METRON - *International Journal of Statistics*
2004, vol. LXII, n. 3, pp. 377-389

NAGI S. ABD-EL-HAKIM – KHALAF S. SULTAN

Maximum likelihood estimation from record-breaking data for the generalized Pareto distribution

Summary - In this paper, we obtain the MLEs of parameters for the generalized Pareto distribution (GPD) based on record-breaking data (record values). Then, we discuss the properties of these estimates. Next, we compare the MLEs of the location and scale parameters with the BLUEs given by Sultan and Moshref (2000). In addition, we use the MLEs to construct confidence intervals for the location and scale parameters of GPD.

Key Words - Upper record values; Maximum likelihood estimates; Biased and unbiased estimates; Best linear unbiased estimates; Interval estimation; Minimum variance bound and relative efficiency.

1. Introduction

A random variable X is said to have the GPD if its probability density function (pdf) is of the following form [see Pickands (1975)]:

$$f(x) = \begin{cases} \dfrac{1}{\sigma}\left\{1+\beta\left(\dfrac{x-\theta}{\sigma}\right)\right\}^{-(1+1/\beta)}, & x \geq \theta,\ \text{for } \beta > 0\,, \\ & \theta < x < \theta - \sigma/\beta \ \text{for } \beta < 0\,, \\ \dfrac{1}{\sigma} e^{-(x-\theta)/\sigma}, & x \geq \theta,\ \text{for } \beta = 0\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{1.1}$$

while the standard form of the GPD is given from (1.1) by substituting $\sigma = 1$ and $\theta = 0$. Some related distributions are listed below [see also Johnson, Kotz and Balakrishnan (1994)].

Received December 2003 and revised December 2004.

1. For $\beta > 0$, GP distribution is known as Pareto type II or Lomax distribution.
2. For $\beta = -1$, GP distribution coincides with the uniform distribution on $(\theta, \theta + \sigma)$.
3. As $\beta \to 0$, GP distribution leads to a two-parameter exponential distribution.

The generalized Pareto distribution was introduced by Pickands (1975). Some of its applications include its uses in the analysis of extreme events, in the modeling of large insurance claims, and to describe the annual maximum flood at river gauging station. Hosking and Wallis (1987) studied the parameter and quantile estimation for the two-parameter generalized Pareto distribution, Smith (1987) has discussed the maximum likelihood estimation for the GPD under simple random sampling. For some interesting graphical representation of the generalized Pareto densities see Reiss (1989).

Record values arise naturally in many real life applications involving data relating to weather, sports, economics and life testing studies. Many authors have studied record values and associated statistics; for example, see Chandler (1952), Ahsanullah (1980, 1988, 1990, 1993, 1995), and Arnold, Balakrishnan and Nagaraja (1992, 1998). Ahsanullah (1980, 1990), Balakrishnan and Chan (1993), and Balakrishnan, Ahsanullah and Chan (1995) have discussed some inferential methods for exponential, Gumbel, Weibull and logistic distributions, respectively. Maximum likelihood estimates of parameters for some useful distributions, including one and two parameter exponential, one and two parameter uniform, normal, logistic and Gumbel distributions are discussed in Arnold, Balakrishnan and Nagaraja (1998). Balakrishnan and Ahsanullah (1994) have established some recurrence relations satisfied by the single and double moments of upper record values from the standard form of the GPD.

In this paper, we derive the MLEs of parameters of GPD given in (1.1) based on record values, then we discuss the efficiency of these estimates. Also, we compare our results by the BLUEs of the location and scale parameters obtained by Sultan and Moshref (2000). Finally, we use the MLEs to construct confidence intervals for the location and scale parameters of GPD.

2. MLEs

Let $X_{U(1)}, X_{U(2)}, \ldots X_{U(n)}$ be the first n upper record-braking values from the GPD given in (1.1), for convenience let us denote $X_{U(i)}$ by $X_i, i = 1, 2, \ldots, n$. Then the pdf of the n-th upper record value is given by

$$f_n(x) = \frac{1}{\Gamma(n)}[-\log\{1 - F(x)\}]^{n-1} f(x)\,, \tag{2.1}$$

where $f(.)$ is given by (1.1) and $F(.)$ is the corresponding cdf.

