Math 545 Final Exam
2nd Semester 1426/1427 H

Ex 1. State Schur's lemma and then prove it.

Ex 2. Let A be an F-algebra and let V be a completely reducible A-module. Let W be an A-submodule of V. Show that V/W is a completely reducible A-module.

Ex 3. Let χ be a character of a group G. Define $\det \chi : G \to \mathbb{C}$ as follows. Choose a representation X that affords χ and set

$$(\det \chi)(g) = \det X(g)$$

(where $\det X(g)$ is the determinant of the matrix $X(g)$). Show that $\det \chi$ is a uniquely defined linear character of G, independent of the choice of X.

Ex 4. Let G be a finite group. Show that G is abelian iff every $\chi \in \mathrm{Irr}(G)$ is linear.

Ex 5. Let G be a group such that $|G| = p^r$ where p is prime and $r \geq 0$. Show that the number of linear characters of G is one iff $r = 0$.

Ex 6. Let $G =< g >$ be a cyclic group of order n. Let

$$a = e^{\frac{2\pi i}{n}} = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},$$

and define $\lambda : G \to \mathbb{C}$ by $\lambda(g^k) = a^k$, $k \in \{0, 1, 2, 3, \ldots\}$.
(a) Show that λ is a well defined linear character of G.
(b) Show that $\mathrm{Irr}(G)$ is a cyclic group of order n generated by λ.

Ex 7. Let χ be a character (not necessarily irreducible) of an abelian group G.
(a) Show that $\sum_{g \in G} |\chi(g)|^2 \geq |G|\chi(1)$.
(b) Show that $\sum_{g \in G} |\chi(g)|^2 = |G|\chi(1)$ iff $\chi = \sum_{i=1}^{r} \lambda_i$, where each λ_i is a linear character and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Ex 8. Let $G = H \times K$. Show that $\mathrm{Irr}(G) = \{\theta \times \eta : \theta \in \mathrm{Irr}(H) \text{ and } \eta \in \mathrm{Irr}(K)\}$.
