Stat 523: Selected topics
First semester, 1428-1429
Assignment 1

Answer the Following Questions:

1. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be order statistics from the exponential distribution
$$f(x) = \frac{1}{\theta} \exp\{-x/\theta\}, \; x \geq 0, \; \theta > 0.$$
   (a) Determine the distribution of $X_{1:3}$.
   (b) Determine the distribution of $X_{3:3}$.
   (c) Determine the distribution of the range $X_{3:3} - X_{1:3}$.
   (d) When $\theta = 1$, prove that $nX_{1:n}$ and $(n-1)(X_{2:n} - X_{1:n})$ are statistically independent.

2. Let X_1 and X_2 be i.i.d Pareto random variables with pdf
$$f(x) = \nu x^{-\nu-1}, \quad x \geq 1, \; \nu > 0.$$
Discuss the distribution of the following random variables:
$$(i)\; X_{1:2}, \quad (ii)\; X_{2:2} - X_{1:2}, \quad (ii)\; X_{2:2}/X_{1:2}.$$

3. Let X_1 and X_2 be i.i.d random variables from the exponential distribution
$$f(x) = \frac{1}{\theta} \exp\{-x/\theta\}, x \geq 0.$$
Define
$$Z = \frac{X_{1:2}}{X_1 + X_2} \text{ and } W = \frac{X_1}{X_1 + X_2}.$$
Determine the distributions of Z and W.

4. For the logistic distribution with density function

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty.$$

(a) Show that the moment-generating function of $X_{i:n}$ is

$$M_{i:n}(t) = E(e^{tX_{i:n}}) = \frac{\Gamma(i+t)\Gamma(n-i+1-t)}{\Gamma(i)\Gamma(n-i+1)}, \; 1 \le i \le n.$$

(b) Show that for $i = 1, 2, \ldots, n$,

$$\mu_{i:n} = \psi(i) - \psi(n-i+1) \text{ and } \sigma_{i,i:n} = \psi'(i) - \psi'(n-i+1),$$

where $\psi(z) = \frac{d}{dz} \log \Gamma(z) = \Gamma'(z)/\Gamma(z)$ is the digamma (or psi) function and $\psi'(z)$ is the derivative of $\psi(z)$ known as the trigamma function.

5. Let $x_1, x_2, \ldots, x_n$ be a random sample from the uniform distribution $U(\mu - \sqrt{3}\sigma,\ \mu + \sqrt{3}\sigma)$. Show that both of the maximum likelihood estimates of μ and σ are unbiased.