RECORD VALUES FROM THE INVERSE WEIBULL LIFETIME MODEL: DIFFERENT METHODS OF ESTIMATION

By

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1 Records

Let $X_1,\ X_2,\ X_3,\ldots \rightarrow$ i.i.d from f(x) and F(x)
with a random sample size

- Upper records: $X_{U(1)},\ X_{U(2)},\ldots,X_{L(n)}$
- Lower records: $X_{L(1)},\ X_{L(2)},\ldots,X_{L(n)}$
- Basic Theory: For more details on the basic theory of records of both continuous and discrete record values, see Chandler (1952), Nagaraja (1988), Ahsanullah (1988), Arnold and Balakrishnan (1989), and Arnold, Balakrishnan and Nagaraja (1992, 1998).
- Applications: Sport, Economics, weather, life testing studies, ...

1.1 The *pdf* of the record values

- The joint density function of $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ is given by

$$f_{1,2,\ldots,n}(x_{U(1)}, x_{U(2)}, \ldots, x_{U(n)}) = f(x_{L(n)}) \prod_{i=1}^{n-1} \frac{f(x_{U(i)})}{1 - F(x_{U(i)})}.$$

- The *pdf* of the $X_{U(n)}$ is

$$f_n(x) = \frac{1}{\Gamma(n)} \{-\log[1 - F(x)]\}^{n-1} f(x), \quad -\infty < x < \infty, n = 1, 2, \ldots.$$

- The joint *pdf* of $X_{U(m)}$ and $X_{U(n)}$ is

$$\begin{aligned} f_{m,n}(x, y) &= \frac{1}{\Gamma(m)\Gamma(n-m)} \{-\log[1 - F(x)]\}^{m-1} \\ &\times \frac{f(x)}{1 - F(x)} \{-\log[1 - F(y)] + \log[1 - F(x)]\}^{n-m-1} \\ &\times f(y), \quad -\infty < x < y < \infty, \; m = 1, 2, \ldots, m < n. \end{aligned}$$

- The joint *pdf* of $X_{U(m)}$, $X_{U(n)}$ and $X_{U(\ell)}$ [see Sultan and Balakrishnan (1999)].

- The joint *pdf* of $X_{U(m)}$, $X_{U(n)}$, $X_{U(\ell)}$ and $X_{U(q)}$ [see Sultan and Balakrishnan (1999)].

- The joint density function of $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ is given by

$$f_{1,2,\ldots,n}(x_{L(1)}, x_{L(2)}, \ldots, x_{L(n)}) = f(x_{L(n)}) \prod_{i=1}^{n-1} \frac{f(x_{L(i)})}{F(x_{L(i)})}.$$

- The *pdf* of the $X_{L(n)}$ is

$$f_n(x) = \frac{1}{\Gamma(n)}\{-\log[F(x)]\}^{n-1} f(x),$$
$$-\infty < x < \infty,\ n = 1, 2, \ldots,$$

- The joint *pdf* of $X_{L(m)}$ and $X_{L(n)}$ is

$$\begin{aligned} f_{m,n}(x, y) &= \frac{1}{\Gamma(m)\Gamma(n-m)} \{-\log[F(x)]\}^{m-1} \\ &\times \{-\log[F(y)] + \log[F(x)]\}^{n-m-1} \frac{f(x)}{F(x)} f(y), \\ &\qquad -\infty < y < x < \infty,\ m = 1, 2, \ldots, m < n. \end{aligned}$$

- The joint *pdf* of $X_{L(m)}$, $X_{L(n)}$ and $X_{L(\ell)}$ [see Sultan and Balakrishnan (1999)].

- The joint *pdf* of $X_{L(m)}$, $X_{L(n)}$, $X_{L(\ell)}$ and $X_{L(q)}$ [see Sultan and Balakrishnan (1999)].

1.2 Inverse Weibull distribution (IW)

Let $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ be the first n lower record values from the IW pdf

$$f(x) = \beta x^{-\beta-1} e^{-x^{-\beta}}, \quad x \geq 0,\ \beta > 0,$$

and cdf

$$F(x) = e^{-x^{-\beta}}, \quad x \geq 0.$$

The relation between the pdf and cdf can be written as:

$$f(x) = \beta\{-\log F(x)\}F(x).$$

The location-scale IW distribution has its density function given by

$$f(y) = \frac{\beta}{\sigma}\left(\frac{\sigma}{y-\theta}\right)^{\beta+1} \exp\{-(\frac{\sigma}{y-\theta})^{\beta}\},\ y \geq \theta, \sigma > 0, \theta \geq 0.$$

Drapella (1993) calls the IW distribution as the complementary Weibull distribution, Jiag, Murthy and Ji (2001) have discussed some useful measures for the IW distribution.

The IW distribution plays an important role in many applications, including the dynamic components of diesel engines and several data set such as the times to breakdown of an insulating fluid subject to the action of a constant tension; see Nelson (1982). Calabria and Pulcini (1990) provide an interpretation of the IW distribution in the context of the load-strength relationship for a component. Recently, Maswadah (2003) has the fitted IW distribution to the flood data. For more details on the IW distribution see for example Murthy, Xie and Jiang (2004).

1.3 Moments of record values from IW

- The single moment of $X_{L(n)}$ is

$$\begin{aligned} \mu_m^{(i)} &= E(X_{L(m)}^i) \\ &= \frac{\Gamma(m - \frac{i}{\beta})}{\Gamma(m)}, \; i < m\beta, \; n = 1, 2, \ldots . \end{aligned}$$

  To develop the Edgeworth approximation we need $\mu_n^{(1)}$, $\mu_n^{(2)}$, $\mu_n^{(3)}$ and $\mu_n^{(4)}$.

