_j)p_i^2p_j$$
$$+580\mathrm{FI4}(p_i)\mathrm{FI2}(p_j)p_i^3p_j - 1872\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_ip_j$$
$$+1488\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_ip_j^2 - 672\mathrm{FI4}(p_i)\mathrm{FI2}(p_j)p_i^2p_j$$
$$-1104\mathrm{FI3}(p_i)\mathrm{FI2}(p_j)p_ip_j + 576p_i\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_j$$
$$+560p_i\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_j - 456p_i\mathrm{FI3}(p_i)\mathrm{FI3}(p_j)p_j^2$$
$$+580\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_ip_j^3 - 1068\mathrm{FI2}(p_i)\mathrm{FI4}(p_j)p_ip_j^2$$
$$-96\mathrm{FI3}(p_i)\mathrm{FI2}(p_j) - 48\mathrm{FI2}(p_i)\mathrm{FI1}(p_j) + 672\mathrm{FI1}(p_i)\mathrm{FI3}(p_j)p_j^2$$
$$-672\mathrm{FI1}(p_i)\mathrm{FI3}(p_j)p_j - 672\mathrm{FI3}(p_i)\mathrm{FI1}(p_j)p_i - 672\mathrm{FI2}(p_i)\mathrm{FI2}(p_j)p_i$$
$$+296p_i^2\mathrm{FI5}(p_i)\mathrm{FI1}(p_j) + 408\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_j + 472\mathrm{FI4}(p_i)\mathrm{FI1}(p_j)p_i$$
$$-360\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_j^2 + 528\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)p_i + 792\mathrm{FI3}(p_i)\mathrm{FI2}(p_j)p_i$$
$$+752\mathrm{FI4}(p_i)\mathrm{FI1}(p_j)p_i^3 - 1128\mathrm{FI4}(p_i)\mathrm{FI1}(p_j)p_i^2 - 488\mathrm{FI4}(p_i)\mathrm{FI2}(p_j)p_i^3$$
$$+244\mathrm{FI5}(p_i)\mathrm{FI1}(p_j)p_i^4 - 488\mathrm{FI5}(p_i)\mathrm{FI1}(p_j)p_i^3 - 1128\mathrm{FI1}(p_i)\mathrm{FI4}(p_j)p_j^2$$
$$+540p_i^2\mathrm{FI4}(p_i)\mathrm{FI2}(p_j) + 360p_i^2\mathrm{FI3}(p_i)\mathrm{FI3}(p_j) + 472\mathrm{FI1}(p_i)\mathrm{FI4}(p_j)p_j$$
$$-1128\mathrm{FI3}(p_i)\mathrm{FI2}(p_j)p_i^2 + 96\mathrm{FI1}(p_i)\mathrm{FI2}(p_j)p_j + 96\mathrm{FI2}(p_i)\mathrm{FI1}(p_j)p_i$$
$$-48\mathrm{FI1}(p_i)\mathrm{FI2}(p_j) + 216\mathrm{FI2}(p_i)\mathrm{FI2}(p_j) + 144\mathrm{FI1}(p_i)\mathrm{FI3}(p_j)$$
$$+672\mathrm{FI3}(p_i)\mathrm{FI1}(p_j)p_i^2 + 296\mathrm{FI1}(p_i)\mathrm{FI5}(p_j)p_j^2 + 752\mathrm{FI1}(p_i)\mathrm{FI4}(p_j)p_j^3$$
$$+244\mathrm{FI1}(p_i)\mathrm{FI5}(p_j)p_j^4 - 488\mathrm{FI1}(p_i)\mathrm{FI5}(p_j)p_j^3 - 336\mathrm{FI2}(p_i)\mathrm{FI2}(p_j)p_j$$

$$+\mathrm{FI7}(p_i)\mathrm{FI1}(p_j)p_i^6+\mathrm{FI1}(p_i)\mathrm{FI7}(p_j)p_j^6+144\mathrm{FI3}(p_i)\mathrm{FI1}(p_j)$$

$$-3p_i^2\mathrm{FI5}(p_i)\mathrm{FI3}(p_j)p_j+3p_i^2\mathrm{FI5}(p_i)\mathrm{FI3}(p_j)p_j^2-4p_i^2\mathrm{FI3}(p_i)\mathrm{FI5}(p_j)p_j$$

$$-3p_i\mathrm{FI3}(p_i)\mathrm{FI5}(p_j)p_j^4-44p_i\mathrm{FI3}(p_i)\mathrm{FI4}(p_j)p_j-32p_i\mathrm{FI2}(p_i)\mathrm{FI5}(p_j)p_j$$

$$-20p_i\mathrm{FI4}(p_i)\mathrm{FI3}(p_j)p_j-6p_i^2\mathrm{FI4}(p_i)\mathrm{FI4}(p_j)p_j-\mathrm{FI1}(p_i)\mathrm{FI7}(p_j)p_j^3$$

$$-p_i^3\mathrm{FI7}(p_i)\mathrm{FI1}(p_j)+108\mathrm{FI3}(p_i)p_i\mathrm{FI4}(p_j)p_j^2-64\mathrm{FI3}(p_i)p_i\mathrm{FI4}(p_j)p_j^3$$

$$-96\mathrm{FI2}(p_i)\mathrm{FI3}(p_j)-276\mathrm{FI3}(p_i)p_i^2\mathrm{FI4}(p_j)p_j^2+136\mathrm{FI3}(p_i)p_i^2\mathrm{FI4}(p_j)p_j^3$$

