JACOBI FIELDS AND THEIR APPLICATIONS

BY

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A THESIS

SUBMITTED IN PARTIAL FULLFILMENT OF THE REQUIRMENTS
FOR THE DEGREE OF

MASTER OF SCIENCE

IN THE DEPARTMENT OF MATHEMATICS AT THE COLLEGE OF SCIENCE

KING SAUD UNIVERSITY

1427-2006

To

My wonderful parents, sisters,

brothers and nephew with love

ACKNOWLEDGEMENTS

Foremost, praise and thanks be to Almighty ALLAH, Most Gracious, Most Merciful, Who enabled me to reach my goal, and peace be upon His Prophet.

I would like to express my gratitude to my teacher Professor Sharief Deshmukh under whose supervision this Thesis is prepared. I am greately endebted to him for his generous and invaluable help, advice and guidance.

I would like to express my thanks to King Saud University for providing necessary support and facilities. Also I wish to extend my thanks to all my teachers and colleagues in Mathematics Department, King Saud Universtiy for their cooperation and kind help.

I am very greatful and very deeply indebted to my parents for their unconditional love, encouragement and support that I could write this Thesis. Ofcourse I cannot forget my brothers and sisters for great assistance they extended to me.

Finally, I would like to thank every one who supported me in any way or form.

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PREFACE

Jacobi vector fields are important in the study of Riemannian geometry, in particular they are a very powerful tool for an elegent study of the extrinsic and intrinsic geometry of geodesic spheres and tubes about curves and submanifolds of a Riemannian manifold. Moreover, many properties of the geometry of a Riemannian manifold (M, g) may be studied using Jacobi vector fields.

Jacobi fields are vector fields defined along a geodesic in a Riemannian manifold which satisfies a second order differential equation involiving curvature operator and are associated to variation of geodesics. The study of Jacobi vector fields leads to the study of the Jacobi differential equation which is very useful in the study of the geometry..

The goal of the present thesis is to give an application of Jacobi vector fields, use Jacobi vector fields to give proofs of some theorems and to show how the Jacobi differential equation can be solved completely in some cases.

The thesis is divided into six chapters and each chapter is divided into subsections and the results in each section are numbered as (a.b.c), for instance Theorem a.b.c, means Theorem number c in the section b of chapter a.

The first chapter is introductory and is basically intended to make the thesis as self-contained as possible. In this chapter we gave basic definitions and summarized the basic formulas and results on Riemannian manifolds, submanifolds, which are essential for other chapters.

In second chapter, we introduce Jacobi fields and state the basic information on Jacobi fields along a geodesic in a Riemannian manifold and their properties.

In third chapter, in first section we first study variation of arc-length and energy, and then we conclude the first and second variation formula. In second section, we use the variation formulas to prove the fundamental theorems of Myers, Sunge and Rauch. In third section, we study the generalizations of Myers theorem and all the results in this section are taken from [33].

One of the interesting questions in geometry of a Riemannian manifolds is to investigate whether given compact Riemannian manifold can be immersed into another Riemannian manifold. Recently Moore [35] using applications of second variation formula obtained most generalized non-immersibility result. In chapter four, we have described his proof in more detail.

The fifth chapter is an application of Jacobi fields, all the results in this chapter are taken from [28] and [52]. In first and second section, we give some preparatory material. In third section, we derive a basic formula for the shape operator of geodesic sphere using the method of Jacobi vector fields. In forth section, we show how the shape operator associated with geodesic sphere can be expressed in terms of associated Jacobi vector fields and then we apply this technique to obtain simple proofs of some theorems. In last section the theorem characterizing Riemannian manifolds of constant sectional curvature is proved.

Chapter six is devoted to study of Sasakian geometry. In first section, we treat some general preliminaries about Sasakian manifolds and we discuss a very interesting example of Sasakian manifold. In second section, we consider the Jacobi equation in particular case, namely when the manifold is Sasakian space form. In this special case, the Jacobi equation takes a diffrenet but nice form and it is still possible to solve it completely. All the results in this section are taken from [7].

The thesis ends with a list of references, which by no means is exhaustive on the subject, but lists only those references which have either been directly used in the thesis or have relevance to our work.

