King Saud University: Math. Dept. Math-109
1st. Midterm Examination Semester I 1427-1428 H
Time: 90 min. Max. Marks 20

Question 1: (3 + 4 + 3)

(a) Solve the inequality and express your answer in terms of intervals

$$\frac{2x+3}{x^2+5x+6} \geq 0.$$

Solution: (a) Let

$$\begin{aligned} \frac{2x+3}{x^2+5x+6} &\geq 0 \\ \frac{2x+3}{x^2+3x+2x+6} &\geq 0 \\ \frac{2x+3}{x(x+3)+2(x+3)} &\geq 0 \\ \frac{2x+3}{(x+3)(x+2)} &\geq 0 \end{aligned}$$

Thus the solution of the given inequality is

$$x \in (-3,-2)U[-\frac{3}{2},\infty)$$

(b) Find $(f \circ g)(1)$ and $(g \circ f)(1)$, where

$$f(x) = x + \frac{1}{x} \qquad \text{and} \qquad g(x) = \frac{x+1}{x+2}.$$

Solution: (b) Since

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x+1}{x+2}\right) \\ &= \frac{x+1}{x+2} + \frac{x+2}{x+1} \\ &= \frac{(x+1)^2+(x+2)^2}{(x+1)(x+2)} \end{aligned}$$

and at $x = 1$

$$(f \circ g)(1) = \frac{(1+1)^2+(1+2)^2}{(1+1)(1+2)} = \frac{36}{6} = 6$$

Solution: (b) Since

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x + \frac{1}{x}) \\ &= \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2} \\ &= \frac{(x^2 + x + 1)/x}{(x^2 + 2x + 1)/x} \\ &= \frac{(x^2 + x + 1)}{(x + 1)^2}\end{aligned}$$

and at $x = 1$

$$g \circ f)(1) = \frac{(1 + 1 + 1)}{(1 + 1)^2} = \frac{3}{4}$$

(c) Determine the value of k such that the given function is continuous on $(-\infty, \infty)$

$$f(x) = \begin{cases} 5x + 2, & \text{if } x \leq 1 \\ kx^2, & \text{if } x > 1. \end{cases}$$

Solution: (c) Let

$$\begin{aligned}(i)\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} kx^2 = k(1)^2 = k \\ (ii)\ \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (5x + 2) = 5(1) + 2 = 7 \\ (iii)\ f(1) &= (5x + 2)|_{x=1} = 7\end{aligned}$$

Hence

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1) = 7 = k$$

Question 2: (4 + 3 + 3)

(a) Find the following limits, if exist

$$(i) \quad \lim_{x \to 0} \frac{x^2}{1 - \cos x}, \qquad (ii) \quad \lim_{x \to 0} \frac{\sqrt{x^2 + x + 4} - 2}{x^2 + x}.$$

Solution: (a)(i) Let

$$\lim_{x \to 0} \left(\frac{x^2}{1 - \cos x} \right) = \lim_{x \to 0} \left(\frac{x^2}{1 - \cos x} \right) \left(\frac{1 + \cos x}{1 + \cos x} \right)$$

$$
\begin{aligned}
&= \lim_{x\to 0}\frac{x^2(1+\cos x)}{1-\cos^2 x}\\
&= \lim_{x\to 0}\frac{x^2(1+\cos x)}{\sin^2 x}\\
&= \lim_{x\to 0}\left(\frac{x^2}{\sin x}\right)^2(1+\cos x)\\
&= (1)^2(1+1) = 2
\end{aligned}
$$

Solution: (a)(ii) Let

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+x+4}-2}{x^2+x} &= \lim_{x\to 0}\frac{(\sqrt{x^2+x+4}-2)(\sqrt{x^2+x+4}+2)}{(x^2+x)(\sqrt{x^2+x+4}+2)}\\
&= \lim_{x\to 0}\frac{(x^2+x+4-4)}{(x^2+x)(\sqrt{x^2+x+4}+2)}\\
&= \lim_{x\to 0}\frac{(x^2+x)}{(x^2+x)(\sqrt{x^2+x+4}+2)}\\
&= \lim_{x\to 0}\frac{1}{(\sqrt{x^2+x+4}+2)}\\
&= \frac{1}{(\sqrt{4}+2)} = \frac{1}{2+2} = \frac{1}{4}.
\end{aligned}
$$

(b) Find the horizontal asymptote and the vertical asymptote for the graph of the function

$$f(x) = \frac{x^2+2}{x^2-1}.$$

Solution: (a) First, we will find the horizontal asymptote as follows:

$$
\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\left(\frac{x^2+2}{x^2-1}\right)\\
&= \lim_{x\to\infty}\frac{x^2(1+\frac{2}{x^2})}{x^2(1-\frac{1}{x^2})}\\
&= \lim_{x\to\infty}\frac{(1+\frac{2}{x^2})}{(1-\frac{1}{x^2})}\\
&= \frac{(1+0)}{(1-0)} = 1
\end{aligned}
$$

Also

$$
\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty}\left(\frac{x^2+2}{x^2-1}\right)
$$

$$= \lim_{x \to -\infty} \frac{x^2(1+\frac{2}{x^2})}{x^2(1-\frac{1}{x^2})}$$

$$= \lim_{x \to -\infty} \frac{(1+\frac{2}{x^2})}{(1-\frac{1}{x^2})}$$

$$= \frac{(1+0)}{(1-0)} = 1$$

Thus $y = 1$ is the only horizontal asymptote.
Now we will find the vertical asymptotes: As $x^2 - 1 = 0$, gives $x = \pm 1$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(\frac{x^2+2}{x^2-1} \right)$$

$$= \lim_{x \to 1^+} \frac{(x^2+2)}{(x+1)(x-1)} = +\infty$$

and

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(\frac{x^2+2}{x^2-1} \right)$$

$$= \lim_{x \to 1^-} \frac{(x^2+2)}{(x+1)(x-1)} = -\infty$$

Hence $x = 1$ is vertical asymptote. Also,

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \left(\frac{x^2+2}{x^2-1} \right)$$

$$= \lim_{x \to -1^+} \frac{(x^2+2)}{(x+1)(x-1)} = -\infty$$

and

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \left(\frac{x^2+2}{x^2-1} \right)$$

$$= \lim_{x \to -1^-} \frac{(x^2+2)}{(x+1)(x-1)} = +\infty$$

Thus $x = -1$ is also vertical asymptote.

(c) Use the definition of the derivative to find $f'(x)$ of the function $f(x) = \dfrac{1}{x+1}$; and find the equation of the tangent line to the graph of this function at the point $P(1, \frac{1}{2})$.

Solution: (c) By definition

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h} \\ &= \lim_{h\to 0}\frac{(x+1)-(x+h+1)}{h(x+h+1)(x+1)} \\ &= \lim_{h\to 0}\frac{-h}{h(x+h+1)(x+1)} \\ &= \frac{-1}{(x+1)^2} \end{aligned}$$

Substituting for $x = 1 = x_1$ in $f'(x)$, we obtain

$$m = f'(1) = \frac{-1}{(1+1)^2} = -\frac{1}{4}$$

Since an equation of the tangent line at a point $P(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

putting the values, we obtain

$$y - \frac{1}{2} = -\frac{1}{4}(x-1)$$

or

$$y = -\frac{1}{4}x + \frac{3}{4}.$$
