MOMENTS OF ORDER STATISTICS FROM RAYLEIGH DISTRIBUTION IN THE PRESENCE OF OUTLIER OBSERVATIONS

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1 Order statistics from IID variables

Let $X_1, ..., X_n$ be IID random variables from a population with cdf F(x) and pdf f(x). Let $X_{1:n} \leq X_{2:n} \leq \; \leq X_{n:n}$ be the order statistics obtained by arranging the n $X_i s$ in increasing order of magnitude.

Then, the pdf of $X_{r:n}$ $(1 \leq r \leq n)$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \;\; x \in R.$$

The joint pdf of $X_{r:n}$ and $X_{s:n}, \;\; 1 \leq r < s \leq n$ is given by

$$\begin{aligned} f_{r,s:n}(x) &= \frac{(n)!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1} \\ &\times \; [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)(y), -\infty < x < y < \infty. \end{aligned}$$

2 Order statistics from a single-outlier model

Barnett and Lewis (1994) have defined an outlier in a set of data to be *"an observation" or subset of observations" which appears to be inconsistent with the remainder of the set of data"*.

The distributions of order statistics presented in the last section, though simple in form, become quite complicated once the assumption of identical distribution of the random variables is lost.

A well-known case in this scenario is the single-outlier model wherein $X_1, ..., X_n$ are independent random variables with $X_1, ..., X_{n-1}$ being from a population with cdf F(x) and pdf f(x) and X_n being an outlier from a different population with cdf $G(x)$ and $g(x)$.

Then the pdf of the $r - th$ order statistic in this case is given by [Arnlod and Balakrishnan (1989)]

$$\begin{aligned} f_{r:n}(x) &= \frac{(n-1)!}{(r-2)!(n-r)!} F(x)^{r-2} G(x) f(x) 1 - F(x)^{n-r} \\ &+ \frac{(n-1)!}{(r-1)!(n-r)!} F(x)^{r-1} g(x) 1 - F(x)^{n-r} \\ &+ \frac{(n-1)!}{(r-1)!(n-r-1)!} F(x)^{r-1} f(x) 1 - F(x)^{n-r-1} 1 - G(x), x \in R. \end{aligned}$$

where the first and last terms vanish when $r = 1$ and $r = n$, respectively.

Proceeding similarly, the joint pdf of $X_{r:n}$ and $X_{s:n}$, $(1 \leq r < s \leq n)$ can be expressed as

$$\begin{aligned}
f_{r,s:n}(x,y) &= \frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!}\{F(x)\}^{r-2}G(x)f(x)\{F(y)-F(x)\}^{s-r-1} \\
&\times f(y)\{1-F(y)\}^{n-s} \\
&+ \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!}\{F(x)\}^{r-1}g(x)\{F(y)-F(x)\}^{s-r-1} \\
&\times f(y)\{1-F(y)\}^{n-s} \\
&+ \frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!}\{F(x)\}^{r-1}f(x)\{F(y)-F(x)\}^{s-r-2} \\
&\times \{G(y)-G(x)\}f(y)\{1-F(y)\}^{n-s} \\
&+ \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!}F(x)^{r-1}f(x)\{F(y)-F(x)\}^{s-r-1} \\
&\times g(y)\{1-F(y)\}^{n-s} \\
&+ \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s-1)!}\{F(x)\}^{r-1}f(x)\{F(y)-F(x)\}^{s-r-1} \\
&\times f(y)\{1-F(y)\}^{n-s-1}\{1-G(y)\}, -1 < x < y < 1,
\end{aligned}$$

where the first, middle and last terms vanish when r = 1, s = r + 1, and s = n, respectively. see Arnold and Balakrishnan (1989).

Balakrishnan (1994) has derived some recurrence relations satisfied by the single and product moments of order statistics from the right truncated exponential distribution.

Also he has deduced the recurrence relations for the multiple outlier model (with slippage of observations), see also Balakrishnan (1994a).

Childs, Balakrishnan and Moshref (2001) have derived some recurrence relations for the single and product moments of order statistics from n independent and non-identically distributed Lomax and the right-truncated Lomax random variables.

