Quality Technology & Quantitative Management
Vol. 2, No. 2, pp. 161-175, 2005

QTQM

Exact Bayesian Prediction of Exponential Lifetime Based on Fixed and Random Sample Sizes

A. H. Abd Ellah[1] and K. S. Sultan[2]

[1]Department of Mathematics, Teachers College, Riyadh, Saudi Arabia
[2]Department of Statistics & Operations Research, College of Science, King Saud University, Riyadh, Saudi Arabia
(*Received* September 2004, *accepted* March 2005)

Abstract: We consider the problem of predictive intervals for future observations from an exponential distribution. We consider the following two cases: (i) fixed sample size (*FSS*), and (ii) random sample size (*RSS*). Further, we derive the predictive function for both *FSS* and *RSS* in closed forms. Next, the upper and lower 1%, 2.5%, 5% and 10% critical points for the predictive functions are calculated. To show the usefulness of our results, we present some simulation examples. Finally, we apply our results to some real data sets in life testing given in Lawless [16].

Keywords: Bayesian prediction, order statistics, predictive function, predictive intervals, random sample size.

---

1. Introduction

In many applications, the sample size could be random rather than fixed, Lingappaiah [19] has discussed the prediction problem for future samples based on *FSS* and *RSS*. Also, he obtained the Bayesian predictive distribution of the range when the parent distribution is one-parameter exponential. Upadhyay and Pandey [27] have provided predicted intervals for future observations from one parameter exponential distribution in the case of *FSS*. Problems for constructing prediction limits have also been considered by Geisser [11], Soliman and Abd-Ellah [26], Soliman [24, 25], Balakrishnan and Basu [3], Dunsmore [10], Aitchison and Dunsmore [1], Hahn and Meeker [15], Lawless [17], Nelson [21], Patel [22], Arnold, Balkrishnan and Nagaraja [2], Geisser [12] and Nagaraja [20]. Balakrishnan and Lin [5] have developed exact prediction intervals for failure times from one-parameter and two-parameter exponential distributions based on doubly Type-II censored samples. Balakrishnan and Lin [4] have developed exact prediction intervals for failure times of the items censored at the last observation from one-parameter and two-parameter exponential distributions based general progressively Type-II censored samples.

In this paper, we consider the Bayesian prediction for future observations from exponential distribution using both *FSS* and *RSS*. Motivation for this work is that in many biological and quality control problems, sample sizes cannot be taken as fixed all the times. In Section 2, we derive the exact Bayesian predictive function for the FSS. The Bayesian predictive function for the *RSS* is considered in Section 3. Some applications of the findings of the papers are given in Section 4. Finally, some concluding remarks are added in Section 5.

Let $x_{1:n} \le x_{2:n} \le \ldots \le x_{r:n}$, be the first r ordered lifetimes fail in a sample of n ($r \le n$) components from the exponential distribution with *pdf*

$$f(x\mid\lambda)=\lambda e^{-\lambda x}, \quad \lambda, x>0, \tag{1}$$

and let $x_{r+1:n} \le x_{r+2:n} \le \ldots \le x_{n:n}$, be the remaining ($n$–$r$) lifetimes from the same distribution.

The joint density function of $X_{r:n}$ and $X_{s:n}$, $r < s$ is

$$f_{r,s:n}(x,y)=C_{r,s:n}F(x)^{r-1}(F(y)-F(x))^{s-r-1}(1-F(y))^{n-s}f(x)f(y), \tag{2}$$

where $f(.)$ and $F(.)$ are respectively, the *pdf* and *cdf* of the exponential distribution given in (1) and

$$C_{r,s:n}=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}. \tag{3}$$

For more details, see David [8], Aronlod, Balarishnan and Nagaraja [2] and David and Nagaraja [9].

Lawless [16] and Lingappaiah [18] have used two different statistics to predict the future observation $x_{s:n}$ based on $x_{1:n} \le x_{2:n} \le \ldots \le x_{r:n}$ when the sample size n is fixed. Their classical approach was based on the following two statistics

$$\text{Statistic-1:} \quad W=\frac{X_{s:n}-X_{r:n}}{S_r}, \text{ (see Lawless [16])}, \tag{4}$$

$$\text{Statistic-2:} \quad U=\frac{X_{s:n}-X_{r:n}}{X_{r:n}}, \text{ (see Lingappaiah [18])}, \tag{5}$$

where

$$S_r=\sum_{i=1}^{r}X_{i:n}+(n-r)X_{r:n}. \tag{6}$$

The distributions of Statistic-1 and Statistic-2 given in (4) and (5), require the *pdf* of $Z = X_{s:n} - X_{r:n}$, which is given by

$$f(z\mid\lambda)=\frac{\lambda(e^{-\lambda z})^{n-s+1}(1-e^{-\lambda z})^{s-r-1}}{\beta(s-r,n-s+1)}. \tag{7}$$