CHAPTER I

INTRODUCTION

In this introductory chapter, we aim at making the thesis as self-contained as possible as well as to fix the notations so as to mention the uniformity in the symbols and formulas. Therefore, we start in section 1 with definition of a smooth function on a smooth manifold and end at smooth forms and exterior differential operator. In section 2, we describe Riemannian manifolds, Riemannian connection, geodesic; parallel transport map, the exponential map, normal coordinates; and properties of curvature tensor fields. In section 3, we define submanifolds of a Riemannian manifold and state the fundamental equations of Gauss, Codazzi and Ricci.

1.1 SMOOTH MANIFOLDS

Definition 1.1.1 Let M be an n-dimensional smooth manifold and $f : M \to R$ be a function. The function f is said to be smooth at $p \in M$ if for each chart (U, ϕ) around p, $f \circ \phi^{-1}$ is smooth at $\phi(p)$.

Theorem 1.1.1 Let M be an n-dimensional smooth manifold. If $f : M \to R$ is such that there is a chart (U, ϕ) around $p \in M$ with $f \circ \phi^{-1}$ is smooth at $\phi(p)$. Then f is smooth at $p \in M$.

In light of above theorem, it is enough check the smoothness with respect to just one chart around p.

A function f is said to be smooth if it is smooth at each point. We will denote by $C^\infty(p)$ the set $\{f : U \to R \mid U$ a neighborhood of p and f is smooth$\}$.

Example 1.1.1 Take $M = S^2$ and $p = (1,1,0) \in S^2$. Then for the function $f : S^2 \to R$, $f(x,y,z) = xy + yz + zx$, we have $f \in C^\infty(p)$ and for $g : S^2 \to R$, $g(x,y,z) = (\sqrt{xy}, y, z)$, we have $g \notin C^\infty(p)$.

Proposition 1.1.1 $C^\infty(p)$ is a ring.

We have associated at each $p \in M$ a ring $C^\infty(p)$, and thus ready to define a tangent vector X_p at $p \in M$ as follows:

Definition 1.1.2 Let M be an n-dimensional smooth manifold. A tangent vector X_p at $p \in M$ is a map $X_p : C^\infty(p) \to R$ satisfying:

1. $X_p(\lambda f + \mu g) = \lambda X_p(f) + \mu X_p(g)$,
2. $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$, $f, g \in C^\infty(p)$, $\lambda, \mu \in R$.

We will denote by T_pM the tangent space to M at $p \in M$.

Proposition 1.1.2 T_pM is an n-dimensional vector space over R.

Next, we shall study differentiability of functions from smooth manifolds to smooth manifolds.

Definition 1.1.3 Let M and N be the smooth manifolds. A map $f : M \to N$ is said to be smooth at p if

1. f is continuous at p.
2. for a chart (U, ϕ) around $p \in M$, there is a chart (V, ψ) around $f(p) \in N$ such that $f(U) \subset V$ and the map $\psi \circ f \circ \phi^{-1}$ is smooth at $\phi(p)$.

A map $f : M \to N$ is said to be smooth if it is smooth at each point of M.

It is easy to see that the natural projection $\pi : S^n \to RP^n$ is smooth map as well as the map $f : S^2 \to S^1$, $f(x, y, z) = (\cos xyz, \sin xyz)$.

Definition 1.1.4 Let $f : M \to N$ be a smooth map and $p \in M$. Then the differential of f at $p \in M$ is a map

$$df_p : T_pM \to T_{f(p)}N,$$

$$df_p(X_p)(g) = X_p(f^*(g)),\ X_p \in T_pM, g \in C^\infty(f(p))$$

where $f^* : C^\infty(f(p)) \to C^\infty(p)$ is ring homomorphism defined by $f^*(g) = g \circ f$.

It is easy to show that df_p is linear map and that we have:

Theorem 1.1.2 (Chain rule). Let $f : M \to N$ and $g : N \to L$ are smooth maps. Then for $p \in M$,

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

Corollary 1.1.1 If $f : M \to N$ is a diffeomorphism then $df_p : T_pM \to T_{f(p)}N$ is an isomorphism satisfying $(df_p)^{-1} = df_q^{-1}$, $f(p) = q$.

Corollary 1.1.2 (Inverse function theorem). If $f : M \to N$ is smooth map such that there is a $p \in M$ with $df_p : T_pM \to T_{f(p)}N$ is an isomorphism. Then there exist neighborhoods U and V of p and $f(p)$ respectively such that $f : U \to V$ is a diffeomorphism.