3 Order statistics from p-outlier model

Let $X_1, X_2, \ldots, X_{n-p}$ are iid from $f(x)$ while $X_{n-p+1}, \ldots, X_n$ are independent (and independent $X_1, X_2, \ldots, X_{n-p}$) from modified version of $f(x)$ as $g(x)$ in which the location and/or scale parameters have been shifted in value.

When we arrange the all n observations, we get
$X_{1:n} \leq X_{2:n} \leq X_{r:n} \leq X_{n:n}$.

Example: Let 1, 6, 5, 9, 3, from f(x) and $\underline{2}, \underline{4}, \underline{3}, \underline{11}$ from g(x)
Order Statistics: 1, $\underline{2}$, 3, $\underline{3}$, $\underline{4}$, 5, 6, 9, $\underline{11}$.

The probability density function of the r^{th} order statistics $X_{r:n}$, under the multiple outlier model can be written as, see Childs (1996)

$$\begin{aligned} f_{r:n}[p](x) &= \sum_{s=max(0,r-p-1)}^{min(n-p-1,r-1)} C_1 f(x)\{F(x)\}^s\{G(x)\}^{r-s-1}\{1-F(x)\}^{n-p-s-1} \\ &\times \{1-G(x)\}^{p-r+s+1} \\ &+ \sum_{s=max(0,r-p)}^{min(n-p,r-1)} C_2 g(x)\{F(x)\}^s\{G(x)\}^{r-s-1} \\ &\times \{1-F(x)\}^{n-p-s}\{1-G(x)\}^{p-r+s}, \\ &\qquad 1 \le r \le n,\ p = 0,1,2,\ldots,n,\ -\infty < x < \infty, \end{aligned}$$

where

$$C_1 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s-1)!(p-r+s+1)!},$$

and

$$C_2 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s)!(p-r+s)!}.$$

Similarly, the joint density function of $X_{r:n}$ and $X_{s:n}(1 \le r < s \le n)$ is given by

$$\begin{aligned}
f_{r,s:n}[p](x,y) &= \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-2)}^{min(n-p-j-2,r-1)} A_1 f(x)f(y)\{F(x)\}^i\{G(x)\}^{r-1-i}\{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j}\{1-F(y)\}^{n-p-i-j-2}\{1-G(y)\}^{p-s+i+j+2} \\
&+ \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 f(x)g(y)\{F(x)\}^i\{G(x)\}^{r-1-i}\{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j}\{1-F(y)\}^{n-p-i-j-1}\{1-G(y)\}^{p-s+i+j+1} \\
&+ \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 g(x)f(y)\{F(x)\}^i\{G(x)\}^{r-1-i}\{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j}\{1-F(y)\}^{n-p-i-j-1}\{1-G(y)\}^{p-s+i+j+1} \\
&+ \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j)}^{min(n-p-j,r-1)} A_3 g(x)g(y)\{F(x)\}^i\{G(x)\}^{r-1-i}\{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j}\{1-F(y)\}^{n-p-i-j}\{1-G(y)\}^{p-s+i+j}, \\
&\qquad 1 \le r < s \le n,\ p = 0, 1, 2, \ldots, n,\ -\infty < x < y < \infty,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \frac{(n-p)!p!}{i!(r-1-i)!j!(s-r-1-j)!(n-p-i-j-2)!(p-s+i+j+2)!}, \\
A_2 &= \frac{(n-p)!p!}{i!(r-1-i)!j!(s-r-1-j)!(n-p-i-j-1)!(p-s+i+j+1)!},
\end{aligned}$$

and

$$A_3 = \frac{(n-p)!p!}{i!(r-1-i)!j!(s-r-1-j)!(n-p-i-j)!(p-s+i+j)!}.$$

Setting $p = 1$ in (1.1) and (1.2), we obtain the corresponding pdf's in case of the single outlier given by Shu(1978) and David and Shu(1978).

Now we consider the case when the variable $X_1, X_2, \ldots, X_{n-p}$ are independent observations from Rayleigh distribution with density

$$f(x) = \frac{x}{\theta^2} e^{\frac{-x^2}{2\theta^2}}, \qquad x \geq 0, \quad \theta > 0$$

and $X_{n-p+1}, \ldots, X_n$ arise from the same distribution with density

$$g(x) = \frac{x}{\tau^2} e^{-\frac{x^2}{2\tau^2}}; \qquad x \geq 0, \quad \tau > 0,$$

where $\tau > \theta$.

