Midterm Exam in <u>Math 545</u>
2nd Semester 1428/1429

Pbm 1. Let $\varphi : A \to B$ be an F-algebra homomorphism, where F is a field.
(1) Show that $\mathrm{Im}(\varphi)$ is a subalgebra of B.
(2) Is $\ker(\varphi)$ always a subalgebra of A?(Explain why!)

Pbm 2. Let A be a semi-simple F-algebra, where F is a field. Prove that

$|\mathcal{M}(A)|$ is finite. ($\mathcal{M}(A)$ is a full set of representatives for the isomorphism classes of irreducible A-modules.)

Pbm 3. Let F be a field and let Γ be a finite group such that $|\Gamma|$ is not divisible by the characteristic of F. Prove that the $F\Gamma$-module $F\Gamma$ is completely reducible.

In the following G is a finite group and all characters are over $\mathbb{C}$.

Pbm 4. Let V be a $\mathbb{C}G$-module and let $U = \{u \in V : ug = u \text{ for all } g \in G\}$.
(1) Show that U is a $\mathbb{C}G$-submodule of V.
(2) Let χ be the character afforded by V and suppose $\chi \in \mathrm{Irr}(G)$ and $\chi \neq 1_G$. Prove that $U = \{0_V\}$.

Pbm 5. Let $N \lhd G$ and $g \in G$. Show that $|C_{G/N}(Ng)| \leq |C_G(g)|$. (Here $C_{G/N}(Ng)$ is the centralizer of the coset Ng in G/N.) (Hint: Use the Second Orthogonality Relation.)

Pbm 6. Let X be a $\mathbb{C}$-representation of G affording the character χ. Define

$$Z = \{g \in G : X(g) = \alpha I \text{ for some } \alpha \in \mathbb{C}\}.$$

(1) Prove that Z is a subgroup of G and that the function $\lambda : Z \to \mathbb{C}$ defined by $X(g) = \lambda(g)I$ is a linear character of Z that satisfies $\ker(\lambda) = \ker(\chi)$.
(2) Show that $Z \subseteq \{g \in G : |\chi(g)| = \chi(1)\}$.