Correlation Goodness-of-Fit Test of Mixture of Two Weibull Distributions

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Abstract:

In this paper, we use the moment of order statistics for testing mixture of two Weibull distrinutions (MTWD) by using correlation-type goodness-of fit test. Also, we show the performance of the test by calculating the power based on some alternative distributions including; normal, gamma, lognormal, Weibull and the mixture of gamma and lognormal distributions. In addition, we discuss a numerical example. Finally, we apply our results to real lifetime data.

1 Introduction

The subject of assessing whether a particular distributional model can be used to analyze a data set is one which has received a good deal of attention in the past several years. This topic is especially important for models such as Weibull that are used to predict lifetimes of products or assess the reliability of systems. The book of Murthy, Xie and Jiang (2004) provides a good survey and foundation for model building involving Weibull models.

The goodness-of-fit tests are the key in selecting the distribution model that best fit the data. Good references for detailed description of many of these tests are D'Agostino and Stephens (1986) and Huber-Carol, et al (2002).

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If the sample had, in fact, come from hypnotized (location and scale unspecified) distribution, then the plot $X_{i:n},\ i = 1, 2, \ldots, n$ (sample order statistics) versus $E(X_{i:n}),\ i = 1, 2, \ldots, n$ will be approximately linear. The statistic of the correlation coefficient test measures the linearity between $X_{i:n}$ and $E(X_{i:n}),\ i = 1, 2, \ldots, n$. This test is computationally simple, readily extendable as a distributional test statistic for other distributions, and conceptually easy to be understood in that it combines two fundamentally simple concepts; the probability plot and the correlation coefficient. The correlation coefficient test have introduced by Filliben (1975). Looney and Gulledge (1985) have updated the tables for testing goodness-of-fit to the normal distribution. Kinnison (1989) has used the correlation coefficient goodness-of-fit test for the Extreme-Value distribution. Sultan (2001) has used the same technique to present tables for testing goodness-of-fit of the logarithmically-decreasing survival distribution.

Finite mixture distributions have provided a mathematical-based approach to the statistical modeling of a wide variety of random phenomena. Because of their usefulness as an extremely flexible method of modeling, finite mixture models have continued to receive increasing attention over the recent years, from both practical and theoretical points of view, and especially for lifetime distributions. Mixture distributions have been considered extensively in literature. For excellent survey of estimation techniques, discussion and applications, see Everitt and Hand (1981), Titterington, Simth and Makov (1985), Maclachlan and Basford (1988), Lindsay (1995), Mclachlan and Krishnan (1997) and McLachlan and Peel (2000). Recently, AL-Hussaini and Sultan (2001) have reviewed some properties of finite mixtures of life time models.

The mixture of two Weibull distributions with location, scale and shape parameters has its pdf as

$$f(x) = w f_1(x) + (1 - w) f_2(x),\ 0 \leq w \leq 1, \tag{1.1}$$

where for $i = 1, 2$

$$f_i(x) = \frac{\alpha_i}{\sigma}\left(\frac{x-\theta}{\sigma}\right)^{\alpha_i - 1} \exp\left[-\left(\frac{x-\theta}{\sigma}\right)^{\alpha_i}\right],\ x \geq \theta,\ \sigma > 0, \alpha_i > 0. \tag{1.2}$$

The mixture of two Weibull distributions with scale and shape parameters can be obtained from (1.1) and (1.2) when $\theta = 0$, while the mixture of two Weibull distributions with shape parameters can also be obtained from (1.1) and (1.2) when $\theta = 0$ and $\sigma = 1$.

From (1.1) and (1.2), the cumulative distribution function (cdf) of the MTWD with shape parameters α_1 and α_2 is given by

$$F(x) = 1 - w\exp\left(-x^{\alpha_1}\right) - (1 - w)\exp\left(-x^{\alpha_2}\right),\ x > 0, \alpha_1, \alpha_2 > 0. \tag{1.3}$$

In Section 2 of this paper, we derive the single moment of order statistics from mixture of two Weibull distributions MTWD. In addition, we calculate the moments of order statistics for some different choices of the parameters and the sample size. In Section 3, we use the correlation coefficient goodness-of-fit for MTWD with scale and shape parameters while in Section 4, we consider the MTWD with location, scale and shape parameters. In addition, we simulate the lower 1%, 2.5% and 5% percentage points of the statistics of the tests. In Section 5, we calculate the power of the tests for both cases based on some alternative distributions. Finally, in Section 6, we apply the test to real life data.