The corresponding cumulative distribution functions $F(x)$ and $G(x)$ are given as

$$F(x) = 1 - e^{\frac{-x^2}{2\theta^2}}, \qquad x \geq 0, \quad \theta > 0$$

and

$$G(x) = 1 - e^{-\frac{x^2}{2\tau^2}}; \qquad x \geq 0, \quad \tau > 0.$$

The relation between $f(x)$ and $F(x)$ is given by

$$f(x) = \frac{x}{\theta^2}\{1 - F(x)\}. \tag{3.1}$$

Similarly, the relation between $g(x)$ and $G(x)$ is

$$g(x) = \frac{x}{\tau^2}\{1 - G(x)\}. \tag{3.2}$$

4 Single moments

In this section we derive the k^{th} moment of the r^{th} order statistic under multiple outlier model (with a slippage of p observations). Let $\mu_{r:n}^{(k)}[p]$; $(1 \leq r \leq n)$ denote the k^{th} single moments of order statistics in the presence of p-outlier observations from Rayleigh distribution. The following theorem gives an explicit form of $\mu_{r:n}^{(k)}[p]$.

Theorem 1

For $1 \leq r \leq n,\ p = 0, 1, \ldots, n$, and $k = 0, 1, \ldots$, the single moment $\mu_{r:n}^{(k)}[p]$ is given by

$$\begin{aligned}\mu_{r:n}^{(k)}[p] &= \frac{\Gamma(\frac{k}{2}+1)}{2^{\frac{k}{2}+2}}\Big\{\frac{1}{\theta^2}\sum_{s=max(0,r-p-1)}^{min(n-p-1,r-1)} C_1 \sum_{l=0}^{s}\tbinom{s}{l}\sum_{m=0}^{r-s-1}\tbinom{r-s-1}{m} \\ &\times \frac{(-1)^{l+m}}{\{\frac{n-p-s+l}{\theta^2}+\frac{p-r+s+1+m}{\tau^2}\}^{\frac{k}{2}+1}} \\ &+ \frac{1}{\tau^2}\sum_{s=max(0,r-p)}^{min(n-p,r-1)} C_2 \sum_{l=0}^{s}\tbinom{s}{l}\sum_{m=0}^{r-s-1}\tbinom{r-s-1}{m}\Big(\frac{(-1)^{l+m}}{\{\frac{n-p-s+l}{\theta^2}\frac{p-r+s+1+m}{\tau^2}\}^{\frac{k}{2}+1}}\Big\}.\end{aligned}$$

where

$$C_1 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s-1)!(p-r+s+1)!},$$

and

$$C_2 = \frac{(n-p)!p!}{s!(r-s-1)!(n-p-s)!(p-r+s)!}.$$

Proof

Starting from (1.1), we have

$$\begin{aligned}
\mu_{r:n}^{(k)}[p] &= \int_0^\infty x^k f_{r:n}[p](x)dx \\
&= \sum_{s=max(0,r-p-1)}^{min(n-p-1,r-1)} C_1 \int_0^\infty x^k f(x)\{F(x)\}^s\{G(x)\}^{r-s-1}\{1-F(x)\}^{n-p-s-1} \\
&\times \ \{1-G(x)\}^{p-r+s+1}dx \\
&+ \sum_{s=max(0,r-p)}^{min(n-p,r-1)} C_2 \int_0^\infty x^k g(x)\{F(x)\}^s\{G(x)\}^{r-s-1}\{1-F(x)\}^{n-p-s} \\
&\times \ \{1-G(x)\}^{p-r+s}dx \\
&= \frac{1}{\theta^2}\sum_{s=max(0,r-p-1)}^{min(n-p-1,r-1)} C_1 \int_0^\infty x^{k+1}\{F(x)\}^s\{G(x)\}^{r-s-1}\{1-F(x)\}^{n-p-s} \\
&\times \ \{1-G(x)\}^{p-r+s+1}dx \\
&+ \frac{1}{\tau^2}\sum_{s=max(0,r-p)}^{min(n-p,r-1)} C_2 \int_0^\infty x^{k+1}\{F(x)\}^s\{G(x)\}^{r-s-1}\{1-F(x)\}^{n-p-s} \\
&\times \ \{1-G(x)\}^{p-r+s+1}dx
\end{aligned} \tag{4.1}$$