- The double moment of $X_{L(m)}$ and $X_{L(n)}$ $(m < n)$ is

$$\begin{aligned} \mu_{m,n}^{(i,j)} &= E(X_{L(m)}^i X_{L(n)}^j) \\ &= \frac{\Gamma(m - \frac{i}{\beta})\Gamma(n - \frac{i+j}{\beta})}{\Gamma(m)\Gamma(n - \frac{i}{\beta})}, \; i + j < n\beta, \end{aligned}$$

  Similarly, Edgeworth approximation needs $\mu_{m,n}^{(1,1)}$, $\mu_{m,n}^{(1,2)}$, $\mu_{m,n}^{(2,1)}$, $\mu_{m,n}^{(3,1)}$, $\mu_{m,n}^{(1,3)}$ and $\mu_{m,n}^{(2,2)}$.

- The triple moments of the m-th, n-th and p-th lower record values form IW distribution as

$$\begin{aligned}\mu_{m,n,p}^{(i,j,k)} &= E(X_{L(m)}^{i} X_{L(n)}^{j} X_{L(p)}^{k}) \\ &= \frac{\Gamma(m-\frac{i}{\beta})\Gamma(n-\frac{i+j}{\beta})\Gamma(p-\frac{i+j+k}{\beta})}{\Gamma(m)\Gamma(n-\frac{i}{\beta})\Gamma(p-\frac{i+j}{\beta})}, i+j+k<p\beta.\end{aligned}$$

  Edgeworth approximation needs
  $\mu_{m,n,p}^{(1,1,1)}$, $\mu_{m,n,p}^{(1,1,2)}$, $\mu_{m,n,p}^{(1,2,1)}$ and $\mu_{m,n,p}^{(2,1,1)}$.

- The required quadruple moment of lower record values

$$\begin{aligned}\mu_{m,n,p,q}^{(i,j,k,l)} &= E(X_{L(m)}^{i} X_{L(n)}^{j} X_{L(p)}^{k} X_{L(q)}^{l}) \\ &= \frac{\Gamma(m-\frac{i}{\beta})\Gamma(n-\frac{i+j}{\beta})\Gamma(p-\frac{i+j+k}{\beta})\Gamma(q-\frac{i+j+k+l}{\beta})}{\Gamma(m)\Gamma(n-\frac{i}{\beta})\Gamma(p-\frac{i+j}{\beta})\Gamma(q-\frac{i+j+k}{\beta})}, \\ &\qquad i+j+k+l<q\beta.\end{aligned}$$

  In this case Edgeworth approximation needs only $\mu_{m,n,p,q}^{(1,1,1,1)}$.

2 BLUEs of the location and scale parameters

Let $Y_{L(1)} \leq Y_{L(2)} \leq \ldots \leq Y_{L(n)}$ be the upper record values from f(y), and let $X_{U(i)} = \left(Y_{U(i)} - \theta\right)/\sigma,\ i = 1, \ldots, n$, be the corresponding record values from the standard form. Let us denote $E(X_{U(i)})$ by μ_i, $Var(X_{U(i)})$ by $\sigma_{i,i}$, and $Cov(X_{U(i)}, X_{U(j)})$ by $\sigma_{i,j}$; further, let

$$\begin{aligned} \boldsymbol{Y} &= \left(Y_{L(1)}, Y_{L(2)}, \ldots, Y_{L(n)}\right)^T, \\ \boldsymbol{\mu} &= (\mu_1, \mu_2, \ldots, \mu_n)^T, \\ \mathbf{1} &= \underbrace{(1, 1, \ldots, 1)^T}_{n}, \\ \text{and} \quad \Sigma &= ((\sigma_{i,j})),\ 1 \leq i,\ j \leq n. \end{aligned}$$

Then, the BLUE's of θ and σ are given by [see Balakrishnan and Cohen (1991)]

$$\theta^* = \left\{ \frac{\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^{\mathbf{2}}} \right\} \boldsymbol{Y} = \sum_{i=1}^{n} A_i Y_{L(i)},$$

and

$$\sigma^* = \left\{ \frac{\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1} - \mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^{\mathbf{2}}} \right\} \boldsymbol{Y} = \sum_{i=1}^{n} B_i Y_{L(i)}.$$

Furthermore, the variances and covariance of these BLUE's are given by [see Balakrishnan and Cohen (1991)]

$$Var(\theta^*) = \sigma^2 \left\{ \frac{\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{(\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^{\mathbf{2}}} \right\} = \sigma^2 V_1,$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^{\mathbf{2}}} \right\} = \sigma^2 V_2,$$

and

$$Cov(\theta^*, \sigma^*) = \sigma^2 \left\{ \frac{-\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{(\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1})^{\mathbf{2}}} \right\} = \sigma^2 V_3;$$

In our model (IW), the $(i, j)-th$ element of Σ can be written as $p_i q_j$, $i \leq j$, where

$$p_i = \frac{\Gamma(i - \frac{1}{\beta})}{\Gamma(i)} \text{ and } q_j = \frac{\Gamma(j - \frac{2}{\beta})}{\Gamma(j - \frac{1}{\beta})} - \frac{\Gamma(j - \frac{1}{\beta})}{\Gamma(j)}.$$

Then, Σ^{-1} can be derived as [using the Lemma in Graybill (1983), Arnold, Blakrishnan and Nagaraja (1992)].

$$
\Sigma^{-1} = \begin{cases} -\frac{\beta^2\Gamma(i+1)}{\Gamma(i-\frac{2}{\beta})}(i-\frac{1}{\beta}), & j=i+1, \\ \frac{\beta^2\Gamma(i)}{\Gamma(i-\frac{2}{\beta})}\{(i-\frac{1}{\beta})^2+(i-1)(i-\frac{2}{\beta}-1)\}, & i=j=2 \text{ to } n-1, \\ \frac{\beta^2(1-\frac{1}{\beta})^2}{\Gamma(1-\frac{2}{\beta})}, & i=j=1, \\ \frac{q_n}{q_{n-1}}\frac{\beta^2\Gamma(n)}{\Gamma(n-\frac{2}{\beta}-1)}(n-\frac{1}{\beta}-1), & i=j=n, \\ 0, & j>i+1. \end{cases}
$$

Table 1 represents the coefficients of the BLUEs A_i and B_i for records of sizes 4, 5, 6, 7 and the shape parameter $\beta = 3$,4 and 5.