Our goal is to predict the future order statistics based on the observed sample, $x_{1:n}$, $x_{2:n}, \ldots, x_{r:n}$ using both *FSS* and *RSS* cases. In the case of *FSS*, the Bayes predictive density function of $y = x_{s:n}$ for given $x = (x_{1:n}, x_{2:n}, \ldots, x_{r:n})$ can be written as

$$h(y\mid x)=\int_0^\infty f(y\mid\lambda)\pi(\lambda\mid x)d\lambda, \tag{8}$$

where $f(y\mid\lambda)$ is the conditional *pdf* of the future observation and $\pi(\lambda\mid x)$ is the posterior *pdf.*

In the case of *RSS*, the predictive distribution function of y when the sample size n is a random variable is given by (see Gupta and Gupta [14]).

$$q(y\mid x,n)=\frac{1}{\Pr(n\ge s)}\sum_{n=s}^{\infty}r(n)h(y\mid x), \tag{9}$$

where $r(n)$ is the *pmf* of n and $h(y\mid x)$ is given in (8).

2. Prediction for Fixed Sample Size

In this section, we derive the predictive distribution function (8) based on the two different statistics given in (4) and (5). The posterior *pdf* $\pi(\lambda \mid x)$ in (8) requires different priors. In the case of Statistic-1, the gamma *pdf* will be a suitable while the *pdf* of the *r-th* order statistic will be suitable prior for Statistic-2 (for more details for choosing the prior distributions, see for example, Gelman *et al.* [13], Upadhyay and Pandey [27] and Nagaraja [20]). Also, obtaining the posterior requires the likelihood function which is

$$L(x \mid \lambda) = \frac{n!}{(n-r)!} \lambda^r e^{-\lambda S_r}, \tag{10}$$

where S_r is given by (6).

2.1. Prediction Based on Statistic-1

Under Statistic-1, we assume the 2-parameter gamma *pdf* as a prior for the exponential parameter λ as

$$g(\lambda \mid a, b) = \frac{1}{\Gamma(a) b^a} \lambda^{a-1} e^{-\lambda / b}, \quad a,\ b > 0, \tag{11}$$

where a and b are known. Combining (11) with the likelihood function in (10), we obtain then the posterior *pdf* of λ¸ as

$$\pi(\lambda \mid x) = \frac{1}{\Gamma(R)} \lambda^{R-1} A^R e^{-\lambda A}, \tag{12}$$

where

$$R = r + a, \quad A = S_r + 1/b, \tag{13}$$

and S_r is given by (6).

By using (7) and (12) in (8), the Bayesian predictive *pdf* of W given in (4) is obtained as (see Appendix A).

$$h_1(w \mid x) = \frac{R A^R}{\beta(s-r, n-s+1)} \sum_{i=0}^{s-r-1} \frac{\binom{s-r-1}{i}(-1)^i}{(A + (n-s+i+1)w)^{R+1}}, \tag{14}$$

while the Bayesian predictive *cdf* is given by

$$\begin{aligned} H_1(t) &= \Pr(W \le t \mid x) \\ &= 1 - \frac{A^R}{\beta(s-r, n-s+1)} \sum_{i=0}^{s-r-1} \frac{\binom{s-r-1}{i}(-1)^i (n-s+i+1)^{-1}}{(A + (n-s+i+1)t)^R}, \end{aligned} \tag{15}$$

where A and R are given in (13). The percentage points of the predictive *cdf* given in (15) can be easily obtained by solving the following nonlinear equation

$$H_1(t) = 1 - \alpha. \tag{16}$$

Then the exact two sided (1- α)100% Bayesian interval for the future observation $x_{s:n}$ can be constructed as

$$\left(t_{\alpha/2} + x_{r:n}, t_{1-\alpha/2} + x_{r:n}\right), \tag{17}$$

where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ are the lower and upper percentage points of $H_1(t)$.

Example 1.

In this example, we generate 5 order statistics from the exponential *pdf* given in (1) as: 0.017, 0.363, 0.365, 0.438 and 0.456. By using these data, we construct a 90% and a 95% predictive intervals for the observation $x_{s:n}$, n = 10, s = 6; 7; 8; 9; 10 by (17), these are given below:

r	s	90% *P.I*	95% *P.I*
5	6	(0.4591, 0.7246)	(0.4574,0.8131)
5	7	(0.4803,0.9697)	(0.4723, 1.1166)
5	8	(0.5180, 1.2877)	(0.5019, 1.5106)
5	9	(0.5765, 1.7695)	(0.5494, 2.1117)
5	10	(0.6815, 2.7888)	(0.6347, 3.4048)

Notice that, the values of a and b in the prior distribution are chosen randomly.