by using the differential equations (1.3) and (1.4) in (2.2), we have

$$\begin{aligned}
\mu_{r:n}^{(k)}[p] &= \frac{1}{\theta^2}\sum_{s=max(0,r-p-1)}^{min(n-p-1,r-1)} C_1 \int_0^\infty x^{k+1}\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m} \\
&\quad \times\{1-F(x)\}^{n-p-s+l}\{1-G(x)\}^{p-r+s+1+m}dx \\
&+ \ \frac{1}{\tau^2}\sum_{s=max(0,r-p)}^{min(n-p,r-1)} C_2 \int_0^\infty x^{k+1}\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{r-s-1}\binom{r-s-1}{m}(-1)^{l+m} \\
&\quad \times\{1-F(x)\}^{n-p-s+l}\{1-G(x)\}^{p-r+s+1+m}dx
\end{aligned} \tag{4.2}$$

It is easy to derive (2.1) by writing $F(x) = 1-(1-F(x))$ and $G(x) = 1-(1-G(x))$ in (2.3) and integrate over x.

Table 1 given below displays the values of the single moments of order statistics in (2.1) when $\theta = 1$, $\tau = 1/4$ and $p = 0, 1, 2$.

Table 1. The
Expected Values and The Variances in the Presence of Multiple Outlier

p	i	$\mu_{i:5}[p]$	Variance	p	i	$\mu_{i:5}[p]$	Variance	p	i	$\mu_{i:5}[p]$	Variance
0	1	0.2000	0.0400	1	1	0.2222	0.0494	2	1	0.2500	0.0625
0	2	0.4500	0.1025	1	2	0.5040	0.1290	2	2	0.5714	0.1667
0	3	0.7833	0.2136	1	3	0.8881	0.2786	2	3	1.0190	0.3742
0	4	1.2833	0.4636	1	4	1.4897	0.6577	2	4	1.7476	0.9743
0	5	2.2833	1.4636	1	5	2.8960	3.1870	2	5	3.4119	4.2996

5 Product moments

In this section, we derive the product moments of order statistics under multiple outlier model (with a slippage of p observations).

Let $\mu_{r,s:n}^{(k,m)}[p]$; $(1 \leq r < s \leq n)$ denote the (k^{th}, m^{th}) product moments of the (r^{th}, s^{th}) order statistics in the presence of p-outlier observations from Rayleigh distribution. The following theorem gives an explicit form of $\mu_{r,s:n}^{(k,m)}[p]$.

Theorem 2

For $1 \leq r < s \leq n$, $p = 0, 1, \ldots, n$ and $k, m = 0, 1, \ldots$, the product the (k^{th}, m^{th}) product moments of the (r^{th}, s^{th}) order statistics in the presence of p-outlier observations from Rayleigh distribution is given by

$$\begin{aligned}
\mu_{r,s:n}^{(k,m)}[p] &= \frac{\Gamma(\frac{m+k}{2}+2)}{2\theta^4} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-2)}^{min(n-p-j-2,r-1)} A_1 \sum_{b=0}^{i} \binom{i}{b} \sum_{d=0}^{r-1-i} \binom{r-1-i}{d} \\
& \sum_{l=0}^{j} \binom{j}{l} \sum_{q=0}^{s-r-1-j} \binom{s-r-1-j}{q} (-1)^{b+d+l+q} \frac{I_{\frac{z_1}{z_1+z_2}}(\frac{k}{2}+1, \frac{m}{2}+1)}{z_1^{(\frac{k}{2}+1)} z_2^{(\frac{m}{2}+1)}} \\
&+ \frac{1}{2\theta^2\tau^2} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \sum_{b=0}^{i} \binom{i}{b} \sum_{d=0}^{r-1-i} \binom{r-1-i}{d} \\
& \sum_{l=0}^{j} \binom{j}{l} \sum_{q=0}^{s-r-1-j} \binom{s-r-1-j}{q} (-1)^{b+d+l+q} \frac{I_{\frac{z_1}{z_1+z_2}}(\frac{k}{2}+1, \frac{m}{2}+1)}{z_1^{(\frac{k}{2}+1)} z_2^{(\frac{m}{2}+1)}} \\
&+ \frac{\Gamma(\frac{m+k}{2}+2)}{2\theta^2\tau^2} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \sum_{b=0}^{i} \binom{i}{b} \sum_{d=0}^{r-1-i} \binom{r-1-i}{d} \\
& \sum_{l=0}^{j} \binom{j}{l} \sum_{q=0}^{s-r-1-j} \binom{s-r-1-j}{q} (-1)^{b+d+l+q} \frac{I_{\frac{z_3}{z_3+k_4}}(\frac{k}{2}+1, \frac{m}{2}+1)}{4 z_3^{(\frac{k}{2}+1)} z_4^{(\frac{m}{2}+1)}} \\
&+ \frac{1}{2\tau^4} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j)}^{min(n-p-j,r-1)} A_3 \sum_{b=0}^{i} \binom{i}{b} \sum_{d=0}^{r-1-i} \binom{r-1-i}{d} \\
& \sum_{l=0}^{j} \binom{j}{l} \sum_{q=0}^{s-r-1-j} \binom{s-r-1-j}{q} (-1)^{b+d+l+q} \frac{I_{\frac{z_3}{z_3+k_4}}(\frac{k}{2}+1, \frac{m}{2}+1)}{z_3^{(\frac{k}{2}+1)} z_4^{(\frac{m}{2}+1)}}
\end{aligned}$$

