Midterm Exam in Math 541
2nd Semester 1428/1429

Pbm 1. Let G be a group of order p^n with p a prime number. Suppose G acts on a finite set X such that $|X|$ is not divisible by p. Show that there exists $x \in X$ for which $xg = x$ for all $g \in G$. (Hint: Partition X into its orbits.)

Pbm 2. Let G be a group of order p^n with p a prime number and let $1 \neq H \triangleleft G$.

(1) Show that $H \cap Z(G) \neq 1$.

(2) Show that every normal subgroup K of G of order p must be contained in $Z(G)$.

(Hint: For (1) let $X = H - \{1\}$ and then use Pbm 1.)

Pbm 3. Let X be a G-set and suppose H is a subgroup of G. Prove that every G-orbit is a disjoint union of H-orbits.

Pbm 4. Let $N \triangleleft G$ and let X be a transitive G-set. View X as an N-set and show that every two stabilizers N_x and N_y for $x, y \in X$ are G-conjugate.

Pbm 5. Let G act on a set X and with respect to this action suppose a subgroup H of G acts transitively on X. Show that $G = HG_x$ for every $x \in X$.

Pbm 6. Let G be a group of order pq with p, q prime numbers. Show that a Sylow p-subgroup of G or a Sylow q-subgroup of G is normal in G.

Pbm 7. Suppose a subgroup K of S_n contains an odd permutation. Prove that $|K|$ is even and that $|K|/2$ is the number of odd permutations contained in K.

(Hint: Show that $KA_n = S_n$ and then consider the quotient group $K/(A_n \cap K)$.)