2 Order statistics

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics from MTWD with shape parameters $\alpha_1,\ \alpha_2$. Then, the pdf of the r-th order statistic is given by

$$f_{i:n}(x) = C_{i:n}\{F(x)\}^{i-1}\{1-F(x)\}^{n-i}f(x),\ x>0, \tag{2.1}$$

where

$$C_{i:n} = \frac{n!}{(i-1)!(n-i)!}. \tag{2.2}$$

For more details see, David (1981 and 2003) and Arnold, Balakrishnan and Nagaraja (1992). In the following theorem, we state and prove an expression of the moment of order statistics from MTWD.

Theorem 1

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics from MTWD given in (1.3), then the single moment of order statistics is given by

$$\begin{aligned}
\mu_{i:n} &= C_{i:n}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k} \\
&\times \Bigg[\frac{w^{n-i+k+1}\Gamma\left(\frac{\alpha_1+1}{\alpha_1}\right)}{(n-i+k+1)^{\frac{\alpha_1+1}{\alpha_1}}} + \frac{(1-w)^{n-i+k+1}\Gamma\left(\frac{\alpha_2+1}{\alpha_2}\right)}{(n-i+k+1)^{\frac{\alpha_2+1}{\alpha_2}}} \\
&+ \sum_{L=1}^{n-i+k}\binom{n-i+k}{L}\frac{w^{n-i+k+1-L}(1-w)^{L}\Gamma\left(\frac{\alpha_1+1}{\alpha_1}\right)}{(n-i+k+1-L)^{\frac{\alpha_1+1}{\alpha_1}}} \\
&+ \sum_{L=0}^{n-i+k-1}\binom{n-i+k}{L}\frac{w^{n-i+k-L}(1-w)^{L+1}\Gamma\left(\frac{\alpha_2+1}{\alpha_2}\right)}{(n-i+k+1)^{\frac{\alpha_2+1}{\alpha_2}}}
\end{aligned}$$

$$+ \sum_{L=1}^{n-i+k} w^{n-i+k-L+1}(1-w)^L \sum_{j=1}^{\infty} \frac{(-1)^j \binom{n-i+k}{L} L^j \Gamma\left(\frac{j\alpha_2+\alpha_1+1}{\alpha_1}\right)}{j!(n-i+k-L+1)^{\frac{j\alpha_2+\alpha_1+1}{\alpha_1}}}$$
$$+ \sum_{L=0}^{n-i+k-1} w^{n-i+k-L}(1-w)^{L+1} \sum_{j=1}^{\infty} \frac{(-1)^j \binom{n-i+k}{L} (n-i+k-L)^j \Gamma\left(\frac{j\alpha_1+\alpha_2+1}{\alpha_2}\right)}{j!(L+1)^{\frac{j\alpha_1+\alpha_2+1}{\alpha_2}}}\Bigg], \tag{2.3}$$

where $C_{i:n}$ is given by (2.2).

Proof

From (2.1), the single moment of the $i-$th order statistics from (1.3) can be written as

$$\mu_{i:n} = C_{i:n} \int_0^{\infty} x\{F(x)\}^{i-1}\{1-F(x)\}^{n-i} f(x)dx, \tag{2.4}$$

where $C_{i:n}$ is given in (2.2), $F(x)$ is given in (1.3) and $f(x)$ is the corresponding pdf.