where

$$
\begin{aligned}
z_1 &= \frac{1}{2}\Big\{\frac{b+j-l+1}{\theta^2}+\frac{d+s-r-1-j-q}{\tau^2}\Big\},\\
z_2 &= \frac{1}{2}\Big\{\frac{n-p-i-j-1+l}{\theta^2}+\frac{p-s+i+j+2+q}{\tau^2}\Big\},\\
z_3 &= \frac{1}{2}\Big\{\frac{b+j-l}{\theta^2}+\frac{d+s-r-j-q}{\tau^2}\Big\}\\
z_4 &= \frac{1}{2}\Big\{\frac{n-p-i-j+l}{\theta^2}+\frac{p-s+i+j+1+q}{\tau^2}\Big\}\\
I_{\frac{z_1}{z_1+z_2}}(\frac{k}{2}+1,\frac{m}{2}+1) &= \int_0^{\frac{z_1}{z_1+z_2}} u^{\frac{k}{2}}(1-u)^{\frac{m}{2}}du,\\
I_{\frac{z_3}{z_3+k_4}}(\frac{k}{2}+1,\frac{m}{2}+1) &= \int_0^{\frac{z_3}{z_3+z_4}} u^{\frac{k}{2}}(1-u)^{\frac{m}{2}}du.
\end{aligned}
$$

Proof: Starting from (1.2), we have

$$\begin{aligned}
\mu_{r,s:n}^{(k,m)}[p] &= \int_0^\infty \int_0^y x^k y^m f_{r,s:n}[p](x,y)dxdy \\
&= \frac{1}{\theta^2} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-2)}^{min(n-p-j-2,r-1)} A_1 \int_0^\infty \int_0^y x^k y^{m+1} \{F(x)\}^i \{G(x)\}^{r-1-i} \{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j} \{1-F(y)\}^{n-p-i-j-1} \{1-G(y)\}^{p-s+i+j+2} f(x)dxdy \\
&+ \frac{1}{\tau^2} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \int_0^\infty \int_0^y x^k y^{m+1} \{F(x)\}^i \{G(x)\}^{r-1-i} \{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j} \{1-F(y)\}^{n-p-i-j-1} \{1-G(y)\}^{p-s+i+j+2} f(x)dxdy \\
&+ \frac{1}{\theta^2} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \int_0^\infty \int_0^y x^k y^{m+1} \{F(x)\}^i \{G(x)\}^{r-1-i} \{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j} \{1-F(y)\}^{n-p-i-j} \{1-G(y)\}^{p-s+i+j+1} g(x)dxdy \\
&+ \frac{1}{\tau^2} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j)}^{min(n-p-j,r-1)} A_3 \int_0^\infty \int_0^y x^k y^{m+1} \{F(x)\}^i \{G(x)\}^{r-1-i} \{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j} \{1-F(y)\}^{n-p-i-j} \{1-G(y)\}^{p-s+i+j+1} g(x)dxdy \\
&= \frac{1}{\theta^4} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-2)}^{min(n-p-j-2,r-1)} A_1 \int_0^\infty \int_0^y x^{k+1} y^{m+1} \{F(x)\}^i \{G(x)\}^{r-1-i} \{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j} \{1-F(y)\}^{n-p-i-j-1} \{1-G(y)\}^{p-s+i+j+2} \{1-F(x)\}dxdy \\
&+ \frac{1}{\theta^2\tau^2} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \int_0^\infty \int_0^y x^{k+1} y^{m+1} \{F(x)\}^i \{G(x)\}^{r-1-i} \{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j} \{1-F(y)\}^{n-p-i-j-1} \{1-G(y)\}^{p-s+i+j+2} \{1-F(x)\}dxdy \\
&+ \frac{1}{\theta^2\tau^2} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2 \int_0^\infty \int_0^y x^{k+1} y^{m+1} \{F(x)\}^i \{G(x)\}^{r-1-i} \{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j} \{1-F(y)\}^{n-p-i-j} \{1-G(y)\}^{p-s+i+j+1} \{1-G(x)\}dxdy \\
&+ \frac{1}{\tau^4} \sum_{j=0}^{s-r-1} \sum_{i=max(0,s-p-j)}^{min(n-p-j,r-1)} A_3 \int_0^\infty \int_0^y x^{k+1} y^{m+1} \{F(x)\}^i \{G(x)\}^{r-1-i} \{F(y)-F(x)\}^j \\
&\times \{G(y)-G(x)\}^{s-r-1-j} \{1-F(y)\}^{n-p-i-j} \{1-G(y)\}^{p-s+i+j+1} \{1-G(x)\}dxdy
\end{aligned} \tag{5.1}$$