$$+20p_i\mathrm{FI4}(p_i)\mathrm{FI3}(p_j)p_j^2-3p_i\mathrm{FI2}(p_i)\mathrm{FI6}(p_j)p_j^2-3p_i\mathrm{FI3}(p_i)\mathrm{FI5}(p_j)p_j^2$$

$$+6p_i\mathrm{FI3}(p_i)\mathrm{FI5}(p_j)p_j^3+9\mathrm{FI2}(p_i)p_i\mathrm{FI6}(p_j)p_j^3+164\mathrm{FI2}(p_j)p_j\mathrm{FI4}(p_i)p_i$$

$$-9\mathrm{FI2}(p_i)p_i\mathrm{FI6}(p_j)p_j^4+3\mathrm{FI2}(p_i)p_i\mathrm{FI6}(p_j)p_j^5+44\mathrm{FI2}(p_j)p_j\mathrm{FI5}(p_i)p_i^2$$

$$+3\mathrm{FI2}(p_j)p_j\mathrm{FI6}(p_i)p_i^3+12\mathrm{FI4}(p_i)p_i^2\mathrm{FI4}(p_j)p_j^2-6\mathrm{FI4}(p_i)p_i^2\mathrm{FI4}(p_j)p_j^3$$

$$+8\mathrm{FI4}(p_i)p_i^3\mathrm{FI4}(p_j)p_j^3-208\mathrm{FI4}(p_i)p_i^2\mathrm{FI3}(p_j)p_j+136\mathrm{FI4}(p_i)p_i^3\mathrm{FI3}(p_j)p_j^2$$

$$+164\mathrm{FI3}(p_i)\mathrm{FI4}(p_j)p_i^2p_j+180\mathrm{FI4}(p_i)p_i^2\mathrm{FI3}(p_j)p_j-132\mathrm{FI4}(p_i)p_i^2\mathrm{FI3}(p_j)p_j^2$$

$$-18\mathrm{FI4}(p_i)p_i^3\mathrm{FI4}(p_j)p_j^2+76\mathrm{FI2}(p_i)p_i\mathrm{FI5}(p_j)p_j^4+15\mathrm{FI3}(p_i)\mathrm{FI5}(p_j)p_i^2p_j^2$$

$$-18\mathrm{FI3}(p_i)\mathrm{FI5}(p_j)p_i^2p_j^3+7\mathrm{FI3}(p_i)\mathrm{FI5}(p_j)p_i^2p_j^4+12\mathrm{FI4}(p_i)\mathrm{FI4}(p_j)p_i^3p_j$$

$$-184\mathrm{FI2}(p_i)p_i\mathrm{FI5}(p_j)p_j^3+140\mathrm{FI2}(p_i)p_i\mathrm{FI5}(p_j)p_j^2+76\mathrm{FI2}(p_j)p_j\mathrm{FI5}(p_i)p_i^4$$

$$+3\mathrm{FI2}(p_j)p_j\mathrm{FI6}(p_i)p_i^5-48\mathrm{FI1}(p_i)\mathrm{FI4}(p_j)-6\mathrm{FI2}(p_j)p_j\mathrm{FI6}(p_i)p_i^4$$

$$+14\mathrm{FI5}(p_i)\mathrm{FI3}(p_j)p_i^3p_j-10\mathrm{FI5}(p_i)\mathrm{FI3}(p_j)p_i^3p_j^2+7\mathrm{FI5}(p_i)\mathrm{FI3}(p_j)p_i^4p_j^2$$

$$-11\mathrm{FI5}(p_i)\mathrm{FI3}(p_j)p_i^4p_j-48\mathrm{FI4}(p_i)\mathrm{FI1}(p_j)-120\mathrm{FI2}(p_j)p_j\mathrm{FI5}(p_i)p_i^3)/(n+2)^4$$

References

Arnold, B.C., Balakrishnan, N., 1989. Relations, Bounds and Approximations for Order Statistics. Lecture Notes in Statistics, Vol. 53. Springer, New York.

Arnold, B.C., Balakrishnan, N., Nagaraja, H.N., 1992. A First Course in Order Statistics. Wiley, New York.

Balakrishnan, N., Chan, P.S., 1992. Order statistics from extreme value distribution, I: tables of means, variances and covariances. Comm. Statist. Simulation Comput. 21, 1199–1217.

Barnett, V.V., 1966. Order statistics estimators of the location of the Cauchy distribution. J. Amer. Statist. Assoc. 61, 1205–1218.

Clark, C.E., Williams, G.T., 1958. Distributions of the members of an ordered sample. Ann. Math. Statist. 29, 862–870.

D'Agostino, R.B., Stephens, M.A., 1986. Goodness-of-Fit Techniques. Marcel Dekker, New York.

David, H.A., 1981. Order Statistics, 2nd Edition. Wiley, New York.

David, F.N., Johnson, N.L., 1954. Statistical treatment of censored data, I. Fundamental formulae. Biometrika 41, 228–240.

Lawless, J.F., 1982. Statistical Models and Methods for Lifetime Data. Wiley, New York.

Maple 6.01 for Microsoft Windows, 2000. Waterloo Maple Software, University of Waterloo.

Scheffé, H., Tukey, J.W., 1945. Non-parametric estimation—I. Validation of order statistics. Ann. Math. Statist. 16, 187–192.

Tietjen, G.L., Kahaner, D.K., Beckman, R.J., 1977. Variances and covariances of the normal order statistics for sample sizes 2 to 50. Select. Tables Math. Statist. 5, 1–74.

van Zwet, W.R., 1964. Convex Transformations of Random Variables. Mathematical Centre Tracts, Vol. 7. Mathematisch Centrum, Amsterdam.