The variance and covariances of the BLUEs are also calculated in Table 2.

Table 1: The Coefficients of the BLUEs

	$\beta=3$		$\beta=4$		$\beta=5$	
n	A_i	B_i	A_i	B_i	A_i	B_i
4	-0.2353	0.3519	-0.5294	0.7182	-0.8421	1.0764
	-1.4118	2.1112	-1.7647	2.3939	-2.1053	2.6909
	-2.1176	3.1668	-2.3529	3.1919	-2.6316	3.3636
	4.7647	-5.6299	5.6471	-6.3040	6.5789	-7.1309
5	-0.1400	0.2284	-0.3383	0.4896	-0.5591	0.7523
	-0.8400	1.3704	-1.1278	1.6319	-1.3978	1.8807
	-1.2600	2.0555	-1.5038	2.1759	-1.7473	2.3509
	-1.6200	2.6428	-1.8045	2.6111	-2.0161	2.7126
	4.8600	-6.2972	5.7744	-6.9085	6.7204	-7.6966
6	-0.0942	0.1647	-0.2395	0.3649	-0.4071	0.5706
	-0.5653	0.9881	-0.7985	1.2162	-1.0178	1.4265
	-0.8479	1.4821	-1.0646	1.6216	-1.2723	1.7831
	-1.0902	1.9055	-1.2776	1.9459	-1.4680	2.0575
	-1.3082	2.2866	-1.4601	2.2239	-1.6311	2.2861
	4.9058	-6.8270	5.8403	-7.3725	6.7964	-8.1238
7	-0.0684	0.1266	-0.1809	0.2875	-0.3143	0.4556
	-0.4104	0.7595	-0.6029	0.9582	-0.7857	1.1391
	-0.6156	1.1393	-0.8038	1.2776	-0.9821	1.4239
	-0.7915	1.4648	-0.9646	1.5331	-1.1332	1.6429
	-0.9498	1.7578	-1.1024	1.7521	-1.2591	1.8255
	-1.0959	2.0282	-1.2249	1.9468	-1.3686	1.9842
	4.9316	-7.2763	5.8794	-7.7552	6.8429	-8.4712

As a check, the entries of Table 1 stratify the identities

$$\sum_{i=1}^{n} A_i = 1 \text{ and } \sum_{i=1}^{n} B_i = 0.$$

Table 2: The variances and covariances of the BLUEs

β	n	$Var(\theta^*)$	$Var(\sigma^*)$	$Cov(\theta^*, \sigma^*)$
3	4	0.3152	0.6834	-0.4713
3	5	0.1875	0.4901	-0.3059
3	6	0.1262	0.3803	-0.2206
3	7	0.0916	0.3103	-0.1696
4	4	0.3128	0.5646	-0.4243
4	5	0.1999	0.4138	-0.2893
4	6	0.1415	0.3255	-0.2156
4	7	0.1069	0.2680	-0.1698
5	4	0.3135	0.5055	-0.4007
5	5	0.2082	0.3739	-0.2801
5	6	0.1516	0.2960	-0.2124
5	7	0.1170	0.2447	-0.1696

2.1 Edgeworth Approximate Inference

By using the moments of record values, we construct some confidence intervals of the location and scale parameters θ and σ of the IW distribution based on the following pivotal quantities:

$$
\begin{aligned}
R_1 &= \frac{\theta^* - \theta}{\sigma\sqrt{V_1}}, \\
R_2 &= \frac{\sigma^* - \sigma}{\sigma\sqrt{V_2}}, \\
R_3 &= \frac{\theta^* - \theta}{\sigma^*\sqrt{V_1}},
\end{aligned}
$$

where θ^* and σ^* are the BLUEs of θ and σ with variances $\sigma^2 V_1$ and $\sigma^2 V_2$, respectively. R_1 can be used to draw inferences on θ when σ is known, while R_3 can be used to draw inference on θ when σ is unknown. Similarly, R_2 can be used to draw inference for σ when θ is unknown.

Notice that R_1 and R_2 can be rewritten as

$$
\begin{aligned}
R_1 &= \frac{1}{\sqrt{V_1}}\left(\sum_{i=1}^{n} A_i X_{L(i)}\right) = \frac{R_1^*}{\sqrt{V_1}}, \\
R_2 &= \frac{1}{\sqrt{V_2}}\left(\sum_{i=1}^{n} B_i X_{L(i)} - 1\right) = \frac{R_2^* - 1}{\sqrt{V_2}},
\end{aligned}
$$

where $X_{L(i)} = (Y_{L(i)} - \theta)/\sigma,\ i = 1, 2, \ldots, n.$

Thus, they are linear functions of record values arising from the one parameter IW distribution.

Since the distribution of a linear function of record values will in general not be known, we consider finding the approximate distribution by using Edgeworth approximation for a statistic T (with mean 0 and variance 1) given by [see Johnson, Kotz and Balakrishnan (1994)]

$$G(t) \approx \Phi(t) \; - \; \phi(t)\Big\{\frac{\sqrt{\beta_1}}{6}(t^2-1) \\ + \; \frac{\beta_2-3}{24}(t^3-3t)+\frac{\beta_1}{72}(t^5-10t^3+15t)\Big\},$$

where $\sqrt{\beta_1}$ and β_2 are the coefficients of skewness and kurtosis of T, respectively, and $\Phi(t)$ is the *cdf* of the standard normal distribution with corresponding *pdf* $\phi(t)$.

By making use of the exact expressions of moments presented in Section 2, and the BLUEs A_i and B_i, we determined the values of the mean, variance and the coefficients of skewness and kurtosis ($\sqrt{\beta_1}$ and β_2) of R_1^* and R_2^*, for $n = 4(1)7$ and $\beta = 5$. Notice that Edgeworth approximate is valid only when $\beta > 4$, that is because of the conditions on the quadruple moments.