2.2. Prediction Based on Statistic-2

Under Statistic-2, we suggest the following prior *pdf* for the exponential parameter λ:

$$g(\lambda) = \frac{n!}{(r-1)!(n-r)!}\lambda(1-e^{-\lambda x})^{r-1}e^{-(n-r+1)\lambda x}. \tag{18}$$

From (10) and (18), the posterior *pdf* of λ, is obtained as

$$\pi(\lambda \mid x) = \frac{1}{K\Gamma(r+2)}\lambda^{r+1}(1-e^{-\lambda x})^{r-1}e^{-\lambda(S_r+(n-r+1)x)}, \tag{19}$$

where

$$K = \sum_{\ell=0}^{r-1}\frac{\binom{r-1}{\ell}(-1)^{\ell}}{(S_r+(n-r+\ell+1)x)^{r+2}}. \tag{20}$$

The Bayesian predictive density function of Statistic-2 can be derived upon using (7) and (19) in (8) as (see Appendix B):

$$h_2(u \mid x) = \frac{(r+2)}{K\beta(s-r,n-s+1)}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\frac{(-1)^{j+i}}{[S_r+(n-r+i+1)x+(n-s+j+1)u]^{r+3}}, \tag{21}$$

with *cdf* given by

$$H_2(t) = \Pr(U \le t \mid x)$$

$$= \frac{1}{K\beta(s-r, n-s+1)} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \frac{\binom{s-r-1}{j}\binom{r-1}{i}(-1)^{j+i}}{(n-s+j+1)}$$

$$\times \left[\frac{1}{[S_r + (n-r+i+1)x + (n-s+j+1)t]^{r+2}} \right.$$

$$\left. - \frac{1}{[S_r + (n-r+i+1)x]^{r+2}} \right], \qquad (22)$$

where S_r and K are given by (6) and (20), respectively.

The percentage points of the predictive *cdf* given in (15) can be easily obtained by solving the following nonlinear equation

$$H_2(t) = 1 - \alpha \,. \qquad (23)$$

Then the exact two sided (1- α)100% Bayesian predictive interval for the future observation $x_{s:n}$ is given in (17), where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ in this case are the lower and upper percentage points of $H_2(t)$.

Example 2.

In this example, we use the same sample as in Example 1, for constructing 90% and 95% predictive intervals for the observation $x_{s:n}$, n = 10, s = 6; 7; 8; 9; 10 using (17), (22) and (23). The results are given below:

r	*s*	90% *P.I*	95% *P.I*
5	6	(0.4608 ,0.7781)	(0.4584 ,0.8659)
5	7	(0.4920 ,1.0507)	(0.4804 ,1.1839)
5	8	(0.5490 ,1.3997)	(0.5256 ,1.5914)
5	9	(0.6388 ,1.9312)	(0.5996 ,2.2189)
5	10	(0.8002 ,3.0781)	(0.7328 ,3.6004)

3. Prediction for Random Sample Size

In this section, we assume the sample size to be random and distributed as (i) Poisson distribution and (ii) binomial distribution. For more details in this aspect see, Raghunandanan and Patil [23], Buhrman [6] and Consul [7].

3.1. Sample Size Has Poisson Distribution

We assume here that, the sample size n has Poisson distribution with *pmf* given by

$$p(n;\theta) = \frac{e^{-\theta}\theta^n}{n!}, \quad n = 0,1,2,\ldots,\theta > 0\,. \qquad (24)$$

Replacing $r(n)$ given in (9) by $p(n;\ \theta)$ given in (24), we obtain

$$q(y \mid x, n) = \frac{1}{1-P(s-1)} \sum_{n=s}^{\infty} \frac{e^{-\theta}\theta^{n}}{n!} h(y \mid x)\,, \tag{25}$$

where $P(.)$ is the *cdf* of the Poisson distribution.

In the following two subsections, we use (25) to derive the Bayesian predictive *pdf* in the case of *RSS* based on both Statistic-1 and Statistic-2.

3.1.1. Prediction Based on Statistic-1

Using Statistic-1 and replacing $h(y\,|\,x)$ given in (25) by $h_1(w\,|\,x)$ given in (14), we obtain the Bayes predictive *pdf* of W when the sample size is distributed as $p(n;\ \theta)$. This is given by (see Appendix C):

$$q_1(w \mid x, n) = \frac{RA^{R}e^{-\theta}}{1-P(s-1)} \sum_{n=s}^{\infty} \sum_{i=0}^{s-r-1} \frac{\theta^{n}\binom{s-r-1}{i}(-1)^{i}\,[A+(n-s+i+1)w]^{-R-1}}{n!\,\beta(s-r, n-s+1)}\,. \tag{26}$$

The *cdf* of W is given by

$$\begin{aligned} Q_1(t) &= \Pr(W \le t \mid x) \\ &= 1 - \frac{1}{1-P(s-1)} \sum_{n=s}^{\infty} \sum_{i=0}^{s-r-1} \frac{A^{R}e^{-\theta}\theta^{n}\binom{s-r-1}{i}(-1)^{i}}{n!\,\beta(s-r, n-s+1)\left(n-s+i+1\right)} \\ &\quad \times \left[\frac{1}{[A+(n-s+i+1)t]^{R}}\right], \end{aligned} \tag{27}$$

where A and R are given in (13).