The likelihood function in this case may be written as

$$L(\theta,\sigma,\beta)=\begin{cases}\dfrac{1}{\sigma^n}\left[1+\beta\left(\dfrac{x_n-\theta}{\sigma}\right)\right]^{-1/\beta}\displaystyle\prod_{i=1}^{n}\left[1+\beta\left(\dfrac{x_i-\theta}{\sigma}\right)\right]^{-1}, & \beta\neq 0\,,\\ \dfrac{1}{\sigma^n}e^{-(x_n-\theta)/\sigma}, & \beta=0\,.\end{cases} \tag{2.2}$$

From (2.2), we discuss the following cases:

1. **When $\beta = 0$ (Two-parameter exponential distribution):** Arnold, Balakrishnan and Nagaraja (1998) have obtained the MLEs of θ and σ to be

$$\hat{\theta} = x_1 \quad \text{and } \hat{\sigma} = (x_n - x_1)/n\,.$$

They also have discussed the unbiasedness and variances. For the sake of completeness and comparisons, we present their results as given below:

$$E(\hat{\theta}) = \theta + \sigma,\ \ \mathrm{Var}(\hat{\theta}) = \sigma^2,\ \text{ and MSE } (\hat{\theta}) = 2\sigma^2\,, \tag{2.3}$$

and

$$E(\hat{\sigma}) = \frac{(n-1)\sigma}{n},\ \ \mathrm{Var}(\hat{\sigma}) = \frac{(n-1)\sigma^2}{n^2},\ \text{ and MSE } (\hat{\sigma}) = \frac{\sigma^2}{n}\,. \tag{2.4}$$

In this case, we propose the unbiased estimate of σ to be

$$\tilde{\sigma} = \frac{x_n - x_1}{n-1}\,, \tag{2.5}$$

and hence

$$\mathrm{Var}(\tilde{\sigma}) = \mathrm{MSE}(\tilde{\sigma}) = \frac{\sigma^2}{n-1}\,. \tag{2.6}$$

The minimum variance bound for the estimate of σ (MVB) is given by $\frac{\sigma^2}{n}$ and the relative efficiency of $\tilde{\sigma}$ (with respect to MVB(σ)) is given by $\frac{n-1}{n}$.

For θ we propose the following MLEs

$$\tilde{\theta}_1 = x_1 - \hat{\sigma} = \frac{(n+1)x_1 - x_n}{n}\,, \tag{2.7}$$

and

$$\tilde{\theta}_2 = x_1 - \tilde{\sigma} = \frac{nx_1 - x_n}{n-1}\,. \tag{2.8}$$

From (2.7) and (2.8), we have

$$E(\tilde{\theta}_1) = \theta + \frac{\sigma}{n}, \quad \mathrm{Var}(\tilde{\theta}_1) = \frac{(n^2+n-1)\sigma^2}{n^2} \quad \text{and} \quad \mathrm{MSE}(\tilde{\theta}_1) = \frac{(n+1)\sigma^2}{n}, \tag{2.9}$$

and

$$E(\tilde{\theta}_2) = \theta \quad \text{and} \quad \mathrm{Var}(\tilde{\theta}_2) = \mathrm{MSE}(\tilde{\theta}_2) = \frac{n\sigma^2}{n-1}\,. \tag{2.10}$$

From the above discussion, we note that $\hat{\sigma}$ represents a biased estimate for σ while $\tilde{\sigma}$ represents an unbiased estimate for σ but $\mathrm{MSE}(\hat{\sigma}) < \mathrm{MSE}(\tilde{\sigma})$, while $\tilde{\theta}_1$ is biased estimate for θ and $\tilde{\theta}_2$ is unbiased estimate for θ but $\mathrm{MSE}(\tilde{\theta}_1) < \mathrm{MSE}(\tilde{\theta}_2)$. Also, we can see that $\hat{\sigma}$ and $\tilde{\sigma}$ are consistent.