by using (1.3) and (1.4) in (3.8), we have

$$
\begin{aligned}
\mu_{r,s:n}^{(k,m)}[p] &= \frac{1}{\theta^4}\sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-2)}^{min(n-p-j-2,r-1)} A_1\sum_{b=0}^{i}\binom{i}{b}\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{l=0}^{j}\binom{j}{l}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}(-1)^{b+d+l+q}\\
&\int_0^\infty\int_0^y x^{k+1}y^{m+1}\{1-F(x)\}^{b+j-l+1}\{1-G(x)\}^{d+s-r-1-j}\\
&\{1-F(y)\}^{n-p-i-j-1+l}\{1-G(y)\}^{p-s+i+j+2+q}dxdy\\
&+ \frac{1}{\theta^2\tau^2}\sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2\sum_{b=0}^{i}\binom{i}{b}\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{l=0}^{j}\binom{j}{l}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}(-1)^{b+d+l+q}\\
&\int_0^\infty\int_0^y x^{k+1}y^{m+1}\{1-F(x)\}^{b+j-l+1}\{1-G(x)\}^{d+s-r-1-j}\\
&\{1-F(y)\}^{n-p-i-j-1+l}\{1-G(y)\}^{p-s+i+j+2+q}dxdy\Big\}\\
&+ \frac{1}{\theta^2\tau^2}\sum_{j=0}^{s-r-1}\sum_{i=max(0,s-p-j-1)}^{min(n-p-j-1,r-1)} A_2\sum_{b=0}^{i}\binom{i}{b}\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{l=0}^{j}\binom{j}{l}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}(-1)^{b+d+l+q}\\
&\int_0^\infty\int_0^y x^{k+1}y^{m+1}\{1-F(x)\}^{b+j-l}\{1-G(x)\}^{d+s-r-j-q}\\
&\{1-F(y)\}^{n-p-i-j+l}\{1-G(y)\}^{p-s+i+j+1+q}dxdy\\
&+ \frac{1}{\tau^4}\sum_{i=max(0,s-p-j)}^{min(n-p-j,r-1)} A_3\sum_{b=0}^{i}\binom{i}{b}\sum_{d=0}^{r-1-i}\binom{r-1-i}{d}\sum_{l=0}^{j}\binom{j}{l}\sum_{q=0}^{s-r-1-j}\binom{s-r-1-j}{q}(-1)^{b+d+l+q}\\
&\int_0^\infty\int_0^y x^{k+1}y^{m+1}\{1-F(x)\}^{b+j-l}\{1-G(x)\}^{d+s-r-j-q}\\
&\{1-F(y)\}^{n-p-i-j+l}\{1-G(y)\}^{p-s+i+j+1+q}dxdy.
\end{aligned}
$$