The percentage points of the predictive *cdf* given in (27) can be easily obtained by solving the following nonlinear equation

$$Q_1(t) = 1-\alpha\,. \tag{28}$$

Then the exact two sided $(1-\alpha)100\%$ Bayesian interval for the future observation $x_{s:n}$ is given in (17), where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ in this case are the lower and upper percentage points of $Q_1(t)$.

Example 3.

In this example, we generate the sample size n from $p(n;\,2)$ to be 7. Then, we generate the first 6 order statistics based on n = 7 from the standard exponential *pdf* they are: .153, .417, .433, .720, .876 and .926. Next, this sample is used to predict the 90% and 95% predictive intervals for the future observations up to 10 as give below:

r	s	90%*P.I*	95% *P.I*
6	7	(0.9520, 3.1091)	(0.9390, 3.7941)
6	8	(1.0538, 3.7831)	(1.0118, 4.5553)
6	9	(1.1603, 4.2142)	(1.0985, 5.0401)
6	10	(1.2547, 4.5361)	(1.1796, 5.4022)

3.1.2. Prediction Based on Statistic-2

Using Statistic-2 and replacing $h(y|x)$ given in (25) by $h_2(u|x)$ given in (21), then the Bayes predictive *pdf* of U when the sample size is distributed as $p(n;\ \theta)$. This is given by (see Appendix D):

$$q_2(u\,|\,x,n) = \frac{r+2}{1-P(s-1)} \sum_{n=s}^{\infty}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} \frac{e^{-\theta}\theta^{n}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{Kn!\,\beta(s-r,n-s+1)}$$

$$\times\frac{1}{[S_r+(n-r+i+1)x+(n-s+j+1)u]^{r+3}}\,. \qquad (29)$$

The *cdf* of U is given by

$$Q_2(t) = \Pr(U \le t\,|\,x)$$

$$= \frac{1}{1-P(s-1)} \sum_{n=s}^{\infty}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} \frac{e^{-\theta}\theta^{n}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{Kn!(n-s+j+1)}$$

$$\times\left[\frac{1}{(S_r+(n-r+i+1)x+(n-s+j+1)t)^{r+2}}\right.$$

$$\left.-\frac{1}{(S_r+(n-r+i+1)x)^{r+2}}\right], \qquad (30)$$

where S_r and K are give in (6) and (20), respectively.

The percentage points of the predictive *cdf* given in (30) can be easily obtained by solving the following nonlinear equation

$$Q_2(t) = 1-\alpha\,. \qquad (31)$$

Then the exact two sided (1-α)100% Bayesian interval for the future observation $x_{s:n}$ is given in (17), where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ in this case are the lower and upper percentage points of $Q_2(t)$.

Example 4.

By using the same data in Example 3, we predict the 90% and 95% predictive intervals for the future observations up to 10 based on Statistic-2 when the sample size is $p(n;\ 2)$. The results are given below:

r	s	90% *P.I*	95% *P.I*
6	7	(0.9494 , 2.6086)	(0.9379 , 3.0815)
6	8	(1.0591 , 3.3841)	(1.0157 , 3.9823)
6	9	(1.1972 , 4.0648)	(1.1264 , 4.7787)
6	10	(1.3485 , 4.6963)	(1.2596 , 5.4864)

3.2. Sample Size Has the Binomial Distribution

In this section, we assume that the sample size n has the binomial distribution, $b(n; M; p)$, with *pmf*

$$b(n;M,p)=\binom{M}{n}p^n q^{M-n}, \quad q = 1 - p, \; n = 0, 1, 2,\ldots, M. \tag{32}$$

Replacing $r(n)$ given in (9) by $b(n; M; p)$ given in (32), we obtain

$$v(y\,|\,x,n)=\frac{1}{1-B(s-1)}\sum_{n=s}^{M}\binom{M}{n}p^n q^{M-n}h(y\,|\,x), \tag{33}$$

where $B(.)$ is the cdf of the binomial distribution. In the following two subsections, we use (33) to derive the Bayesian predictive pdf in the case of RSS based on both Statistic-1 and Statistic-2.