Remark. when $\beta = 0$, the estimators in (2.3) and (2.4) are neither asymptotically centered nor consistent, while the estimators in (2.9) and (2.10) are not consistent.

2. **When** $\beta \neq 0$: maximizing the logarithm of the likelihood function in (2.2) with respect to θ, σ and β, respectively, gives

$$\hat{\theta} = x_1\,, \tag{2.11}$$

$$\hat{\sigma} = \frac{\hat{\beta}}{e^{n\hat{\beta}} - 1}(x_n - \hat{\theta})\,, \tag{2.12}$$

$$\sum_{i=1}^{n}\left[e^{n\hat{\beta}} + \frac{x_n - x_i}{x_i - \hat{\theta}}\right]^{-1} = \frac{n}{e^{n\hat{\beta}} - 1} - \frac{1}{\hat{\beta}e^{n\hat{\beta}}}\,. \tag{2.13}$$

In order to discuss the efficiency of the MLEs of θ and σ, we consider the following cases:

(a) σ **and** β **are known:** from (2.11), it is easy to show that

$$E(\hat{\theta}) = \theta + \frac{\sigma}{1-\beta}, \quad \beta < 1\,, \tag{2.14}$$

with variance given by

$$\mathrm{Var}(\hat{\theta}) = \frac{\sigma^2}{(1-2\beta)(1-\beta)^2}, \quad \beta < 1/2\,, \tag{2.15}$$

and

$$\mathrm{MSE}(\hat{\theta}) = \frac{2\sigma^2}{(1-2\beta)(1-\beta)}, \quad \beta < 1/2\,. \tag{2.16}$$

From (2.14), we may propose the unbiased estimate of θ as

$$\tilde{\theta} = x_1 - \frac{\sigma}{1-\beta}\,, \tag{2.17}$$

with the same variance given in (2.15).

Notice that, the results given in (2.3) can be easily obtained from (2.14), (2.15) and (2.16) by letting $\beta \to 0$.

(b) θ **and** β **are known:** if θ and β are known, then from (2.12), we have

$$E(\hat{\sigma}) = \frac{(1-\beta)^{-n}-1}{e^{n\beta}-1}\sigma\,, \tag{2.18}$$

$$\mathrm{Var}(\hat{\sigma}) = \frac{(1-2\beta)^{-n}-(1-\beta)^{-2n}}{(e^{n\beta}-1)^2}\sigma^2\,, \tag{2.19}$$

and

$$\mathrm{MSE}(\hat{\sigma}) = \frac{(1-2\beta)^{-n}-2(1-\beta)^{-n}e^{n\beta}+e^{2n\beta}}{(e^{n\beta}-1)^2}\sigma^2\,. \tag{2.20}$$

In this case, we propose the unbiased estimate of σ to be

$$\tilde{\sigma} = \frac{\beta(x_n-\theta)}{(1-\beta)^{-n}-1}\,. \tag{2.21}$$

It can be shown that

$$\mathrm{Var}(\tilde{\sigma}) = \frac{(1-2\beta)^{-n}-(1-\beta)^{-2n}}{((1-\beta)^{-n}-1)^2}\sigma^2\,. \tag{2.22}$$

(c) β **is known:** if θ is unknown and β is known, then from (2.12), we have

$$E(\hat{\sigma}) = \frac{(1-\beta)^{-n}-(1-\beta)^{-1}}{e^{n\beta}-1}\sigma\,, \tag{2.23}$$

$$\begin{aligned}\mathrm{Var}(\hat{\sigma}) = \Big[&(1-2\beta)^{-n}-(1-\beta)^{-2n}-2\beta^2(1-2\beta)^{-1}(1-\beta)^{-(n+1)}\\ &+\beta^2(1-2\beta)^{-1}(1-\beta)^{-2}\Big]\frac{\sigma^2}{(e^{n\beta}-1)^2}\,,\end{aligned} \tag{2.24}$$