Definition 1.1.5 Let $f : M \to N$ be a smooth map. Then

1. f is said to be an immersion if $df_p : T_pM \to T_{f(p)}N$ is injection, for all $p \in M$.
2. f is said to be an imbedding if f is one-to-one immersion.

Definition 1.1.6 Let M be an n-dimensional smooth manifold. We denote by $TM = \cup_{p \in M} T_pM$ and call TM the tangent bundle of M.

We denote the elements of TM by (p, X_p), where $X_p \in T_pM$. Thus $TM = \{(p, X_p) : p \in M, X_p \in T_pM\}$ and we have natural map $\pi : TM \to M$, $\pi(p, X_p) = p$, which is on to map called projection map. Also, T_p^*M is the dual space of the tangent space T_pM called the cotangent space of M at p and $T^*M = \cup_{p\in M} T_p^*M$ called the cotangent bundle of M. It can be shown that for each chart (U, ϕ), $(\pi^{-1}(U), \Phi)$ is a chart on TM where $\Phi : \pi^{-1}(U) \to R^{2n}$ is defined by $\Phi(p, X_p) = (\phi(p), d\phi_p(X_p))$ which makes TM a $2n$-dimensional smooth manifold. Similarly, T^*M is also a $2n$-dimensional smooth manifold.

Definition 1.1.7 Let M be a smooth manifold. Then a smooth map $X : M \to TM$ satisfying $\pi \circ X = i_d : M \to M$ is called a smooth vector field. We denote by $\mathfrak{X}(M)$ the set of smooth vector fields.

Remark 1.1.1 Given a $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we define $X(f) : M \to R$ by $X(f)(p) = X_p(f)$, where $X : p \to X_p$. Using differentiable structure of TM it can be shown that $X(f)$ is a smooth map and consequently a vector field X is also a derivation $X : C^\infty(M) \to C^\infty(M)$. Conversely given $X : C^\infty(M) \to C^\infty(M)$ a derivation of ring $C^\infty(M)$, it can be shown that $X : M \to TM$ is smooth satisfying $\pi \circ X = i_d$ that is $X \in \mathfrak{X}(M)$.

Definition 1.1.8 For $X, Y \in \mathfrak{X}(M)$, we define the map

$$[\ ,\ ] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), (X, Y) \to [X, Y]$$

by $$[X, Y](f) = X(Y(f)) - Y(X(f)) \in C^\infty(M) \qquad \forall f \in C^\infty(M).$$

The vector field $[X,Y] \in \mathfrak{X}(M)$ called Lie-bracket of X and Y.

Proposition 1.1.3 $\mathfrak{X}(M)$ is a lie-algebra over $C^{\infty}(M)$.

Definition 1.1.9 A smooth curve on M is a smooth map $\alpha : (a,b) \to M$. If for $t_\circ \in (a,b)$, $p = \alpha(t_\circ) \in M$, then we say that α passes through $p \in M$.

Definition 1.1.10 Given a vector field X on a manifold M and a smooth curve $\alpha : (a,b) \to M$, we say that α is an integral curve of X if $X(\alpha(t)) = \alpha^{\cdot}(t)$, where $\alpha^{\cdot}(t) = d\alpha_t\left(\frac{d}{dt}\,|_t\right)$ is the tangent vector to the curve α at $t \in (a,b)$.

Theorem 1.1.3 Let $\sigma : I \to M$ and $\rho : J \to M$ be integral curves of $X \in \mathfrak{X}(M)$ such that $\sigma(0) = \rho(0) = p \in M$. Then $\sigma(t) = \rho(t), \quad t \in I \cap J$.

Definition 1.1.11 Let M be a smooth manifold. A collection of diffeomorphism $\{\phi_t\}$, $t \in R$ is called a one parameter group of transformation of M if it satisfies the following:

1. $\phi_\circ = i_d : M \to M$.
2. $\phi_t \circ \phi_s = \phi_{t+s}, \quad t,s \in R$.
3. The map $\phi : R \times M \to M, \quad \phi(t,p) = \phi_t(p)$ is smooth.

Definition 1.1.12 Let $\{\phi_t\}$ be a one parameter group of transformation of M . For a fixed $p \in M$, we define a smooth map $\sigma : R \to M$ by $\sigma(t) = \phi_t(p)$ such that $\sigma(0) = p$, that is, it is a smooth curve passing through p called orbit of $\{\phi_t\}$ at p.