From (1.3) and (2.4) and by expanding $\{F(x)\}^{i-1}$ binomially, we get

$$\begin{aligned} \mu_{i:n} &= C_{i:n} \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} \int_0^{\infty} x \left[w \exp\left(-x^{\alpha_1}\right) + (1-w)\exp\left(-x^{\alpha_2}\right)\right]^{n-i+k} \\ &\times \left[w\alpha_1 x^{\alpha_1-1} \exp\left(-x^{\alpha_1}\right) + (1-w)\alpha_2 x^{\alpha_2-1} \exp\left(-x^{\alpha_2}\right)\right] dx. \end{aligned} \tag{2.5}$$

Again, by using the binomial expansion in (2.5), we get

$$\mu_{i:n} = C_{i:n} \sum_{k=0}^{i-1} \sum_{L=0}^{n-i+k} (-1)^k \binom{i-1}{k} \binom{n-i+k}{L} w^{n-i+k-L}(1-w)^L \{I_1 + I_2\} dx, \tag{2.6}$$

where

$$I_1 = \int_0^{\infty} w\alpha_1 x^{\alpha_1} \left[\exp\left(-x^{\alpha_1}\right)\right]^{n-i+k-L+1} \left[\exp\{-x^{\alpha_2}\}\right]^L dx, \tag{2.7}$$

and

$$I_2 = \int_0^{\infty} (1-w)\alpha_2 x^{\alpha_2} \left[\exp\left(-x^{\alpha_1}\right)\right]^{n-i+k-L} \left[\exp\{-x^{\alpha_2}\}\right]^{L+1} dx. \tag{2.8}$$

By integrating (2.7) and (2.8), we get

$$I_1 = w \sum_{j=0}^{\infty} \frac{(-1)^j L^j \Gamma\left(\frac{j\alpha_2+\alpha_1+1}{\alpha_1}\right)}{j!\,(n-i+k-L+1)^{\frac{j\alpha_2+\alpha_1+1}{\alpha_1}}}, \tag{2.9}$$

and

$$I_2 = (1-w)\sum_{j=0}^{\infty} \frac{(-1)^j (n-i+k-L)^j \Gamma\left(\frac{j\alpha_1+\alpha_2+1}{\alpha_2}\right)}{j!\,(L+1)^{\frac{j\alpha_1+\alpha_2+1}{\alpha_2}}}. \tag{2.10}$$

Substituting (2.7) - (2.10) into (2.6) and avoid the terms 0^0, we get (2.3).

As a check, at $w = 0$ and $w = 1$, the single moment in (2.3) reduces to the corresponding single moments of $f_1(x)$ and $f_2(x)$, respectively, see Balakrishnan and Cohen (1991).

The following table represents the moment of order statistics from the MTWD given in (2.3) based on different choices of the parameters and the sample size.

Table 1: Single moment of order statistics from MTWD

$\mu_{i:20}, i = 1,\ldots,20,\quad w = 0.5, \alpha_1 = 1.5, \alpha_2 = 2.5$									
0.1705	0.2763	0.3615	0.4358	0.5037	0.5674	0.6284	0.6878	0.7466	0.8056
0.8655	0.9273	0.9922	1.0616	1.1378	1.2242	1.3273	1.4604	1.6588	2.0615
$\mu_{i:30}, i = 1,\ldots,30,\quad w = 0.5, \alpha_1 = .75, \alpha_2 = 3$									
0.0334	0.0800	0.1354	0.1962	0.2596	0.3233	0.3855	0.4453	0.5023	0.5564
0.6079	0.6571	0.7046	0.7507	0.7961	0.8410	0.8862	0.9322	0.9796	1.0295
1.0832	1.1429	1.2126	1.2997	1.4179	1.59217	1.8663	2.3214	3.1411	5.0748

The moments of order statistics given in Table 1 are checked by using the identity

$$\sum_{i=1}^{n} \mu_{i:n} = nE(X), \tag{2.11}$$

for more details see, Arnold, Balakrishnan and Nagaraja (1992).

More extensive tables of order statistics from MTWD up to $n = 25$ and different choices of the parameters are available from the authors upon request.

