Final Exam in Math 541
2nd Semester 1428/1429

In all that follows G denotes a finite group.

Pbm 1. Show that a sharply k-transitive G-set ($k \geq 1$) cannot be $(k+1)$-transitive.

Pbm 2. Suppose that G is abelian and let X be a faithful transitive G-set. Prove that X is regular.

Pbm 3. Let p be a prime number.

(1) Show that G has a unique maximal normal p-subgroup L.(Hint: Show that if H and K are normal p-subgroups of G then HK is a normal p-subgroup of G.)

(2) Show that L char G.

(3) Suppose $M \triangleleft G$ and let N be the unique maximal normal p-subgroup of M. Show that $N \triangleleft G$.

Pbm 4. Suppose that G is an abelian p-group (p is prime) and let H be the set of all elements in G of order $\leq p$. Prove that H is a subgroup of G and that H char G.

Pbm 5. Let H and K be normal nilpotent subgroups of G.

(1) Show that for every prime p, if $P \in \mathrm{Syl}_p(H)$, then $P \triangleleft G$.

(2) Show that for every prime p, if $P \in \mathrm{Syl}_p(H)$ and $Q \in \mathrm{Syl}_p(K)$, then $PQ \in \mathrm{Syl}_p(HK)$. Deduce that HK is a (normal) nilpotent subgroup of G. (Hint: Prove that $P \cap Q \in \mathrm{Syl}_p(H \cap K)$.)

(3) Show that G contains a unique maximal normal nilpotent subgroup. (Hint: Use (2).)

Pbm 6. Prove that a finite group is nilpotent if and only if it is a direct product of its Sylow subgroups.

Pbm 7. Suppose H and K are solvable subgroups of G with $H \triangleleft G$. Show that HK is a solvable subgroup of G.

Pbm 8. Assume G has order pq where p and q are prime numbers and $p > q$. Show that G is a semidirect product of $\mathbb{Z}_p$ by $\mathbb{Z}_q$. (Do not use the Schur-Zassenhaus lemma.)

Pbm 9. Suppose G is cyclic of order p^n, with p prime and $n > 0$. Prove that G does not contain any subgroup isomorphic to a group of the form $L_1 \times L_2 \times \cdots \times L_r$ where $r > 1$ and all L_i are nontrivial and cyclic.

Pbm 10. Let p and q be distinct primes and let K and Q be abelian groups with $|K| = p^n$ and $|Q| = q^m$ $(n \geq 1, m \geq 1)$. Show that, up to isomorphism, there exists a unique abelian extension of K by Q.