Definition 1.1.13 Define $X_p \in T_pM$ by $X_p = \sigma^{\cdot}(0)$. This gives us $X : M \to TM,\ p \to X_p = \sigma^{\cdot}(0)$ a smooth vector field called vector field induced by $\{\phi_t\}$.

Definition 1.1.14 A vector field $X \in \mathfrak{X}(M)$ induced by a one parameter group of transformation of M is called complete vector field.

Each $X \in \mathfrak{X}(M)$ need not be complete. However if we modify definition from global M and R to nieghborhoods in M and open intervals in R we get what is called local one parameter group of transformations, which also induce a vector field X and the converse also holds.

Definition 1.1.15 A smooth map $\omega : M \to T^*M$ satisfying $\pi \circ \omega = i_d$, is called a smooth 1-form.

For a smooth manifold we denote by $\Lambda^1(M)$ the space of smooth 1-form, we also call $\Lambda^0(M) = C^\infty(M)$ the space of 0-forms.

As in the case of smooth vector fields a smooth 1-form $\omega \in \Lambda^1(M)$ gives rise to a map $\omega : \mathfrak{X}(M) \to C^\infty(M)$ where $\omega(X)$ is defined by $\omega(X)(p) = \omega_p(X_p) \in R, X \in \mathfrak{X}(M)$.

Definition 1.1.16 An alternating (skew-symmetric) bilinear map

$$\alpha : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k-times} \to C^{\infty}(M)$$

is called smooth k-form and we denote by $\Lambda^k(M)$ the space of smooth k-forms.

Definition 1.1.27 Let M be a smooth manifold. Define

$$d : \Lambda^k(M) \to \Lambda^{k+1}(M), \qquad \omega \to d\omega$$

by

$$\begin{aligned} d\omega(X_0, ..., X_k) &= \sum_{i=0}^{k} (-1)^i X_i \left(\omega\left(X_0, ..., \widehat{X_i}, ..., X_k\right)\right) \\ &+ \sum_{0 \le i < j \le k}^{k} (-1)^{i+j} \omega\left([X_i, X_j], X_0, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_k\right), \end{aligned}$$

where $\widehat{X_i}$ means X_i is removed. This map is called the exterior differential operator.

Definition 1.1.18 Let $\alpha \in \Lambda^k(M), \beta \in \Lambda^l(M)$. Then we define the wedge product of α and β, $\alpha \wedge \beta \in \Lambda^{k+l}(M)$ by

$$\begin{aligned} (\alpha \wedge \beta)(X_1, ..., X_{k+l}) &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^{\sigma} \alpha\left(X_{\sigma(1)}, ..., X_{\sigma(k)}\right) \\ &\quad \beta\left(X_{\sigma(k+1)}, ..., X_{\sigma(k+l)}\right) \end{aligned}$$

where S_{k+l} is the symmetric group of permutations on the set $\{1, 2, ..., k+l\}$.

Theorem 1.1.4 The wedge product satisfies the following properties:

1. $\alpha \wedge \alpha = 0$, $\alpha \in \Lambda^k(M)$ where k is odd.
2. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$, $\alpha \in \Lambda^k(M), \beta \in \Lambda^m(M), \gamma \in \Lambda^l(M)$.
3. $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$, $\alpha \in \Lambda^k(M), \beta, \gamma \in \Lambda^l(M)$.
4. $\alpha \wedge \beta = (-1)^k \beta \wedge \alpha$, $\alpha \in \Lambda^k(M), \beta \in \Lambda^l(M)$.

Theorem 1.1.5 The exterior differential operator d satisfies:

1. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, $\alpha \in \Lambda^k(M), \beta \in \Lambda^l(M)$.
2. $d \circ d = 0$.

1.2 RIEMANNIAN MANIFOLDS

Definition 1.2.1 A connection ∇ on a smooth manifold M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad (X, Y) \to \nabla_X Y$$

which satisfies the following properties:

1. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

2. $\nabla_{(X+Y)}Z = \nabla_X Z + \nabla_Y Z$.

3. $\nabla_{fX} Y = f\ \nabla_X Y$.