$$\begin{aligned}\mathrm{MSE}(\hat{\sigma}) = \Big[&(1-2\beta)^{-n}-(1-\beta)^{-2n}-2\beta^2(1-2\beta)^{-1}(1-\beta)^{-(n+1)}\\ &+\beta^2(1-2\beta)^{-1}(1-\beta)^{-2}+[(1-\beta)^{-n}-(1-\beta)^{-1}-e^{n\beta}+1]^2\Big]\\ &\times\frac{\sigma^2}{(e^{n\beta}-1)^2}\,.\end{aligned} \tag{2.25}$$

From (2.23), (2.24) and (2.25), we have

$$\lim_{\beta \longrightarrow 0} E(\hat{\sigma}) = \frac{n-1}{n}\sigma\,, \tag{2.26}$$

$$\lim_{\beta \longrightarrow 0} \mathrm{Var}(\hat{\sigma}) = \frac{n-1}{n^2}\sigma\,, \tag{2.27}$$

and

$$\lim_{\beta \longrightarrow 0} \mathrm{MSE}(\hat{\sigma}) = \frac{\sigma^2}{n}\,, \tag{2.28}$$

which are the same as the results given in (2.4).
In this case, we consider the unbiased estimates of θ and σ to be

$$\tilde{\sigma} = \frac{\beta(x_n - x_1)}{(1-\beta)^{-n} - (1-\beta)^{-1}}\,, \tag{2.29}$$

and

$$\tilde{\theta} = \left(1 + \frac{\beta}{(1-\beta)^{1-n} - 1)}\right) x_1 - \left(\frac{\beta}{(1-\beta)^{1-n} - 1}\right) x_n\,. \tag{2.30}$$

Hence

$$\begin{aligned}\mathrm{Var}(\tilde{\sigma}) = \mathrm{MSE}(\tilde{\sigma}) = \Big[&(1-2\beta)^{-n} - (1-\beta)^{-2n} \\ &- 2\beta^2(1-2\beta)^{-1}(1-\beta)^{-(n+1)} + \beta^2(1-2\beta)^{-1}(1-\beta)^{-2}\Big]\,, \\ &\times \left[\frac{\sigma}{(1-\beta)^{-n} - (1-\beta)^{-1}}\right]^2,\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}\mathrm{Var}(\tilde{\theta}) = \sigma^2\Big[&(1-\beta)^{2n-2} + (1-\beta)^{2n-2}(1-2\beta)^{n-1} \\ &-2(1-\beta)^{n-1}(1-2\beta)^{n-1}\Big] \Big/ \left(\left[1-(1-\beta)^{n-1}\right]^2 (1-2\beta)^n\right)\,.\end{aligned} \tag{2.32}$$

(d) θ **and** σ **are known and** β **is unknown:** solving the equation (2.13) gives the MLE of β.

In the following two theorems, we discuss the minimum variance bound (MVB) of the MLEs of both σ and β:

Theorem 1. *For positive β, the lower bound of the variance of $\hat{\beta}$ is given by*

$$\mathrm{MVB}(\hat{\beta}) = \frac{2\beta^3}{2n\beta - 3 + 4(1+\beta)^{-n} - (1+2\beta)^{-n}}\,, \tag{2.33}$$

and

$$\mathrm{MVB}(\hat{\beta}) = \begin{cases} \dfrac{3}{n(n+1)(n+2)}, & \text{as } \beta \to 0\,, \\ 0, & \text{as } n \to \infty\,. \end{cases} \tag{2.34}$$

Proof. See Appendix B.