3 Test for MTWD with scale parameters

Let $X_{1:n}, X_{2:n}, \ldots, X_{n-r:n}$ denote a Type-II right-censored sample from the MTWD with scale and shape parameters, and let $Z_{i:n} = X_{i:n}/\sigma, i = 1, 2, \ldots, n-r$, be the corresponding order statistics from the MTWD with shape parameters. Let us denote $E(Z_{i:n})$ by $\mu_{i:n}$, then $E(X_{i:n}) = \sigma\mu_{i:n}, i = 1, 2, \ldots, n-r$. The correlation-type goodness-of-fit test in this case may be formed as

H_0 : F is correct, that is $X_1, X_2, \ldots, X_n$ has MTWD with shape parameters versus,
H_1 : F is not correct, that is $X_1, X_2, \ldots, X_n$ has another pdf,

and the statistic used to run the test is given by

$$T_1 = \frac{\sum_{i=1}^{n-r} X_{i:n}\mu_{i:n}}{\sqrt{\sum_{i=1}^{n-r} X_{i:n}^2 \sum_{i=1}^{n-r} \mu_{i:n}^2}}, \tag{3.1}$$

this statistic represents the correlation between $X_{i:n}$ and $\mu_{i:n}, i = 1, 2, \ldots, n-r$. By using the moments $\mu_{i:n}, i = 1, 2, \ldots, n-r$ given in (2.3), the statistic T_1 is simulated through Monte Carlo method based on 10,001 simulations. Table 2 represents the lower 1%, 2.5%, 5% and 10% percentage points of T_1 for sample sizes $n = 5, 10, 15, 20, 25, 30,\ \ (w, \alpha_1, \alpha_2) = (0.5, 0.75, 3.0)$ and different censoring ratios $p = \frac{n-r}{n} = 1.0, 0.8, 0.6$.

Table 2: Percentage Points of T_1

p	n	1%	2.5%	5%	10%
1.0	5	0.8841	0.8975	0.9095	0.9229
	10	0.8809	0.8973	0.9082	0.9228
	15	0.8782	0.8967	0.9092	0.9239
	20	0.8843	0.8997	0.9121	0.9282
	25	0.8848	0.9007	0.9143	0.9311
	30	0.8836	0.9043	0.9196	0.9376
0.8	5	0.8716	0.9075	0.9253	0.9419
	10	0.9113	0.9354	0.9524	0.9642
	15	0.9407	0.9548	0.965	0.9738
	20	0.9531	0.9646	0.9723	0.9793
	25	0.9636	0.9720	0.9787	0.9836
	30	0.9712	0.9774	0.9821	0.9862
0.6	5	0.8383	0.8675	0.9000	0.9351
	10	0.8610	0.8931	0.9204	0.9448
	15	0.9085	0.9313	0.9463	0.9605
	20	0.9186	0.9419	0.9563	0.9677
	25	0.9400	0.9542	0.9645	0.9741
	30	0.9456	0.9586	0.9682	0.9772

As we can see from Table 2, the percentage points of T_1 increases as the sample size increases as well as the significance level increases for censoring ratios $p = 1.0, 0.8, 0.6$.

4 Test for MTWD with location-scale parameters

Let $X_{1:n}, X_{2:n}, \ldots, X_{n-r:n}$ denote a Type-II right-censored sample from the MTWD given in (1.1) and let $Z_i = X_{i+1:n} - X_{1:n}$ and $\nu_i = \mu_{i+1:n} - \mu_{1:n}, i = 1, 2, \ldots, n-$

$r-1$, where $\mu_{i:n}$ be the corresponding moments of order statistics from the MTWD with shape parameters. The correlation-type goodness-of-fit test in this case may be formed as

$$\begin{aligned} H_0 &: F \text{ is correct, that is } X_1, X_2, \ldots, X_n \\ &\quad \text{has MTWD with location and scale parameters given in (1.1) versus,} \\ H_1 &: F \text{ is not correct, that is } X_1, X_2, \ldots, X_n \text{ has another pdf,} \end{aligned}$$

and the statistic used to run the test is given by

$$T_2 = \frac{\sum_{i=1}^{n-r-1} (Z_i)(\nu_i)}{\sqrt{\sum_{i=1}^{n-r-1} Z_i^2 \sum_{i=1}^{n-r-1} \nu_i^2}}, \tag{4.1}$$

this statistic represents the correlation between Z_i and $\nu_i, i = 1, 2, \ldots, n-r$.