4. $\nabla_X fY = X(f)\,Y + f\ \nabla_X Y$, $\forall X, Y, Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

Example 1.2.1 Let $\{x_1, ..., x_n\}$ be the coordinate system on R^n. Then for $X, Y \in \mathfrak{X}(R^n)$, we have $X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}$, $Y = \sum_{i=1}^{n} g^i \frac{\partial}{\partial x^i}$, $f^i, g^i \in C^\infty(R^n)$. Define $\nabla : \mathfrak{X}(R^n) \times \mathfrak{X}(R^n) \to \mathfrak{X}(R^n)$ by $\nabla_X Y = \sum_{i=1}^{n} X\left(g^i\right) \frac{a}{ax^i}$, then it can be easily verified that ∇ satisfies the requirement for a connection. This connection ∇ on R^n is known as the Euclidean connection.

Definition 1.2.2 A Riemannian metric g on a smooth manifold M is a tensor of type $(0, 2)$ which satisfies:

1. $g(X, Y) = g(Y, X)$.

2. $g(fX + hY, Z) = fg(X, Z) + hg(Y, Z)$.

3. If $g(X, Y) = 0, \forall X \in \mathfrak{X}(M)$, then $Y = 0$.

4. $g(X, X) > 0, \forall X \neq 0, \quad X, Y, Z \in \mathfrak{X}(M)$, and $f, h \in C^\infty(M)$.

A smooth manifold M together with a given Riemannian metric g is called a Riemannian manifold, and is denoted by (M, g).

Remark 1.2.1 An immersion $f : M \to \overline{M}$ where $\left(\overline{M}, \overline{g}\right)$ is Riemannian manifold, gives $g = f^{*}\overline{g}$ on M which it self is a Riemannian metric, called the induced metric by the immersion.

Examples 1.2.2

1. $(R^n, <,>)$ is a Riemannian manifold called Euclidean space.
2. (S^n, g) is a Riemannian manifold where g is induced by $i : S^n \to R^{n+1}$.

Theorem 1.2.1 Given a Riemannian manifold M, there exists a unique connection ∇ on M satisfying the conditions:

1. $\nabla_X Y - \nabla_Y X = [X, Y]$.
2. $X.g\,(Y, Z) = g\,(\nabla_X Y, Z) + g\,(Y, \nabla_X Z)\,,\ X, Y, Z \in \mathfrak{X}(M)$.

The unique connection ∇ on the Riemannian manifold (M, g) satisfying (1), (2) in above theorem is called Riemannian connection.

Definition 1.2.3 Let M be a smooth manifold with a connection ∇. A vector field V along a curve $\alpha : I \to M$ is called parallel when $\frac{DV}{dt} = 0$, for all $t \in I$, where if $V\,(t) = X\,(\alpha\,(t))$, $\frac{DV}{dt}\,(t) = (\nabla_{\alpha^{\cdot}} X)\,(t)$.

Theorem 1.2.2 Let M be a smooth manifold with a connection ∇ and $\alpha : I \to M$ be a smooth curve with $\alpha\,(t_\circ) = p \in M$. If $X_p \in T_pM$ then there exists a unique vector field $V\,(t)$ parallel along α such that $V\,(t_\circ) = X_p$.

The unique vector field $V(t)$ generated by above theorem is called parallel translation of X_p along α.

Definition 1.2.4 Let ∇ be a connection on a smooth manifold M and $\alpha : I \to M$ be a smooth curve. We define a map

$$\begin{aligned} P^{\alpha}_{t_\circ t} &: T_{\alpha(t_\circ)}M \to T_{\alpha(t)}M,\ t_\circ, t \in I \\ \text{by } P^{\alpha}_{t_\circ t}(X_p) &= V(t) \in T_{\alpha(t)}M,\ p = \alpha(t_\circ) \end{aligned}$$

where $V(t)$ is the unique parallel vector field along α such that $V(t_\circ) = X_p$. This map $P^{\alpha}_{t_\circ t}$ is called parallel transport map.

Definition 1.2.5 Let ∇ be a connection on M. Then a smooth curve $\alpha : I \to M$ is said to be a geodesic w.r.t ∇ if $\frac{D\alpha^{\cdot}}{dt} = 0$, that is, the tangent vector field $\alpha^{\cdot}(t)$ is parallel along α.

Remark 1.2.2 Consider a chart (U, ϕ) around $p \in M$ with local coordinates $\left(x^1, ..., x^n\right)$ the equation of a geodesic $\frac{D\alpha^{\cdot}}{dt} = 0$ is equivalent to the system of second order differential equations

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{n} \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma^k_{ij} = 0,\ k = 1, ..., n$$

where the connection coeffiecients Γ^k_{ij} are defined as

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

Proposition 1.2.1 If (M, g) be a Riemannian manifold, $p \in M$ and $X_p \in$

T_pM then there is a number $\epsilon > 0$ and a unique geodesic $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$, $\gamma^{\cdot}(0) = X_p$.