3.2.1. Prediction Based on Statistic-1

Using Statistic-1 and replacing $h(y\,|\,x)$ given in (33) by $h_1(w\,|\,x)$ given in (14), we obtain the Bayes predictive *pdf* of W when the sample size is distributed as $b(n; M, p)$. This is given by

$$v_1(w\,|\,x,n)=\frac{RA^R}{1-B(s-1)}\sum_{n=s}^{M}\sum_{j=0}^{s-r-1}\frac{(-1)^j\binom{M}{n}\binom{s-r-1}{j}[A+(n-s+j+1)w]^{-R-1}}{p^{-n}q^{n-M}\beta(s-r,n-s+1)}. \tag{34}$$

The *cdf* of V is given by

$$\begin{aligned}V_1(t)&=\Pr(W\le t)\\&=1-\frac{1}{1-B(s-1)}\sum_{n=s}^{M}\sum_{j=0}^{s-r-1}\frac{A^R\binom{M}{n}p^n q^{M-n}\binom{s-r-1}{j}(-1)^j}{(n-s+j+1)\beta(s-r,n-s+1)}\\&\quad\times\frac{1}{[A+(n-s+j+1)t]^R},\end{aligned} \tag{35}$$

where A, R are given in (13).

Example 5.

In this example, we generate the sample size n from $b(n; 15; 0{:}3)$, then based on the generated n, we generate the first 6 order statistics from the standard exponential as: .177, .195, .493, 1.262, 1.376 and 1.444. By using this data we predict the 90% and 95% predictive intervals for the future observations up to 10, based on Statistic-1 when the sample size is $b(n; 15; 0{:}3)$. This is given below:

r	s	90% $P.I$	95% $P.I$
6	7	(1.4724 , 4.0975)	(1.4581 , 5.0142)
6	8	(1.5975 , 5.0538)	(1.5473 , 6.0555)
6	9	(1.7426 , 5.6908)	(1.6646 , 6.7684)
6	10	(1.8745 , 6.1762)	(1.7738 , 7.3189)

3.2.2. Prediction Based on Statistic-2

Again using Statistic-2 and replacing $h(y|x)$ given in (33) by $h_2(u|x)$ given in (21), we obtain the Bayes predictive *pdf* of U when the sample size is distributed as $b(n; M; p)$. This is given by

$$v_2(u \mid x, n) = \frac{r+2}{1-B(s-1)} \sum_{n=s}^{M} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \frac{\binom{M}{n}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{K\beta(s-r, n-s+1)}$$

$$\times \frac{1}{[S_r + (n-r+i+1)x + (n-s+j+1)u]^{r+3}} . \tag{36}$$

The *cdf* of U is given by

$$V_2(t) = \Pr(U \le t \mid x)$$

$$= \frac{1}{1-B(s-1)} \sum_{n=s}^{M} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \frac{\binom{M}{n} p^n q^{M-n} \binom{s-r-1}{j}\binom{r-1}{i}(-1)^{i+j}}{K\beta\left(s-r, n-s+1\right)(n-s+j+1)}$$

$$\times \left[\frac{1}{(S_r + (n-r+i+1)x + (n-s+j+1)t)^{r+2}} - \frac{1}{(S_r + (n-r+i+1)x)^{r+2}} \right],$$

where S_r and K are given in (6) and (20), respectively.

Example 6.

By using the same data in Example 5, we predict the 90% and 95% predictive intervals for the future observations up to 10 based on Statistic-2 when the sample size is $b(n; 15; 0{:}3)$ as shown below:

r	s	90% $P.I$	95% $P.I$
6	7	(1.4742 , 3.7642)	(1.4589 , 4.4340)
6	8	(1.6288 , 4.9745)	(1.5701 , 5.8308)
6	9	(1.8394 , 6.0728)	(1.7394 , 7.1167)
6	10	(2.0763 , 7.0983)	(1.9372 , 8.2842)

4. Application

In this section, we apply our technique to some real data which follow the exponential distribution as presented in Lawless [16] in which the test is terminated after the following 4 failures: 30, 90, 120 and 170 hours. By using these four times, we calculate the percentage points for the predictive functions presented in Sections 2 and 3 based on FSS and RSS up to 10 failures as given below:

Table 1. Percentage points based on Statistic-1 when n = 10 is fixed.

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	0.5	1.2	2.5	5.1	139.4	195.8	260	360.3
4	6	7.2	11.8	17.5	26.7	273.6	365.3	469.9	630.4
4	7	22.1	31.9	43	59.5	435.4	570.2	722.7	957.9
4	8	45.5	61.7	79.4	105	650.3	843.2	1061.1	1396.1
4	9	80.8	106	133	171.6	976.6	1261.6	1582.1	2077.8
4	10	141.6	183	227.2	290.6	1655	2146.4	2703.3	3565.9

Table 2. Percentage points based on Statistic-2 when n = 10 is fixed.