Once again by using the moments $\mu_{i:n}, i = 1, 2, \ldots, n-r$ given in (2.3), the statistic T_2 is simulated through Monte Carlo method based on 10,001 simulations. Table 3 represents the lower 1%, 2.5%, 5% and 10% percentage points of T_2 for sample sizes $n = 5, 10, 15, 20, 25, 30$, $(w, \alpha_1, \alpha_2) = (0.5, 0.75, 3.0)$ and different censoring ratios p.

From Table 3, we see that, the percentage points of T_2 increases as the sample size increases as well as the significance level increases for censoring ratios $p = 1.0, 0.8, 0.6$.

Table 3: Percentage Points of T_2

p	n	1%	2.5%	5%	10%
1.0	5	0.8680	0.8850	0.8994	0.9159
	10	0.8767	0.8895	0.9022	0.9156
	15	0.8748	0.8918	0.9054	0.9204
	20	0.8826	0.8971	0.9097	0.9260
	25	0.8820	0.8983	0.9130	0.9300
	30	0.8837	0.9031	0.9179	0.9364
0.8	5	0.8779	0.9022	0.9243	0.9411
	10	0.9019	0.9319	0.9503	0.9635
	15	0.9386	0.9546	0.9648	0.9740
	20	0.9542	0.9659	0.9733	0.9801
	25	0.9652	0.9733	0.9788	0.9840
	30	0.9722	0.9781	0.9824	0.9868
0.6	5	0.8956	0.9025	0.9124	0.9324
	10	0.8630	0.9013	0.9246	0.9447
	15	0.9068	0.9312	0.9478	0.9620
	20	0.9224	0.9439	0.9565	0.9686
	25	0.9409	0.9556	0.9660	0.9749
	30	0.9473	0.9591	0.9692	0.9773

5 Power

In this section, we calculate the power of the test by replacing the generator of random variate from MTWD in the simulation program with generators from the alternative distributions including; normal, lognormal, gamma, Weibull and mixture of gamma and lognormal distributions. Based on different sample sizes, different censoring ratios and 10,001 simulations, the power is calculated to be

$$\text{Power} = \frac{\#\text{ of rejection of } H_0}{10,001},$$

where H_0 is rejected if $T_1(T_2) \geq$ the corresponding percentage points given in Table 2 (Table 3), and $T_1(T_2)$ is evaluated from the alternative distributions.

Tables 4 and 5 represent the power of the test for the two MTWD cases, respectively. The different alternative distributions considered are:

1. Normal distribution $N(\mu, \sigma)$
2. Lognormal $LN(\mu, \sigma)$
3. Weibull distribution with shape parameter a, scale parameter σ and location parameter μ, $W(\mu, \sigma, a)$,
4. Gamma distribution with shape parameter k, scale parameter σ and location parameter μ, $G(\mu, \sigma, k)$.
5. Mixture of gamma and lognormal distribution,
   $MGLN(a_1, b_1, k, a_2, b_2) = 0.5G(a_1, b_1, k) + 0.5LN(a_2, b_2)$.

Tables 4 and 5 indicate that the correlation test has good power to reject samples from the chosen alternative distributions. Also, the power increases as the sample sizes increase for all given censoring ratios $p = 1.0, 0.8$ and 0.6 as well as when the significance level increases.

Table 4: Power of The Test Based on MTWD with scale parameter case when ($\sigma = 1$).