Remark 1.2.3 In light of above proposition, we shall denote the unique geodesic with initial point p and initial tangent vector X_p by γ_{X_p}. Thus $\gamma_{X_p} : (-\epsilon, \epsilon) \to M$ is the geodesic satisfying $\gamma_{X_p}(0) = p, \dot{\gamma_{X_p}} = X_p$.

Theorem 1.2.3 (Rescalling Lemma). Let $a > 0$ and $X_p \in T_pM$. Then $\gamma_{aX_p}(t) = \gamma_{X_p}(at)$ holds for all $t \in Dom\gamma_{aX_p}$.

Remark1.2.4 The advantage of this lemma is that, if we decrease the length of $X_p \in T_pM$, the domain of corresponding geodesic increases. Then we can always choose an open ball $B_\epsilon(0) \subset T_pM$, $\epsilon > 0$ such that $\forall X_p \in B_\epsilon(0), 1 \in Dom\gamma_{X_p}$.

Definition 1.2.6 Let (M, g) be a Riemannian manifold and $p \in M$. Define a map $\exp_p : B_\epsilon(0_p) \to M$ by $\exp_p(X_p) = \gamma_{X_p}(1)$ where $B_\epsilon(0_p)$ is an open ball in T_pM around $0_p \in T_pM$ and γ_{X_p} is the unique geodesic satisfying $\gamma_{X_p}(0) = p$, $\dot{\gamma_{X_p}}(0) = X_p$. This map is called the exponential map at $p \in M$.

It can be easily verified using the smooth structure on the tangent bundle TM of M that the exponential map $\exp_p$ is smooth and $d\left(\exp_p\right)_{0_p} = i_d : T_pM \to T_pM$ is an isomorphism, by inverse function theorem there exist neighborhoods $N_\circ$ and N_p of 0_p and p respectively such that $\exp_p$ is a diffeomorphism.

The neighborhood N_p of $p \in M$ obtained from the diffeomorphism $\exp_p : N_0 \to N_p$ is called the normal neighborhood. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM and take $F : T_pM \to R^n$ as $F(X_p) = (\lambda^1, \ldots, \lambda^n)$, where $X_p = \sum_i \lambda^i e_i$.Then the chart (N_p, ϕ) where $\phi = F \circ \exp_p^{-1}$, is called normal chart and its local coordinates are called the normal coordinates.

Lemma 1.2.1 If $x^1, \ldots, x^n$ are normal coordinates around $p \in M$, then

1. The geodesic $\gamma_{X_p} : I \to M, X_p \in T_pM$ is given by

$$\gamma_{X_p}(t) = \phi^{-1}\left(t\lambda^1, \ldots, t\lambda^n\right)$$

   where (N_p, ϕ) is normal chart around $p \in M$ and $X_p = \sum_i \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p$.

2. $g_{ij}(p) = \delta_{ij}$.
3. $\Gamma^k_{ij}(p) = 0$.
4. $\left(\frac{\partial}{\partial x^i}\right)_p g_{jk} = 0$.

Theorem 1.2.4 Let N_p be the normal neighborhood of $p \in M$. Then for $q \in N_p$, there exists a unique geodesic joining p to q.

Theorem 1.2.5 Let N_p be the normal neighborhood of $p \in M$. Then for $q \in N_p$, the geodesic segment joining p to q is the shortest of all smooth curves in N_p joining p to q.

Definition 1.2.7: For a connection ∇ on a smooth manifold M, there is associated a tensor field R of type $(1,3)$ called the curvature tensor field of the connection ∇, defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X,Y,Z \in \mathfrak{X}(M).$$

Theorem 1.2.6 The curvature tensor R of the Riemannian connection ∇ on a Riemannian manifold (M,g) has the following properties:

1. $R(X,Y)Z + R(Y,X)Z = 0.$
2. $R(X+Y,Z)W = R(X,Z)W + R(Y,Z)W.$

   $R(X,Y+Z)W = R(X,Y)W + R(X,Z)W.$

   $R(X,Y)(Z+W) = R(X,Y)Z + R(X,Y)W.$
3. $R(fX,gY)hZ = fghR(X,Y)Z, \quad X,Y,Z \in \mathfrak{X}(M).$

Theorem 1.2.7 (Bianchi's Identity).