Theorem 2. *For $\beta > -1/2$, the lower bound of the variance of $\hat{\sigma}$ is given by*

$$\mathrm{MVB}(\hat{\sigma}) = \frac{2\beta\sigma^2}{1-(1+2\beta)^{-n}}\,, \tag{2.35}$$

and

$$\mathrm{MVB}(\hat{\sigma}) = \begin{cases} \dfrac{\sigma^2}{n}, & \text{as } \beta \to 0\,, \\ 2\beta\sigma^2, & \text{as } n \to \infty,\ \beta > 0\,, \\ 0, & \text{as } n \to \infty,\ \beta \le 0\,. \end{cases} \tag{2.36}$$

Proof. See Appendix A.

3. Simulation and comparisons

In order to show the efficiency of our results, we calculate the variances of the MLEs of the location and scale parameters of GPD and compare them with those of the BLUEs θ^* and σ^* obtained by Sultan and Moshref (2000). Table 1 gives the variances of the BLUEs and MLEs for $n = 3, 4, 5, 6$ and 7.

TABLE 1: *Variances of the BLUEs and MLEs when $\theta = 0$ and $\sigma = 1$*

		Location Parameter		Scale Parameter	
β	n	$\mathrm{Var}(\theta^*)$	$\mathrm{Var}(\tilde{\theta})$	$\mathrm{Var}(\sigma^*)$	$\mathrm{Var}(\tilde{\sigma})$
-0.1	3	1.149	1.150	0.390	0.391
	4	1.024	1.025	0.239	0.240
	5	0.963	0.964	0.165	0.166
	6	0.927	0.928	0.121	0.123
	7	0.903	0.905	0.093	0.095
0.1	3	2.118	2.121	0.716	0.718
	4	1.890	1.896	0.531	0.536
	5	1.779	1.788	0.441	0.449
	6	1.715	1.728	0.389	0.399
	7	1.674	1.691	0.356	0.369

From Table 1, we can see that the variances of the BLUEs and MLEs decrease as n increases, and increase when β increases. In conclusion, we can say that the variances of BLUEs obtained by Sultan and Moshref (2000) and the unbiased MLEs presented in this paper are very close, but the MLEs are simpler to evaluate than the BLUEs. Also, as we can see from Table 1, if $n \to \infty$, $\beta = -0.1$ and $\sigma = 1$, then $\mathrm{Var}(\hat{\theta}) = 0.833$ and $\mathrm{Var}(\hat{\sigma}) = 0.00833$ that is because when $\beta < 0$ we have

$$\lim_{n\to\infty} \mathrm{Var}(\hat{\theta}) = \frac{\sigma^2}{1-2\beta} \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Var}(\hat{\sigma}) = \frac{\beta^2\sigma^2}{1-2\beta}\,.$$

4. Interval estimation

In this section, we construct confidence intervals for the location and scale parameters of GPD given in (1.1).

4.1. A confidence interval for θ when σ and β are known

Confidence interval for θ when σ and β are known may be constructed through the statistic

$$T = \frac{\tilde{\theta} - \mu_{\tilde{\theta}}}{\sigma_{\tilde{\theta}}}\,, \tag{4.1}$$

where $\mu_{\tilde{\theta}}$ and $\sigma_{\tilde{\theta}}$ represent the mean and the standard deviation of the unbiased estimate of θ given in (2.17).

It is easy to show that the distribution of T is the GPD with location parameter $-\sqrt{1-2\beta}$, scale parameter $(1-\beta)\sqrt{1-2\beta}$ and shape parameter β. A $(1-\alpha)100\%$ confidence interval for θ in this case is obtained to be

$$\left(x_1 - \frac{\sigma}{\beta}\left[(\alpha/2)^{-\beta} - 1\right],\ x_1 - \frac{\sigma}{\beta}\left[(1-\alpha/2)^{-\beta} - 1\right]\right), \tag{4.2}$$

where x_1 is the first upper record.

4.2. *A confidence interval for σ when θ and β are known*

Confidence interval for σ when θ and β are known may be constructed using the statistic

$$\tau = \frac{\tilde{\sigma} - \mu_{\tilde{\sigma}}}{\sigma_{\tilde{\sigma}}}, \tag{4.3}$$

where $\mu_{\tilde{\theta}}$ and $\sigma_{\tilde{\theta}}$ represent the mean and the standard deviation of the unbiased estimate of σ given in (2.21).