		$N(1,5)$		$LN(1,5)$		$W(0,1,25)$		$MGLN(0,1,.1,1,5)$	
p	n	5%	10%	5%	10%	5%	10%	5%	10%
1.0	10	0.9010	0.9005	0.8911	0.9307	1.0000	1.0000	0.8911	0.9307
	20	0.9208	0.9208	0.9604	0.9703	1.0000	1.0000	0.9604	1.0000
	30	0.9703	0.9901	0.9901	1.0000	1.0000	1.0000	1.0000	1.0000
0.8	10	0.9901	0.9901	0.9406	0.9901	0.9703	1.0000	0.9505	0.9703
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	30	1.0000	1.0000	0.9901	1.0000	1.0000	1.0000	1.0000	1.0000
0.6	10	1.0000	1.0000	0.6040	0.7723	0.4158	1.0000	0.8416	0.9208
	20	1.0000	1.0000	0.8812	0.9208	1.0000	1.0000	0.9901	1.0000
	30	1.0000	1.0000	0.9604	1.0000	1.0000	1.0000	1.0000	1.0000

Table 5: Power of The Test Based on MTWD with locaton and scale parameter case when $\theta = 0, \sigma = 1$.

		$N(1,5)$		$G(0,1,.1)$		$LN(1,5)$		$MGLN(0,1,.1,1,5)$	
p	n	5%	10%	5%	10%	5%	10%	5%	10%
1.0	10	0.9208	0.9300	0.6832	0.7228	0.9010	0.9109	0.8218	0.9406
	20	0.9210	0.9307	0.7129	0.8119	0.9604	0.9703	0.9703	1.0000
	30	0.9703	0.9901	0.7327	0.9505	0.9901	1.0000	0.9901	1.0000
0.8	10	0.9901	0.9901	0.9802	0.9901	0.9901	0.9901	0.8812	0.9703
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	30	1.0000	1.0000	1.0000	1.0000	0.9901	1.0000	1.0000	1.0000
0.6	10	1.0000	1.0000	0.7624	0.8218	0.5050	0.6139	0.5347	0.8911
	20	1.0000	1.0000	1.0000	1.0000	0.9208	0.9703	0.9703	0.9901
	30	1.0000	1.0000	1.0000	1.0000	0.9802	1.0000	1.0000	1.0000

Example:

In order to illustrate and show the performance of the correlation coefficient goodness-of-fit test for the MTWD in both cases (location-scale and scale parameters), we simulate a set of order statistics from MTWD of size $n = 30$ with $w = 0.5, \alpha_1 = 0.75, \alpha_2 = 3, \sigma = 1$ and $\theta = 0$ as

0.10436, 0.23712, 0.47129, 0.6551, 0.7838, 0.85331, 0.90699, 0.91556, 0.94153, 1.00929,1.01352, 1.11506, 1.12124, 1.12503, 1.19943, 1.23229, 1.23496, 1.40603, 1.40732, 1.41572, 1.46125, 1.51862, 1.73545, 2.25243, 4.01181, 5.04637, 5.52511, 5.5266, 5.83865, 6.17341.

The above order statistics and the corresponding moments of order statistics given in Table 1, are used to calculate the test statistic T_1 given in (3.1). Table 1 is used to run the test and results of the test at 5% significance level are:

	Decision			
P	1%	2.5%	5%	10%
1.0	Acept	Acept	Acept	Acept
0.8	Acept	Acept	Acept	Reject
0.6	Acept	Acept	Acept	Acept

From the above table, we see that the test is performed well for MTWD.

6 Application

The data set is from Lawless (1982, p.28). The data given arose in tests on endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in life test. The order statistics of the first 20 observations are:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 127.92, 128.04, 173.40.

Lawless (2003) has examined the data using the Nelson-Aalen cumulative hazard function estimate plot and Kaplan-Meier estimate for both Weibull and Lognormal distributions. He has concluded that the Lognormal fits the data slightly better than the Wiebull distribution.

For the purpose of using our test to the above lifetime data, we use the moments of order statistics $\mu_{i:20}, i = 1, 2, \ldots, 20$ given in Table 1 and life data for calculating the statistic T_1 given in (2.1) as follows:

n	p	Calculated T_1	Simulated T_1
20	1.0	0.9957	0.9817
	0.8	0.9935	0.9695
	0.6	0.9924	0.9999

As we can see the MTWD distribution fits the data at 5%. So, we recommend the MTWD for the given life data.

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