Final Exam in Math 545
2nd Semester 1428/1429

**Pbm 1.** Let $A$ be a semisimple $F$-algebra where $F$ is algebraically closed. Suppose that $A$ is commutative (i.e., $ab = ba$ for all $a, b \in A$). Show the following.
(1) Every simple $A$-module has dimension 1.
(2) If $X$ is any representation of $A$, then $X$ is similar to a representation $Y$ of $A$ such that $Y(a)$ is a diagonal matrix for every $a \in A$.

In all that follows, $G$ is a finite group and all characters are over $\mathbb{C}$.

**Pbm 2.** Let $V$ and $W$ be $\mathbb{C}G$-modules. Prove that $V$ and $W$ are isomorphic if and only if they afford equal characters.

**Pbm 3**. Suppose $G$ is abelian and write $\widehat{G} = \mathrm{Irr}(G)$.

(1) Show that $\widehat{G}$ is an abelian group under multiplication of characters.

(2) Assume $G = \langle g \rangle$ is cyclic with $|G| = n$. Let $\lambda : G \to \mathbb{C}^*$ be the map defined by $\lambda(g^k) = e^{i2\pi k/n} = \cos(2\pi k/n) + i\sin(2\pi k/n)$ for all $k \in \mathbb{Z}$. Show that $\lambda$ is a linear character of $G$, $\widehat{G} = \langle \lambda \rangle$ and that $\widehat{G} \cong G$.

**Pbm 4.** Let $\chi$ be a faithful character of $G$. Show that a subgroup $H$ of $G$ is abelian if and only if every irreducible constituent of $\chi_H$ is linear.

**Pbm 5.** Let $g \in G$. Prove that $g$ is $G$-conjugate to $g^{-1}$ if and only if $\chi(g) \in \mathbb{R}$ (the real numbers) for all characters $\chi$ of $G$.

**Pbm 6.** Let $\chi \in \mathrm{Irr}(G)$ be nonlinear. Prove that the principal character of $G'$ (the commutator subgroup of $G$) is not an irreducible constituent of $\chi_{G'}$.

**Pbm 7.** Suppose $G$ is nonabelian of order 8.

(1) Show that $G$ has a unique nonlinear irreducible character $\chi$. What's $\chi(1)$?

(2) Show that $|G'| = 2$ and that $2\chi = \lambda^G$ where $\lambda$ is the nonprincipal linear character of $G'$.(Hint: For $2\chi = \lambda^G$,consider $\chi_{G'}$.)

(3) Prove that $\chi(x) = 0$ for all $x \in G - G'$, and that $\chi(y) = -2$ for $y \in G' - \{1\}$.

**Pbm 8.** Let $H \lhd G$ and suppose $\theta \in \mathrm{Irr}(H)$. Let $\theta^G = \sum_{i=1}^m r_i\chi_i$ where $r_i \in \mathbb{Z}^+$ and $\chi_i \in \mathrm{Irr}(G)$ for all $i$.

(1) Show that $\sum_{i=1}^m r_i^2 = |G : H|$ in case $\theta$ is $G$-invariant.

(2) Deduce from (1) that, in general, $\sum_{i=1}^m r_i^2 = |T : H|$ where $T$ is the inertial group of $\theta$ in $G$.

**Pbm 9.** Assume $|\mathrm{Irr}(G)| = 2$. Prove that there is $\chi \in \mathrm{Irr}(G)$ for which $\chi(g) = -\chi(1)^{-1}$ for all $g \neq 1_G$.

**Pbm 10.** Let $H \triangleleft G$ and $\chi \in \mathrm{Irr}(G)$. Suppose $\theta$ is an irreducible constituent of $\chi_H$ and let $\theta = \theta_1, \theta_2, \ldots, \theta_r$ be all the distinct $G$-conjugates of $\theta$. Show that $\chi_H = e \sum_{i=1}^{r} \theta_i$ where $e \in \mathbb{Z}^+$ and $r = |G : T|$ where $T$ is the inertial group of $\theta$ in $G$.