The coefficients of skewness and kurtosis of R_1^* are given in the following Lemma

Lemma 1

$$\begin{aligned}
\sqrt{\beta_1}(R_1^*) &= \frac{L_3 - 3L_2L_1 - 2L_1^2}{(L_2 - L_1^2)^{3/2}}, \\
\beta_2(R_1^*) &= \frac{L_4 - 3L_1^4 + 6L_2L_1^2 - 4L_1L_3}{(L_2 - L_1^2)^2},
\end{aligned}$$

where

$$\begin{aligned}
L_1 &= E(R_1^*) = \sum_{i=1}^{n} A_i \mu_{i:n}^{(1)}, \\
L_2 &= E(R_1^*)^2 = \sum_{i=1}^{n} A_i^2 \mu_{i:n}^{(2)} + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} A_i A_j \mu_{i,j:n}^{(1,1)}, \\
L_3 &= E(R_1^*)^3 = \sum_{i=1}^{n} A_i^3 \mu_{i:n}^{(3)} + 3 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} A_i^2 A_j \mu_{i,j:n}^{(2,1)} \\
&+ 3 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} A_i A_j^2 \mu_{i,j:n}^{(1,2)} + 6 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} A_i A_j A_k \mu_{i,j,k:n}^{(1,1,1)}, \\
L_4 &= E(R_1^*)^4 = \sum_{i=1}^{n} A_i^4 \mu_{i:n}^{(4)} + 4 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} A_i^3 A_j \mu_{i,j:n}^{(3,1)} \\
&+ 4 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} A_i A_j^3 \mu_{i,j:n}^{(1,3)} + 6 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} A_i A_j \mu_{i,j:n}^{(2,2)}, \\
&+ 12 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} A_i^2 A_j A_k \mu_{i,j,k:n}^{(2,1,1)} \\
&+ 12 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} A_i A_j^2 A_k \mu_{i,j,k:n}^{(1,2,1)} \\
&+ 12 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} A_i A_j A_k^2 \mu_{i,j,k:n}^{(1,1,2)} \\
&+ 24 \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} A_i A_j A_k A_l \mu_{i,j,k,l:n}^{(1,1,1,1)}.
\end{aligned}$$

The coefficients of skewness and kurtosis for $R_2^* = \Sigma_{i=1}^{n} B_i Z_{i:n}$ can be obtained following steps similar to those in R_1^* and replacing A_i by B_i.

Table 3 displays the values of the mean, variance and the coefficients of skewness and kurtosis ($\sqrt{\beta_1}$ and β_2) of R_1^* and R_2^*.

Table 3: Mean, Variance and Coefficients of Skewness and Kurtosis of R_1^* and R_2^* when β, θ and σ are 5.0, 0.0 and 1.0

	R_1^*				R_2^*			
n	Mean	V_1	$\sqrt{\beta_1}$	β_2	Mean	V_2	$\sqrt{\beta_1}$	β_2
4	.000	.314	-2.081	13.305	1.000	.505	2.115	13.305
5	.000	.208	-1.720	9.543	1.000	.374	1.751	9.575
6	.000	.152	-1.494	7.716	1.000	.296	1.524	7.756
7	.000	.117	-1.337	6.662	1.000	.245	1.365	6.702

An examination of the ($\sqrt{\beta_1}, \beta_2$) values in Table 3 reveals that the distribution of R_2^* (and hence of R_2) is positively skewed, while the distribution of R_1^* (and hence of R_1) is negatively skewed. In addition, $\sqrt{\beta_1}$ for R_1 increases as n decreases while $\sqrt{\beta_1}$ for R_2 decreases as n increases. Also, the coefficient of kurtosis β_2 of both R_1^* and R_2^* decrease as n increases. Further, β_2 of both R_1^* and R_2^* are almost equal.

By making use of the entries in Table 3, we determined the lower and upper 1%, 2.5%, 5% and 10% points of R_1 and R_2 through the Edgeworth approximation. These values, for $n = 4(1)7$ and $\beta = 5$ are presented in Tables 4 and 5. For the purpose of comparison, these percentage points were also determined by Monte Carlo simulations (based on 10001 runs) and they are presented along with the Edgeworth percentage points in Tables 4 and 5 when $\beta = 5$.

From Tables 4 and 5, we see that the Edgeworth approximation of the distribution of R_1 and R_2 are in close agreement with the simulated percentage points in most of the cases.

Table 4: [Edgeworth] Approximate Values and Simulated values of Percentage Points of R_1 when θ and σ are 0.0 and 1.0

β	n	1%	2.5%	5%	10%	90%	95%	97.5%	99%
5	4	[-3.421]	[-3.190]	[-2.762]	[-1.647]	[.871]	[.974]	[1.714]	[2.461]
		-3.473	-2.524	-1.856	-1.239	.955	1.084	1.177	1.287
	5	[-3.320]	[-3.060]	[-2.691]	[-.906]	[.967]	[1.096]	[2.173]	[2.639]
		-3.319	-2.410	-1.830	-1.238	1.010	1.156	1.284	1.403
	6	[-3.297]	[-3.041]	[-2.315]	[-.862]	[1.022]	[1.171]	[2.625]	[2.889]
		-3.297	-2.488	-1.775	-1.214	1.072	1.238	1.364	1.484
	7	[-3.198]	[-2.951]	[-2.055]	[-.750]	[1.058]	[1.222]	[3.329]	[3.409]
		-3.109	-2.245	-1.736	-1.170	1.123	1.303	1.423	1.569

Table 5: [Edgeworth] Approximate Values and (Simulated values) of Percentage Points of R_2 when θ and σ are 0.0 and 1.0

β	n	1%	2.5%	5%	10%	90%	95%	97.5%	99%
5	4	[-2.876]	[-1.731]	[-.980]	[-.879]	[1.961]	[2.773]	[3.660]	[3.887]
		-1.541	-1.665	-1.084	-.965	1.261	1.904	2.570	3.547
	5	[-2.652]	[-1.174]	[-1.098]	[-.972]	[1.317]	[2.699]	[3.251]	[3.728]
		-1.503	-1.471	-1.167	-1.027	1.258	1.866	2.501	3.445
	6	[-2.593]	[-1.161]	[-1.171]	[-1.025]	[1.169]	[2.318]	[3.101]	[3.607]
		-1.373	-1.372	-1.263	-1.098	1.206	1.811	2.423	3.362
	7	[-1.402]	[-1.324]	[-1.221]	[-1.059]	[1.154]	[2.052]	[2.962]	[3.509]
		-1.278	-1.247	-1.332	-1.158	1.187	1.760	2.301	3.172