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	0.5	1.3	2.7	5.6	138.6	187.8	241	317.9
4	6	8.2	13.5	19.9	30	266	341	420.8	534.4
4	7	25.9	37.1	49.6	67.9	417.8	523.4	634.9	793.4
4	8	54	72.7	92.6	120.8	618.8	766	921	1141
4	9	97	125.9	156.1	198.5	925.1	1139.6	1365.6	1687
4	10	171.4	218.3	267.5	336.4	1569.7	1946.4	2346.8	2919.6

Table 3. Percentage points based on Statistic-1 when $n \sim p(n;2)$.

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	2	5.1	10.4	21.7	707.7	1014.8	1370	1926.7
4	6	21	34.9	52.4	81.3	1002.3	1367.5	1786	2436
4	7	48.4	70.9	97	137.1	1196.8	1599.3	2058.6	2770.3
4	8	75.8	104.6	136.8	184.8	1343.7	1773.5	2263.3	3022.2
4	9	101.2	134.7	171.5	225.5	1461.7	1915.4	2430.6	3224.2
4	10	124.1	161.6	201.9	260.5	1560.6	2033.2	2569.4	3397.1

Table 4. Percentage points based on Statistic-2 when $n \sim p(n;2)$.

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.6	4.2	8.5	17.6	501.5	690.6	899.2	1223.8
4	6	19.2	31.8	47.2	72.6	753.2	982.9	1233.7	1589.4
4	7	48	69.7	95.6	132.3	953.9	1226.1	1498.9	1974
4	8	81	113.1	144.8	192	1141.6	1446.4	1767.7	2255.2
4	9	118.3	153.4	194.3	251.3	1318.3	1646.4	2016.7	2555.4
4	10	152.4	198.2	244.2	308	1490.6	1846.5	2241.9	2815.6

Table 5. Percentage points based on Statistic-1 when $n \sim b(n;15,0.3)$

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.4	3.7	7.5	15.6	569.3	844.9	1148.2	1602.3
4	6	16.9	28.3	42.3	66.3	887.3	1229.1	1613	2235.4
4	7	42.3	62.4	86.5	121.5	1115.2	1506.7	1930.8	2548.4
4	8	70.3	98.2	128	172.7	1288.2	1712.5	2163.9	2901.1
4	9	98.5	132.7	167.1	219.8	1434.4	1886.4	2395.7	3281.3
4	10	124.6	160.5	202.3	260.8	1556.2	2044.7	2590.3	3487.2

Table 6. Percentage points based on Statistic-2 when $n \sim b(n;15,0.3)$.

r	s	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.2	3.2	6.5	13.5	411.2	582.4	776.2	1056.9
4	6	16.3	27	40	61.7	674.6	895.4	1139.4	1492.5
4	7	43.6	63.3	86.8	120.3	895.9	1159.1	1439.1	1831.7
4	8	76.9	107.3	137.4	182.4	1103.4	1402.5	1720.6	2193.2
4	9	115.8	150.2	190.2	246.1	1299	1629.9	1965.5	2506.7
4	10	155.1	198.5	245.4	308.1	1487.4	1854.9	2254.8	2822.2

From the above Tables 1-6, we find the 90% and 95% predictive intervals for the fifth failure *FSS* and *RSS* when $n \sim p(n;2)$ and $n \sim b(n;15,0.3)$ using (17). These are given below:

$1-\alpha$	Sample size	Statistic-1	Statistic-2
90%	Fixed (n = 10)	(172.5 , 365.8)	(172.7 , 357.8)
	$p(n;2)$	(180.4 , 1184.8)	(178.5 , 860.6)
	$b(n;15;0:3)$	(177.5 , 1115)	(176.5 , 752.4)
95%	Fixed (n = 10)	(171.2 , 430.0)	(171.3 , 411.0)
	$p(n;2)$	(175.1 , 1540.0)	(174.2 , 1069.2)
	$b(n;15;0:3)$	(173.7 , 1318.2)	(173.2 , 946.2)

From the above table we see that the width of the Bayesian predictive intervals based on random and fixed sample sizes are close for most cases.

To show the efficiency of our Bayesian technique, we calculate the width of the 95% predictive intervals the results given in the above table analogues with those of the classical technique by Lawless [16] and Lingappaiah [18] as given below:

	Classical Approach	Bayesian Approach
Statistic-1	262.6	258.8
Statistic-2	261.9	239.7

It is clear that the Bayesian approach gives narrower predictive intervals than the corresponding classical intervals.