Upon, we put $F(x) = 1-(1-F(x))$,$G(x) = 1-(1-G(x))$ $F(y)-F(x) = (1-F(x))-(1-F(y))$ and $G(y)-G(x) = (1-G(x))-(1-G(y))$ and using binomial theorem, where $1-F(x) = e^{\frac{-x^2}{2\theta^2}}$, $1-F(y) = e^{\frac{-y^2}{2\theta^2}}$, $1-G(x) = e^{\frac{-x^2}{2\tau^2}}$, and $1-G(y) = e^{\frac{-y^2}{2\tau^2}}$, we get (3.1).

Table 2 given below displays the product moments the corresponding covariance of order statistics in (3.1) when $\theta = 1$,
$tau = 1/4$ and $p = 0, 1, 2$.

Table 2. The Covariances in the Presence of Multiple Outlier

p	r	s	$\sigma_{r,s:5}^{(1,1)}[p]$	p	r	s	$\sigma_{r,s:5}^{(1,1)}[p]$	p	r	s	$\sigma_{r,s:5}^{(1,1)}[p]$
0	1	1	0.04000	1	1	1	0.04940	2	1	1	0.06250
0	1	2	0.04000	1	1	2	0.03224	2	1	2	0.03891
0	2	2	0.10250	1	2	2	0.12900	2	2	2	0.16670
0	1	3	0.04001	1	1	3	0.12819	2	1	3	0.19331
0	2	3	0.10252	1	2	3	-0.04588	2	2	3	0.12131
0	3	3	0.21360	1	3	3	0.27860	2	3	3	0.37420
0	1	4	0.04001	1	1	4	-0.00914	2	1	4	-0.05337
0	2	4	0.10252	1	2	4	0.63753	2	2	4	0.74541
0	3	4	0.21368	1	3	4	-0.54883	2	3	4	-0.00569
0	4	4	0.46360	1	4	4	0.65770	2	4	4	0.97430
0	1	5	0.04001	1	1	5	0.09121	2	1	5	0.04401
0	2	5	0.10252	1	2	5	0.00518	2	2	5	-0.24084
0	3	5	0.21371	1	3	5	1.84174	2	3	5	2.65328
0	4	5	0.46373	1	4	5	-2.35723	2	4	5	-0.89709
0	5	5	1.46360	1	5	5	3.18700	2	5	5	4.29960

6 Special cases

In this section, we deduce some special cases from the single and product moments given in (2.1) and (2.2) as follows:

1. Setting $p = 0$, we get the single and the product moments of order statistics when $x_1, x_2, \ldots, x_n$ have Rayleigh distribution with parameter θ.

$$\mu_{r:n}^{(k)}[0] = \frac{n!}{(r-1)!(n-r)!} \frac{\Gamma(\frac{k}{2}+1)}{2^{\frac{k}{2}+2}} \sum_{l=0}^{r-1} \binom{r-1}{l} (-1)^l \frac{\theta^k}{(n-r+1+l)^{\frac{k}{2}+1}}$$

and

$$\begin{aligned} \mu_{r,s:n}^{(k,m)}[0] &= \frac{n!}{(r-1)!(s-r-1)!(n-r)!} \frac{\Gamma(\frac{k+m}{2}+2)}{2\theta^4} \\ &\times \sum_{b=0}^{r-1} \binom{r-1}{b} \sum_{l=0}^{s-r-1} \binom{s-r-1}{l} (-1)^{b+l} \frac{I_{\frac{z_1}{z_1+z_2}}(\frac{k}{2}+1, \frac{m}{2}+1)}{z_1^{-(\frac{k}{2}+1)} z_2^{-(\frac{m}{2}+1)}} \end{aligned}$$

where $z_1 = \frac{b+s-r-l}{2\theta^2}$ and $z_2 = \frac{n-s-1+l}{2\theta^2}$.

2. If we put $p = n$, we have the same relations above but with parameter τ.

3. If we put $p = 1$, we have the relations for single outlier.

7 Conclusion

1. Rayleigh Distribution under Contaminations (INID) is intoduced
2. Exact form of the single moment of order Statistic from INID Rayleigh is derived.
3. Exact form of the product moment of order Statistic from INID Rayleigh is derived.

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