Table 6: Simulated Percentage Points of R_3 when θ and σ are 0.0 and 1.0

β	n	1%	2.5%	5%	10%	90%	95%	97.5%	99%
5.0	4	-.997	-.901	-.795	-.661	3.044	4.675	6.640	9.937
	5	-1.087	-.974	-.860	-.704	2.674	3.997	5.478	8.165
	6	-1.172	-1.054	-.925	-.750	2.631	3.945	5.174	7.121
	7	-1.217	-1.065	-.934	-.762	2.586	3.729	4.940	6.824

In conclusion, we observe that the Edgeworth approximations of the distributions of R_1 and R_2 both work quite satisfactorily; this is also clear from the probability coverages and the average width of the confidence intervals based on R_1 and R_2 which are presented in Tables 8 and 8, respectively.

Table 7: Probability Coverages of C.I.'s Based on R_1 and R_2 Using Edgeworth Percentage Points when $\theta = 0$ and $\sigma = 1$

	R_1		R_2		R_1 (using $\sigma = \sigma^*$)	
n	95%	90%	95%	90%	95%	90%
4	.8940	.8526	.9452	.8487	.8843	.6850
5	.9421	.9133	.9514	.9053	.7421	.7276
6	.9217	.9012	.9358	.8927	.7759	.7345
7	.9445	.8969	.9345	.8864	.7534	.7321

Table 8: Average Width of the Simulated and [Edgeworth] C.I.'s Based on R_1 and R_2 when $\theta = 0.0$ and $\sigma = 1.0$

		R_1 (Simulated)		R_2 (Simulated)		R_1 (using $\sigma = \sigma^*$) (simulated using R_3)	
c	n	95%	90%	95%	90%	95%	90%
5	4	[1.531]	[2.186]	[2.865]	[1.614]	[1.536]	[2.192]
		1.646	2.072	3.948	5.484	3.071	4.233
	5	[1.428]	[2.023]	[2.660]	[3.532]	[1.724]	[2.017]
		1.363	1.686	3.016	4.081	2.211	2.936
	6	[1.357]	[1.932]	[2.262]	[2.754]	[1.327]	[2.003]
		1.212	1.500	2.648	3.445	1.855	2.372
	7	[1.121]	[1.464]	[1.928]	[2.372]	[1.066]	[1.392]
		1.040	1.255	2.279	2.898	1.517	1.953

It should also be pointed out here that a similar Edgeworth approximation can not be developed for the percentage points of the pivotal quantity R_3 since it is not a linear function of record values. However, as displayed in Tables 7 and 8, we do not recommend drawing approximate inference based on R_1 with σ replaced by σ^* since it does not provide close results to those based on R_3. For this purpose, we have presented in Table 6 some selected percentage points of R_3 determined by Monte Carlo simulations (based on 10001 runs).

3 Example

In order to illustrate the usefulness of the inference procedures discussed in the previous sections, we consider here simulated data sets of size $n = 4, 5, 6$ and 7 (with $\theta = 0.0,\ \sigma = 1.0$). The BLUEs were calculated by making use of the entries in Table 1. The observed record values and the estimates obtained are presented in the following table:

β	n	Records	θ^*	σ^*
5	4	1.176, .975, .794, .780	.000854	.996102
	5	1.257, .983, .798, .754, .743	.002342	.996655
	6	.984, .882, .862, .850, .739, .714	.001434	.998260
	7	1.012, .971, .848, .830, .709, .702, .688	-.002229	1.000068

With these estimates and the use of Tables 2 and 4, we can determine the confidence intervals for θ (when σ is known to be

1.0) based on the Edgeworth approximation as well as using the simulated percentage points, based on the pivotal quantity R_1 through the formula

$$P\left(\theta^* - \sigma\sqrt{V_1}\,(R_1)_{1-\alpha/2} \leq \theta \leq \theta^* - \sigma\sqrt{V_1}\,(R_1)_{\alpha/2}\right) = 1 - \alpha.$$

For example, when $n = 7$ and $\beta = 5$, we have 90% C.I's of θ as

Edgeworth	Simulated
(-0.420 , 0.701)	(-0.448 , 0.592)

It is clear that the confidence intervals based on the Edgeworth approximation and those determined by simulation are quite close to those determined through the exact probabilities.

Similarly, with the use of Tables 2 and 5, we determined the confidence intervals for σ, through the formula

$$P\left(\frac{\sigma^*}{1+\sqrt{V_2}\,(R_2)_{1-\alpha/2}} \leq \sigma \leq \frac{\sigma^*}{1+\sqrt{V_2}\,(R_2)_{\alpha/2}}\right) = 1 - \alpha.$$

For example, when $n = 7$ and $\beta = 5$, we have 90% C.I's of σ as

Edgeworth	Simulated
(0.496 , 2.528)	(0.534, 2.528)

Once again, we observe that the confidence intervals based on the Edgeworth approximation are somewhat close to those based on the exact results except for the small sample size $m = 3$.

In the case when σ is unknown, the Edgeworth approximation method can not be used to draw inference for θ using R_3.

So, we computed the confidence intervals for θ based on the simulated percentage points of the pivotal quantity R_3 (given in Table 6) through the formula

$$P\left(\theta^* - \sigma^*\sqrt{V_1}\,(R_3)_{1-\alpha/2} \le \theta \le \theta^* - \sigma^*\sqrt{V_1}\,(R_3)_{\alpha/2}\right) = 1-\alpha,$$

For example, when $n = 7$ and $\beta = 5$, we have 90% C.I's of θ when $\sigma^* = 1.000068$ as

$$(-0.448, 1.040)$$

As we can see from all the above tables, all confidence intervals become narrower as n increases.

