King Saud University
College of Science
Department of Statistics and OR

Stat 559 (Theory of Statistics –II)
Second semester, 1428-1429
Time: 3 Hours, Final Exam

Try on the following problems

1. Suppose that $X_1,\ldots,X_n$ is a random sample from Poisson distribution unknown mean $\theta > 0$ and suppose that interest is in the quantity
$$\tau(\theta) = (1+\theta)e^{-\theta},$$

(a) Find the Cramer-Rao lower bound for the variance unbiased estimator $\tau(\theta)$.

(b) Show that $\hat{\tau}(\theta) = \begin{cases} 1, & X_1 \le 1, \\ 0, & X_1 \ge 2, \end{cases}$ is an unbiased estimate for $\tau(\theta)$.

(c) Find the minimum variance unbiased estimator UMVUE of $\tau(\theta)$.

(d) Find the maximum likelihood estimate of $\tau(\theta)$.

2. Suppose that $X_1,\ldots,X_n$ is a random sample from the exponential family $f(x,\theta) = a(\theta)b(x)\exp[c(\theta)d(x)]$, where $c(\theta)$ is a monotone function.

(a) Show that the Fisher information of the parameter θ is given
$$I(\theta) = \left[\frac{a'(\theta)}{a(\theta)}\right]^2 - \frac{a''(\theta)}{a(\theta)} + \frac{c''(\theta)a'(\theta)}{c'(\theta)a(\theta)}.$$

(b) Apply part (a) when X is distributed as $Gamma(5, \theta), \quad \theta > 0.$

(c) Show that $f(x,\theta)$ has a monotone likelihood ratio and hence find he uniformly most powerful test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$.

3. Let $X_1,\ldots,X_n$ be a random sample from $N(\theta,\theta^2)$ distribution and let $\overline{X}$ and S^2 be the usual mean and variance.

(a) Show that the pair $(\overline{X}, S^2)$ is a minimal sufficient statistic.

(b) Show that estimators of the form $a\overline{X} + (1-a)kS$, where S is the sample standard deviation and $k = \sqrt{\frac{n-1}{2}} \frac{\Gamma(n-1/2)}{\Gamma(n/2)}$ are unbiased for θ. Find the value of a which gives the minimum variance of all such estimators.

(c) Investigate the efficiency of the estimator found in (b) as a function of the sample size.

4. Let $x_1, \ldots, x_n$ is a random sample from $N(\theta_1, \theta_3)$ and $y_1, \ldots, y_m$ another random sample from $N(\theta_2, \theta_3)$, where θ_1 and θ_3 are unknown. Assume that the two samples are independent.

(a) Construct $(1-\alpha)100\%$ confidence interval for the difference $\theta_1 - \theta_2$ and justify the use of the tabulated value.

(b) Find the likelihood ratio test of size α for testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$.

5. Let $x_1, \ldots, x_n$ is a random sample from Poisson with mean λ and the prior distribution of λ is $Gamma(\alpha, \beta)$, where α and β are known parameters.

(a) Find the Bayesian estimate of λ^k , where k is a constant.

(b) Find the Bayesian test of $H_0 : \lambda \leq 2$ versus $H_0 : \lambda > 2$ when $\sum_{i=1}^{5} x_i = 7$, $\alpha = 5$ and $\beta = 1$.