It is easy to show that the distribution of τ is the $n-$th record value of the GPD given in (2.1) with location parameter θ', scale parameter σ' and shape parameter β, where

$$\theta' = \frac{1-(1-\beta)^{-n}}{\sqrt{(1-2\beta)^{-n} - (1-\beta)^{-2n}}} \text{ and } \sigma' = \frac{\beta}{\sqrt{(1-2\beta)^{-n} - (1-\beta)^{-2n}}}. \tag{4.4}$$

Then $(1-\alpha)100\%$ confidence interval for σ in this case is obtained to be

$$\begin{aligned}\Bigg(&\frac{\beta(x_n-\theta)}{(1-\beta)^{-n} - 1 + \sqrt{(1-2\beta)^{-n} - (1-\beta)^{-2n}}\,\tau_{1-\alpha/2}}, \\ &\frac{\beta(x_n-\theta)}{(1-\beta)^{-n} - 1 + \sqrt{(1-2\beta)^{-n} - (1-\beta)^{-2n}}\,\tau_{\alpha/2}}\Bigg)\end{aligned} \tag{4.5}$$

where x_n is the $n-$th upper record and the percentage point τ_α is the solution of the nonlinear equation

$$\alpha\Gamma(n) = \Gamma\left(n, \frac{1}{\beta}\log\left[1 + \frac{\beta}{\sigma'}(\tau - \theta')\right]\right), \tag{4.6}$$

where θ' and σ' are given by (3.4) and $\Gamma(n,a)$ is the incomplete gamma function defined by

$$\Gamma(n,a) = \int_0^a x^{n-1}\exp(-x)dx\,.$$

4.3. Confidence intervals for θ and σ when β is known

In this case, the $(1-\alpha)100\%$ confidence interval for θ and σ are given, respectively, by

$$\left(x_1 - \frac{\tilde{\sigma}}{\beta}\left[(\alpha/2)^{-\beta} - 1\right],\ x_1 - \frac{\tilde{\sigma}}{\beta}\left[(1-\alpha/2)^{-\beta} - 1\right]\right), \tag{4.7}$$

and

$$\begin{aligned}\Bigg(&\frac{\beta(x_n - \tilde{\theta})}{(1-\beta)^{-n} - 1 + \sqrt{(1-2\beta)^{-n} - (1-\beta)^{-2n}}\tau_{1-\alpha/2}}, \\ &\frac{\beta(x_n - \tilde{\theta})}{(1-\beta)^{-n} - 1 + \sqrt{(1-2\beta)^{-n} - (1-\beta)^{-2n}}\tau_{\alpha/2}}\Bigg)\end{aligned} \tag{4.8}$$

where τ_α is the solution of the equation (4.6) and $\tilde{\theta}$ and $\tilde{\sigma}$ are given, respectively, by (2.29) and (2.30).

APPENDICES

A. Proof of Theorem 2. The pdf of the i-th record value from (1.1) can be written as

$$f_i(x) = \frac{1}{\sigma\Gamma(i)}\left[\frac{\log(1+\beta y)}{\beta}\right]^{i-1}(1+\beta y)^{-(1+1/\beta)},\ x > \theta \text{ for } \beta > 0, \quad \theta < x < \theta - \sigma/\beta \text{ for } \beta < 0, \tag{A.1}$$

where $y = (x-\theta)/\sigma$. From (A.1), it is easy to prove that

$$E\,(\log[1+\beta Y_i]) = i\beta, \tag{A.2}$$

$$E\left(\frac{1}{1+\beta Y_i}\right) = \frac{1}{(1+\beta)^i},\ \beta > -1, \tag{A.3}$$

and

$$E\left(\frac{1}{1+\beta Y_i}\right)^2 = \frac{1}{(1+2\beta)^i},\ \beta > -1/2. \tag{A.4}$$