4 Maximum Likelihood Estimates

Let $x = (x_{L(1)}, x_{L(2)}, \ldots, x_{L(n)})$ be the first n lower record values from the IW distribution

1. pdf

$$f(x) = \alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}},\ x \geq 0,\ \beta, \alpha > 0,$$

2. cdf

$$F(x) = e^{-\alpha x^{-\beta}}, \quad x \geq 0,\ \beta, \alpha > 0.$$

3. Reliability function

$$R(x) = 1 - e^{-\alpha x^{-\beta}}, \quad x \geq 0,\ \beta, \alpha > 0,$$

4. Hazard function

$$H(x) = \frac{\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}}}{1 - e^{-\alpha x^{-\beta}}},\ x \geq 0,\ \beta, \alpha > 0,$$

5. The likelihood function of the first n lower record values $x_{L(1)}, x_{L(2)}, \ldots, x_{L(n)}$ is given by

$$\begin{aligned} L_(\alpha, \beta) &= f(x_{L(n)}) \prod_{i=1}^{n-1} \frac{f(x_{L(i)})}{F(x_{L(i)})} \\ &= \alpha^n \beta^n \exp\left(-\alpha x_{L(n)}^{-\beta}\right) \prod_{i=1}^{n} x_{L(i)}^{-\beta-1}. \end{aligned}$$

- When α unknown and β known:

$$\hat{\alpha} = n x_{L(n)}^{\beta}.$$

- When both of α and β are unknown:

$$\hat{\beta} \;=\; n\left(\sum_{i=1}^{n} \log x_{L(i)} - n \log x_{L(n)}\right)^{-1},$$

and

$$\hat{\alpha} \;=\; n x_{L(n)}^{\beta}.$$

- The MLEs of the reliability and hazard functions $\hat{R}(.)_{ML}$ and $\hat{H}(.)_{ML}$ can be obtained by replacing α and β by their MLEs.

5 Bayes Estimates of One-parameter case

5.1 Estimates based on squared error loss function

Let the conjugate prior of α is

$$g(\alpha) = \frac{b^a \alpha^{a-1}}{\Gamma(a)} \exp\left(-b\alpha\right),\; \alpha > 0,\; a, b > 0.$$

The posterior pdf of α is

$$\begin{aligned}\pi\left(\alpha \mid x\right) &= \frac{L(\alpha; x) g(\alpha)}{\int_0^\infty L(\alpha; x) g(\alpha) d\alpha} \\ &= \frac{1}{\Gamma(n+a)} v^{n+a} \alpha^{n+a-1} \exp\left(-\alpha v\right),\end{aligned}$$

where $x = (x_{L(1)}, x_{L(2)}, \ldots, x_{L(n)})$ and

$$v = b + x_{L(n)}^{-\beta}.$$

The Bayes estimates of α, $R(t)$ and $H(t)$ based on the squared error loss function are given, respectively, as

$$\begin{aligned}
\hat{\alpha}_{BS} &= E(\alpha \mid x) = \frac{a+n}{v}, \\
\hat{R}(t)_{BS} &= E\left([1-\exp(-\alpha t^{-\beta})] \mid x\right) \\
&= 1-\left(\frac{v}{v+t^{-\beta}}\right)^{-(n+a)}, \; t>0, \\
\hat{H}(t)_{BS} &= E\left(\frac{\alpha\beta t^{-\beta-1}\exp(-\alpha t^{-\beta})}{1-\exp(-\alpha t^{-\beta})} \mid x\right) \\
&= \beta(n+a)v^{n+a}t^{\beta(n+a)-1}\zeta\left(n+a+1, 1+v/t^{-\beta}\right),
\end{aligned}$$

where $\zeta(p,q)$ is the generalized Rieman's zeta function given by [see Gradshteyn and Ryzhik (1980), pp. 1072]

$$\zeta(p,q) = \sum_{i=0}^{\infty}(q+i)^{-p}, \; p>0, \; q \neq 0, -1, -2, \ldots .$$

5.2 Estimates based on Linex loss function

The Bayes estimates of α, $R(t)$ and $H(t)$ based on the Linex loss function are given, respectively, as

$$
\begin{aligned}
\hat{\alpha}_{BL} &= \frac{-1}{c}\log\left(E\left[\exp(-c\alpha)\mid x\right]\right) \\
&= \frac{n+a}{c}\log\left(\frac{c+v}{v}\right),
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}(t)_{BL} &= \frac{-1}{c}\log\Big(E\Big[-c\exp[1-\exp(-\alpha t^{-\beta})]\mid x\Big]\Big) \\
&= \frac{-1}{c}\log\Big[v^{n+a}\exp(-c)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^j}{i!}\binom{i}{j}v^j c^{i-j} \\
&\times \Big[(i-j)t^{-\beta}\Big]^{-n-a}\Big], \quad t>0,
\end{aligned}
$$

$$
\begin{aligned}
\hat{H}(t)_{BL} &= \frac{-1}{c}E\left(\exp\left[\frac{\alpha\beta t^{-\beta-1}\exp(-\alpha t^{-\beta})}{1-\exp(-\alpha t^{-\beta})}\right]\mid x\right) \\
&= \frac{-1}{c}\log\Big[\frac{v^{n+a}}{\Gamma(n+a)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^i}{i!}\binom{i+j-1}{j} \\
&\times \frac{(c\beta t^{-\beta-1})^i\Gamma(n+a+i)}{[(i+j)t^{-\beta}+v]^{n+a+i}}\Big], \ t>0.
\end{aligned}
$$

Table 9: MSEs of the estimates of α, $R(t)$ and $H(t)$ when $t = 5$

n	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{BL}$ $c = 1.0$	$\hat{R}_{ML}$	$\hat{R}_{BS}$	$\hat{R}_{BL}$ $c = 0.0$	$\hat{H}_{ML}$	$\hat{H}_{BS}$	$\hat{H}_{BL}$ $c = 1.5$
3	3.946	.456	.197	.318	.323	.323	.000025	.000003	.000003
4	1.826	.418	.205	.316	.318	.318	000011	.000002	.000002
5	1.058	.367	.196	.315	.317	.317	.000006	.000002	.000002
6	.722	.320	.185	312	.313	.313	.000004	.000002	.000002
7	.539	.280	.169	.310	.311	.311	.000003	.000002	.000002