5. Concluding Remarks

The Baysian predictive function for future observations from the exponential distribution based on the fixed and random sample size are investigated. For the finite populations, the binomial distribution is considered to be a suitable model for the sample sizes, while for the large populations, Poisson distribution can be considered as a suitable model. To show the usefulness of the proposed procedure, simulation experiments are carried out and an application is discussed. Finally, we conclude the following remarks:

1. The random samples from the exponential distribution are generated by using the double precision subroutine RNEXP from the IMSL library.
2. The nonlinear equations are solved by using the double precision subroutine ZREAL from the IMSL library.
3. The parameters of the prior a and b are positive values and selected randomly which do not affect the calculations.
4. The sample size n is generated randomly from binomial and Poisson distributions by using the subroutines RNBIN and RNPOI, respectively, from the IMSL library.
5. The proportion of future responses that can be predicted using the proposed predictive intervals is investigated in the sense of probability coverage. It gives probabilities close to their significance levels.
6. The Bayesian approach gives better results than the classical approach in the sense of the predictive average width of the predictive intervals.

Acknowledgments

The authors would like to thank the Editor and the anonymous referees for their helpful comments, which improved the presentation of the paper. The second author would like to thank the Research Center, College of Science, King Saud University for funding the project (Stat/1422/27).

References

1. Aitchison, J. and Dunsmore, I. R. (1975). *Statistical Prediction Analysis.* Cambridge University press, Cambridge.
2. Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1992). *A First Course in Order Statistics.* John Wiley & Sons, New York.
3. Balakrishnan, N. and Basu, A. P. (1995). *The Exponential Distribution, Theory, Methods and Applications.* Gordon and Breach publishers, Amsterdam.
4. Balakrishnan, N. and Lin, C. T. (2002). Exact linear inference and prediction for exponential distributions based on general progressively Type-II censored samples. *Journal of Statistical Computational and Simulation,* 72 (8), 677-686.
5. Balakrishnan, N. and Lin, C. T. (2003). Exact prediction intervals for exponential distributions based on doubly Type-II censored samples. *Journal of Applied Statistics,* 30 (7), 783-8001.
6. Buhrman, J. M. (1973). On order statistics when the sample size has a binomial distribution. *Statistica Neerlandica,* 27, 125-126.
7. Consul, P. C. (1984). On the distribution of order statistics for a random sample size, *Statistica Neerlandica,* 38, 249-256.

8. David, H. A. (1981). *Order Statistics,* 2nd edition. John Wiley & Sons, New York.

9. David, H. A. and Nagaraja, H. N. (2003). *Order Statistics,* 3rd edition. John Wiley & Sons, New York.

10. Dunsmore, I. R. (1974). The Bayesian predictive distribution in life testing model. *Technometrics,* 16, 455-460.

11. Geisser, S. (1986). Predictive analysis, In *Encyclopedia of Statistical Sciences,* 7 (Edited by S. Kotz, N. L. Johnson and C. B. Read). Wiley & Sons, New York.

12. Geisser, S. (1993). *Predictive Inference: An Introduction.* Chapman & Hall, London.

13. Gelman, A., Garlin, J. B., Stern, H. S and Rubin, D. B. (2004). *Bayesian Data Analysis*. Chapman and Hall, London.

14. Gupta, D. and Gupta, R. C. (1984). On the distribution of order statistics for a random sample size. *Statistica Neerlanddica,* 38, 13-19.

15. Hahn, J. G. and Meeker, W. Q. (1991). *Statistical Intervals A Guide for Practitioners.* Wiley & Sons, New York.

16. Lawless, J. F. (1971). A prediction problem concerning samples from the exponential distribution with application in life testing. *Technometrics,* 13, 725-730.

17. Lawlees, J. F. (1982). *Statistical Models & Methods For Lifetime Data.* Wiley & Sons, New York.

18. Lingappaiah, G. S. (1973). Prediction in exponential life testing. *Canadian Journal of Statistics,* 1, 113-117.

19. Lingappaiah, G. S. (1986). Bayes prediction in exponential life-testing when Sample Size is random variable. *IEEE Transactions on Reliability,* 35 (1), 106-110.

20. Nagaraja, H. N. (1995). Prediction problems, In *The Exponential Distribution, Theory, Methods and Applications,* (Edited by N. Balakrishnan and A. P. Basu). Gordon and Breach publishers, Amsterdam.

21. Nelson, W. (1982). *Applied Life Data Analysis.* Wiley & Sons, New York.

22. Patel, J. K. (1989). Prediction intervals - A review. *Communications in Statistics - Theory Methods,* 18, 2393-2465.

23. Raghunandanan, K. and Patil, S. A. (1972). On order statistics for random sample size. *Statistica Neerlandica,* 26, 121-126.

24. Soliman, A. Ahmed (2000a). Bayes prediction in a Pareto lifetime model with random sample size. *The Statisticians,* 49 (1), 51-62.

25. Soliman, A. Ahmed (2000b). Bayes 2-sample prediction for the Pareto distribution. *Journal of The Egyptian Mathematical Society,* 8 (1), 95-109.

26. Soliman,A.A. and Abd Ellah, A. H. (1993). Prediction of s^{th} ordered observations in doubly Type-II censored sample from one parameter exponential distribution. *The Egyptian Statistical Journal,* 37, 346-352.