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \quad X,Y,Z \in \mathfrak{X}(M).$$

Definition 1.2.8 Let R be the curvature tensor field of the Riemannian connection on a Riemannian manifold (M,g). Then define the Riemannian curvature tensor field as

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Theorem 1.2.8 The Riemannian curvature tensor field has the properties:

1. $R(X, Y, Z, W) + R(Y, X, Z, W) = 0.$
2. $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0.$
3. $R(X, Y, Z, W) = R(Z, W, X, Y).$
4. $R(X, Y, Z, W) + R(X, Y, W, Z) = 0.$

Proposition 1.2.2 Let $\sigma \subset T_pM$ be a 2-dimensional subspace of T_pM and let $x, y \in \sigma$ be two linearly independent vectors. Then

$$K(x, y) = \frac{R(x, y, y, x)}{| x \wedge y |^2}$$

does not depend on the choice of the vectors $x, y \in \sigma$.

Definition 1.2.9 Given a point $p \in M$ and a 2-dimensional subspace $\sigma \subset T_pM$, the real number $K(x, y) = K(\sigma)$, where $\{x, y\}$ is any basis of σ, is called the sectional curvature of σ at p.

Definition 1.2.10 If all sectional curvature at all points of M are equal to constant C, then M is said to have constant curvature C.

Theorem 1.1.9 A Riemannian manifold (M, g) is of constant curvature C if and only if its Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = C\{g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\}, \quad X,Y,Z,W \in \mathfrak{X}(M).$$

Example 1.2.3 Let $(R^n, <,>)$ be the Euclidean space and ∇ be the Euclidean connection on R^n. Then $R(X,Y)Z = 0$ and $R(X,Y,Z,W) = 0, \forall X,Y,Z,W \in \mathfrak{X}(R^n)$. Hence $\forall p \in R^n$ and all $\pi \in T_pR^n$, $K(\pi) = 0$ which means that R^n is of constant sectional curvature zero.

Remark 1.2.5 The Riemannian curvature tensor filed of

1. (R^n, g) is given by $R(X,Y,Z,W) = 0$.
2. (S^n, g) is given by $R(X,Y,Z,W) = \{g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\}$.
3. (H^n, g) is given by $R(X,Y,Z,W) = \{g(X,Z)\,g(Y,W) - g(Y,Z)\,g(X,W)\}$.

Definition 1.2.11 Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on a Riemannian manifold (M,g). Then the Ricci tensor field Ric of M is defined by

$$Ric(X,Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i), \qquad X,Y \in \mathfrak{X}(M)$$

and the scalar curvature S of M is the trace of the Ricci tensor, that is S is defined by

$$S = \sum_{i=1}^{n} Ric(e_i, e_i).$$

Definition 1.2.12 A Riemannian manifold (M, g) is called an Einstein manifold if for all $X, Y \in \mathfrak{X}(M)$, $Ric\,(X, Y) = \lambda g\,(X, Y)$, where $\lambda : M \to R$ is a smooth function.

Definition 1.2.13 For a Riemannian manifold (M, g) of dimension n, with a local orthonormal frame $\{e_1, \ldots, e_n\}$, we define the following:

(1) The gradient of a function $f \in C^\infty\,(M)$ is a smooth vector field grad f (or ∇f) on M defined by

$$g\,(grad\ f, X) = X\,(f)\,, \forall X \in \mathfrak{X}(M).$$

(2) The divergence of a vector field $X \in \mathfrak{X}(M)$ is a smooth function $div\ X$ on M defined by

$$divX = \sum_{i=1}^{n} g\,(\nabla_{e_i} X, e_i)\,.$$

(3) The Hessian, H_f of a function $f \in C^\infty\,(M)$ is the symmetric $(0, 2)$ tensor field such that

$$H_f\,(X, Y) = g\,(\nabla_X grad\ f, Y) = XY\,(f) - (\nabla_X Y)\,(f)\,, \quad X, Y \in \mathfrak{X}(M).$$

We also use notation $d^2 f$ for the Hessian of f.