6 Bayes Estimates of the two-parameter case

Suppose that β is restricted to the values $\beta_1, \beta_2, \ldots, \beta_k$ with prior probabilities $\ell_1, \ell_2, \ldots, \ell_k$, that is

$$p(\beta) = Pr(\beta = \beta_j) = \ell_j,\ j = 1, 2, \ldots, k.$$

Further, suppose that the conditional α upon $\beta = \beta_j,\ j = 1, 2, \ldots, k$ has natural conjugate prior as $gamma(a_j, b_j)$ with pdf

$$\pi(\alpha \mid \beta = \beta_j) = \frac{b_j^{a_j} \alpha^{a_j - 1}}{\Gamma(a_j)} \exp(-b_j \alpha),\ \alpha > 0,\ a_j,\ b_j > 0.$$

Then the conditional posterior of $\alpha|\beta = \beta_j$ is

$$\begin{aligned} \pi^* \left(\alpha|\beta = \beta_j, x\right) &= \frac{L(\alpha; x)\pi(\alpha|\beta = \beta_j)}{\int_0^\infty L(\alpha; x)\pi(\alpha|\beta = \beta_j) d\alpha} \\ &= \frac{1}{\Gamma(n + a)} v_j^{n + a_j} \alpha^{n + a_j - 1} \exp\left(-\alpha v_j\right), \end{aligned}$$

where

$$v_j = b_j + x_{L(n)}^{-\beta}.$$

In view of the discrete version of Bayes theorem, we obtain the marginal posterior of β as

$$\begin{aligned} h(\beta|x) = \pi_j &= Pr\left(\beta = \beta_j|x\right) \\ &\propto \int_\alpha \pi^*(\alpha|\beta = \beta_j) p(\beta) L(\alpha; x) d\alpha, \end{aligned}$$

hence we get

$$h(\beta|x) = \pi_j = \frac{b_j^{a_j}\beta_j^n u_j \ell_j \Gamma(a_j+n)}{Q\Gamma(a_j)v_j^{n+a+j}},$$

where

$$\begin{aligned} Q &= \sum_{j=1}^{k} \frac{b_j^{a_j}\beta_j^n u_j \ell_j \Gamma(a_j+n)}{Q\Gamma(a_j)v_j^{n+a+j}}, \\ u_j &= \prod_{i=1}^{n} x_{L(i)}^{-\beta_j-1}. \end{aligned} \tag{6.1}$$

6.1 Estimates based on squared error loss function

The Bayes estimates of α, β, $R(t)$ and $H(t)$ based on the square error loss function are given, respectively, as

$$\begin{aligned}
\hat{\alpha}_{BS} &= \int_{\alpha} \sum_{j=1}^{k} \pi_j \alpha \pi^*(\alpha|\beta=\beta_j) d\alpha = \sum_{j=1}^{k} \frac{\pi_j(n+a_j)}{v_j}, \\
\hat{\beta}_{BS} &= E(\beta) = \sum_{j=1}^{k} \pi_j \beta_j, \\
\hat{R}(t)_{BS} &= \sum_{j=1}^{k} \pi_j \int_{\alpha} \left(1 - \exp(-\alpha t^{-\beta_j})\right) \pi^*(\alpha|\beta=\beta_j) d\alpha \\
&= 1 - \sum_{j=1}^{k} \pi_j \left(\frac{v_j}{v_j + t^{-\beta_j}}\right)^{n+a_j}, \\
\hat{H}(t)_{BS} &= \sum_{j=1}^{k} \pi_j \int_{0}^{\infty} H(t) \pi^*(\alpha|\beta=\beta_j) \\
&= \sum_{k=1}^{k} \pi_j \beta_j (n+a_j+1) v_j^{n+a_j} t^{\beta_j(n+a_j)-1} \\
&\times \ \zeta\left(n+a_j+1, 1+v_j/t^{-\beta_j}\right).
\end{aligned}$$

6.2 Estimates based on linex loss function

The Bayes estimate of a function $\psi(\alpha,\beta)$ based on linex loss function is given by

$$\begin{aligned}\hat{\psi}(\alpha,\beta) &= \frac{-1}{c}\log\left[E\left(\exp\{-c\psi(\alpha,\beta)\}\right)\right],\\ &= \frac{-1}{c}\log\left[\sum_{j=1}^{k}\pi_j\int_0^{\infty}\exp\{-c\psi(\alpha,\beta)\}\pi^*(\alpha|\beta=\beta_j)d\alpha\right],\end{aligned}$$

The Bayes estimates of α, β, $R(t)$ and $H(t)$ based on the Linex loss function are given, respectively, as

$$\begin{aligned}\hat{\alpha}_{BL} &= \frac{-1}{c}\log\left[\sum_{j=1}^{k}\frac{\pi_j v_j^{a_j+n}}{(c+v_j)^{a_j+n}}\right],\\ \hat{\beta}_{BL} &= \frac{-1}{c}\log\left[\sum_{j=1}^{k}\pi_j\exp\{-c\beta_j\}\right],\\ \hat{R}(t)_{BL} &= \frac{-1}{c}\log\left[\sum_{j=1}^{k}\sum_{i=0}^{\infty}\sum_{\ell=0}^{i}\binom{i}{\ell}\frac{\pi_j(-1)^{\ell}c^{i-\ell}v_j^{n+a_j+\ell}\exp(-c)}{i!\left[(i-\ell)t^{-\beta_j}\right]^{a_j+n}}\right],\\ \hat{H}(t)_{BL} &= \frac{-1}{c}\log\Big[\sum_{j=1}^{k}\sum_{i=0}^{\infty}\sum_{\ell=0}^{i}\binom{i+\ell-1}{\ell}\\ &\times\ \frac{\pi_j(-1)^{i}(c\beta_j)^{i}t^{-(\beta_j+1)i}v_j^{a_j+n}\Gamma(a_j+n+i)}{i!\Gamma(a_j+n)[v_j+(i+\ell)t^{-\beta_j}]^{a_j+n+i}}\Big].\end{aligned}$$