27. Upadhyay, S. K. and Pandey, M. (1989). Prediction limits for an exponential distribution: A Bayes predictive distribution approach. *IEEE Transactions on Reliability,* 38 (5), 599-602.

Appendix

A. Proof of Equation (14).

The *pdf* of W given in (4) can be derived by using the transformation $W = Z/S_r$. This gives

$$f_W(w\,|\,\lambda) = f_Z(wS_r\,|\,\lambda)\left|\frac{\partial Z}{\partial W}\right|$$

$$= \frac{S_r \lambda (e^{-\lambda w S_r})^{n-s+1}(1-e^{-\lambda w S_r})^{s-r-1}}{\beta(s-r, n-s+1)}. \tag{A1}$$

From (8), the predictive function $h_1(w\,|\,x)$ can be written as

$$h_1(w\,|\,x) = \int_0^\infty f_W(w\,|\,\lambda)\pi(\lambda\,|\,x)d\lambda, \tag{A2}$$

where $f_W(.\,|\,\lambda)$ and $\pi(\lambda\,|x)$ are given in (A1) and (12), respectively.

By using (A1) in (A2), and expanding the term $\left(1-e^{-\lambda w S_r}\right)^{s-r-1}$ binomially and integrating out of λ, we get

$$h_1(w\,|\,x) = \frac{S_r R A^R}{\beta(s-r, n-s+1)} \sum_{i=0}^{s-r-1} \frac{\binom{s-r-1}{i}(-1)^i}{(A+(n-s+i+1)wS_r)^{R+1}}, \tag{A3}$$

which upon using the transform $ws_r = y$ gives the same distribution given in (14). In fact both of (A3) and (14) give the same cumulative distribution function given in (15).

B. Proof of Equation (21).

The *pdf* of U given in (5) can be derived by using the transformation $U = Z/x_{r:n}$. This gives

$$f_U(u\,|\,\lambda) = f_Z(ux_{r:n}\,|\,\lambda)\left|\frac{\partial Z}{\partial U}\right|$$

$$= \frac{x_{r:n}\lambda(e^{-\lambda u x_{r:n}})^{n-s+1}(1-e^{-\lambda u x_{r:n}})^{s-r-1}}{\beta(s-r, n-s+1)}. \tag{A4}$$

From (8), the predictive function $h_2(u\,|\,x)$ can be written as

$$h_2(u\,|\,x) = \int_0^\infty f_U(u\,|\,\lambda)\pi(\lambda\,|\,x)d\lambda, \tag{A5}$$

where $f_U(.\,|\,\lambda)$ and $\pi(\lambda\,|x)$ are given are given in (A6) and (19), respectively.

By using (A4) in (A5), and expanding the terms $\left(1-e^{-\lambda x_{r:n}}\right)^{r-1}$ and $\left(1-e^{-\lambda u x_{r:n}}\right)^{s-r-1}$ binomially and simplifying, we get

$$h_2(u\,|\,x) = \frac{x_{r:n}(r+2)}{K\beta(s-r, n-s+1)} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times \frac{(-1)^{j+i}}{[S_r+(n-r+i+1)x+(n-s+j+1)ux_{r:n}]^{r+3}}, \tag{A6}$$

which upon using the transform $wsr = y$ gives the same distribution given in (21). In fact both of (A5) and (21) give the same cumulative distribution function given in (22).

C. Proof of Equation (26).

Starting from the predictive function in the of RSS given in (9), we can write the predictive function of Statistic-1 based on RSS as

$$q_1(w \mid x, n) = \frac{1}{1 - P(s-1)} \sum_{n=s}^{\infty} p(n) h_1(w \mid x) \,, \tag{A7}$$

where $p(.)$ and $P(.)$ are the *pmf* and *cdf* of the Poisson distribution, respectively, and $h_1(. \mid x)$ is given in (14). Upon using (14), (24) and integrating out of λ, we get (26).

D. Proof of Equation (29).

Following the same technique as in Appendix C and replacing $h_1(. \mid \lambda)$ in (A7) by $h_2(. \mid \lambda)$ in (21) and simplifying, we get (29).

Authors' Biographies:

Khalaf S. Sultan is an Associate Professor in the Department of Statistics and Operations Research, King Saud University. His research interests include statistical inference, order statistics, goodness of fit tests and mixtures. He is the co-author of several book chapters in Handbook of Statistics (Eds. N. Balakrishnan and C.R. Rao) and in Stochastic Simulation Methods (Eds. N. Balakrishnan, S. Ermakov and V. Melas). His publications have appeared in the statistical literature.

Ahmed H. Abd Ellah is an Assistant Professor in the Department of Mathematics, Teachers College, Riyadh, Saudi Arabia. His interests are in the areas of statistical inference, distribution theory, and prediction. His publications have appeared in the statistical literature.