From the likelihood equation given in (2.1), we may write

$$E\left(\frac{\partial^2 \log L}{\partial\sigma^2}\right) = \frac{1}{\sigma^2}E\left[\sum_{i=1}^{n}\frac{1}{(1+\beta Y_i)^2} + \frac{1}{\beta}\frac{1}{(1+\beta Y_n)^2} - \frac{1}{\beta}\right]. \tag{A.5}$$

By using (A.3) and (A.4) in (A.5), we get

$$E\left(\frac{\partial^2 \log L}{\partial \sigma^2}\right) = -\frac{1-(1+2\beta)^{-n}}{2\beta\sigma^2}\,, \tag{A.6}$$

hence

$$\mathrm{MVB}(\sigma) = \frac{2\beta\sigma^2}{1-(1+2\beta)^{-n}}\,, \tag{A.7}$$

and

$$\lim_{\beta\to 0}\mathrm{MVB}(\sigma) = \frac{\sigma^2}{n}\,, \tag{A.8}$$

which gives the bound in case of two parameters exponential distribution.

Also, from (A.6), we have

$$\lim_{n\to\infty}\mathrm{MVB}(\sigma) = \begin{cases} 2\beta\sigma^2, & \beta > 0, \\ 0, & \dfrac{-1}{2} < \beta \le 0\,. \end{cases} \tag{A.9}$$

B. Proof of Theorem 1. From the likelihood equation given in (2.1), we may write

$$\begin{aligned} E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) &= E\left[\frac{1}{\beta^2}\sum_{i=1}^{n}\left(1-\frac{1}{1+\beta Y_i}\right)^2 + \frac{1}{\beta^3}\left(1-\frac{1}{(1+\beta Y_n}\right)^2\right. \\ &\quad \left. + \frac{2}{\beta^3}\left(1-\frac{1}{1+\beta Y_n}\right) - \frac{2}{\beta^3}\log(1+\beta Y_n)\right]\,, \\ Y_i &= (X_i-\theta)/\sigma\,. \end{aligned} \tag{B.1}$$

By using (A.2), (A.3) and (A.4) in (B.1), we get

$$\begin{aligned} E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) &= \frac{1}{\beta^2}\sum_{i=1}^{n}\left(1-\frac{2}{(1+\beta)^i}+\frac{1}{(1+2\beta)^i}\right) \\ &\quad + \frac{1}{\beta^3}\left(1-\frac{2}{(1+\beta)^n}+\frac{1}{(1+2\beta)^n}\right) \\ &\quad + \frac{2}{\beta^3}\left(1-\frac{1}{(1+\beta)^n}\right) - \frac{2n}{\beta^2} \\ &= \frac{-1}{\beta^3}\left(n\beta - \frac{3}{2} + \frac{2}{(1+\beta)^n} - \frac{1}{2(1+2\beta)^n}\right)\,, \end{aligned} \tag{B.2}$$

hence for positive β, we get

$$\mathrm{MVB}(\beta) = \frac{\beta^3}{n\beta - \dfrac{3}{2} + 2(1+\beta)^{-n} - \dfrac{1}{2}(1+\beta)^{-n}}\,, \tag{B.3}$$

and

$$\lim_{\beta \to 0} \mathrm{MVB}(\beta) = \frac{3}{n(n+1)(n+2)}\,. \tag{B.4}$$

Also, from (B.3), we have

$$\lim_{n \to \infty} \mathrm{MVB}(\beta) = 0\,. \tag{B.5}$$

ACKNOWLEDGMENTS

The authors would like to thank the referees for their helpful comments, which improved the presentation of the paper. The second author would like to thank the Research Center, College of Science, King Saud University for funding the project (Stat/1422/27).

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NAGI S. ABD-EL-HAKIM
Department of Mathematics
El-Minia University
El-Minia (Egypt)

KHALAF S. SULTAN
Department of Statistics
and Operations Research
College of Science
King Saud University
P.O.Box 2455
Riyadh 11451 (Saudi Arabia)
ksultan@ksu.du.sa