Table 10: MSE's of the estimates of α and β.

n	$\hat{\alpha}_{ML}$	$\tilde{\alpha}_{BS}$	$\tilde{\alpha}_{BL}$	$\hat{\beta}_{ML}$	$\tilde{\beta}_{BS}$	$\tilde{\beta}_{BL}$
		simulated records with ($\alpha = 1$, $\beta = 0.5$)				
			$c = 2$			$c = 2$
3	3.7×10^9	4.338	0.331	10.481	0.044	0.033
5	1.762	0.427	0.232	0.446	0.035	0.026
7	0.655	0.239	0.188	0.149	0.029	0.022
		simulated records with ($\alpha = 1$, $\beta = 1$)				
			$c = 2$			$c = -1$
3	8.8×10^{18}	2.624	0.172	27.217	0.016	0.013
5	0.831	0.551	0.148	1.847	0.015	0.012
7	0.629	0.405	0.125	0.592	0.014	0.010
		simulated records with ($\alpha = 1$, $\beta = 3$)				
			$c = 1.5$			$c = -1.5$
3	1.4×10^{71}	0.929	0.204	299.7	0.009	0.005
5	0.927	0.457	0.172	17.03	0.008	0.004
7	0.583	0.284	0.130	5.269	0.005	0.003

Table 11: MSE's of the estimates of $R(t)$ and $H(t)$ with $t = 0.5$.

n	$\hat{R}_{ML}$	$\tilde{R}_{BS}$	$\tilde{R}_{BL}$	$\hat{H}_{ML}$	$\tilde{H}_{BS}$	$\tilde{H}_{BL}$
		simulated records with ($\alpha = 1$, $\beta = 0.5$)				
			$c = 2$			$c = 2$
3	0.121	0.043	0.039	4.4×10^{15}	3.873	0.235
5	0.094	0.034	0.027	1.418	0.312	0.125
7	0.077	0.027	0.022	0.255	0.141	0.085
		simulated records with ($\alpha = 1$, $\beta = 1$)				
			$c = -2$			$c = 1$
3	0.092	0.026	0.022	1.3×10^6	2.694	0.318
5	0.074	0.020	0.016	0.708	0.487	0.219
7	0.064	0.019	0.015	0.487	0.317	0.171
		simulated records with ($\alpha = 1$, $\beta = 3$)				
			$c = -2$			$c = 2$
3	0.022	0.008	0.006	95.86	0.515	0.117
5	0.021	0.005	0.004	0.642	0.268	0.097
7	0.020	0.004	0.003	0.575	0.178	0.072

7 Conclusion and suggestions

1. Record values from IW distribution is introduced.
2. Moments of record values from IW distribution is derived.
3. The BLUEs of the location and scale parameters of IW distribution are obtained.
4. MLE and Bayes estimate of the shape and scale parameters are obtained.
5. Mont Carlo simulations are carried out to show the usefulness of the findings.

6. The bivariate records is still open problem:

   Consider the following sequence of 10 observations on (X, Y) [see Arnold, Balakrishnan and Nagaraja (1998)]

   $$\begin{pmatrix}1.2\\1.3\end{pmatrix}, \begin{pmatrix}0.7\\1.9\end{pmatrix}, \begin{pmatrix}4.1\\3.6\end{pmatrix}, \begin{pmatrix}4.2\\0.5\end{pmatrix}, \begin{pmatrix}9.1\\0.6\end{pmatrix}, \begin{pmatrix}1.2\\3.7\end{pmatrix}, \begin{pmatrix}1.9\\4.0\end{pmatrix}, \begin{pmatrix}4.3\\3.7\end{pmatrix}, \begin{pmatrix}11.4\\6.5\end{pmatrix}, \begin{pmatrix}10.4\\7.1\end{pmatrix}.$$

   Four types of bivariate records can be defined as follows:

   (a) Type 1:

   $$\begin{pmatrix}1.2\\1.3\end{pmatrix}, \begin{pmatrix}1.2\\1.9\end{pmatrix}, \begin{pmatrix}4.1\\3.6\end{pmatrix}, \begin{pmatrix}4.2\\3.6\end{pmatrix}, \begin{pmatrix}9.1\\3.6\end{pmatrix}, \begin{pmatrix}9.1\\3.7\end{pmatrix}, \begin{pmatrix}9.1\\4.0\end{pmatrix}, \begin{pmatrix}11.4\\6.5\end{pmatrix}, \begin{pmatrix}11.4\\7.1\end{pmatrix}.$$

   (b) Type 2:

   $$\begin{pmatrix}1.2\\1.3\end{pmatrix}, \begin{pmatrix}0.7\\1.9\end{pmatrix}, \begin{pmatrix}4.1\\3.6\end{pmatrix}, \begin{pmatrix}4.2\\0.5\end{pmatrix}, \begin{pmatrix}9.1\\0.6\end{pmatrix}, \begin{pmatrix}1.2\\3.7\end{pmatrix}, \begin{pmatrix}1.9\\4.0\end{pmatrix}, \begin{pmatrix}4.3\\3.7\end{pmatrix}, \begin{pmatrix}11.4\\6.5\end{pmatrix}, \begin{pmatrix}10.4\\7.1\end{pmatrix}.$$

   (c) Type 3:

   $$\begin{pmatrix}1.2\\1.3\end{pmatrix}, \begin{pmatrix}4.1\\3.6\end{pmatrix}, \begin{pmatrix}4.3\\3.7\end{pmatrix}, \begin{pmatrix}11.4\\6.5\end{pmatrix}.$$

   (d) Type 4:

   $$\begin{pmatrix}1.2\\1.3\end{pmatrix}, \begin{pmatrix}4.1\\3.6\end{pmatrix}, \begin{pmatrix}11.4\\6.5\end{pmatrix}.$$

Mathematical formulations of the bivariate *pdf* and properties based on some useful models are still open problems?.

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