(4) The Laplacian, Δf of a function $f \in C^{\infty}(M)$ is defined by

$$\Delta f = \sum_{i=1}^{n} H_f(e_i, e_i).$$

Lemma 1.2.2 On a Riemannian manifold (M, g) the following hold

1. $\Delta f = div(\nabla f)$.
2. $div(fX) = X(f) + f div(X)$.
3. $\Delta(fh) = f\Delta h + h\Delta f + 2g(\nabla f, \nabla h)$.

1.3 SUBMANIFOLDS

Given two smooth manifold M and $\overline{M}$, if there exists a smooth immersion $f : M \to \overline{M}$, then we say that M is a submanifold of $\overline{M}$. If f an imbedding, then M is said to be an imbedded submanifold of $\overline{M}$.

If $\overline{M}$ is a Riemannian manifold with a Riemannian metric $\overline{g}$, then the submanifold M becomes Riemannian manifold (M, g) where g is the induced metric by the immersion.

Remark 1.3.1 (1) Since an immersion $f : M \to \overline{M}$ (by an implicit function theorem) is a local imbedding. Therefore, when we are dealing with local expressions on a submanifold M of $\overline{M}$, we shall identify $df\,(X)$ with $X \in \mathfrak{X}(M)$.

(2) Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\overline{M}$, the difference $m-n$ is called the codimension of M in $\overline{M}$.

For a $p \in M$, let $T_p^{\perp}M = \{N_p \in T_p\overline{M} \mid \overline{g_p}\,(N_p, X_p) = 0, \forall X_p \in T_pM\}$, where T_pM is the tangent space of M at p then $T_p^{\perp}M$ is a subspace of $T_p\overline{M}$, precisely it is the orthogonal complement of T_pM in $T_p\overline{M}$, in fact we have $T_p\overline{M} = T_pM \oplus T_p^{\perp}M$, where $\dim T_p^{\perp}M = m-n$. The subspace $T_p^{\perp}M$ is called the normal space of M at p. Moreover we call $\nu = \underset{p\in M}{\cup} T_p^{\perp}M$ the normal bundle of M, and so $T\overline{M}\mid_M = TM \oplus \nu$ and $\mathfrak{X}\left(\overline{M}\right)\mid_M = \mathfrak{X}(M) \oplus \Gamma\,(\nu)$, where $\Gamma\,(\nu)$ is the space of normal vector fields on M. Thus if $\overline{\nabla}$ be the Riemannian connection on $\overline{M}$, then for $X, Y \in \mathfrak{X}(M)$ we can express $\overline{\nabla}_XY$ as follows

$$\overline{\nabla}_XY = \nabla_XY + h\,(X,Y) \tag{1.3.1}$$

where $\nabla_XY \in \mathfrak{X}(M)$ is the tangential component of $\overline{\nabla}_XY$ to M and $h\,(X,Y)$ is the normal component of $\overline{\nabla}_XY$ to M. Using the properties of $\overline{\nabla}$ we can prove that ∇ defines the Riemannian connection on M with respect to the induced Riemannian metric g on M.

The normal component $h\,(X,Y)$ defines a map $h : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma\,(\nu)$ which satisfies:

(i) $h(X, Y+Z) = h(X,Y) + h(X,Z)$.

(ii) $h(X,Y) = h(Y,X)$.

(iii) $h(fX,Y) = fh(X,Y), \qquad \forall X,Y,Z \in \mathfrak{X}(M), f \in C^{\infty}(M)$.

This map h is called the second fundamental form of the submanifold and the formula (1.3.1) is called the Gauss formula.

Similarly for $X \in \mathfrak{X}(M)$ and $N \in \Gamma(\nu)$, we have $\overline{\nabla}_X N \in \mathfrak{X}\left(\overline{M}\right)$ and thus we can express $\overline{\nabla}_X N$ as

$$\overline{\nabla}_X N = -S_N X + \nabla^{\perp}_X N \tag{1.3.2}$$

where $-S_N X$ is the tangential component of $\overline{\nabla}_X N$ to M and $\nabla^{\perp}_X$ is the normal component of $\overline{\nabla}_X N$ to M.

The formula (1.3.2) is called the Weingarten formula and the map $S_N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is called the Weingarten map or the shape operator with respect to $N \in \Gamma(\nu)$ which has the following properties:

1. $S_N fX + gY = fS_N X + gS_N Y$.
2. $S_{fN_1+gN_2} X = fS_{N_1} X + gS_{N_2} X$.
3. $g(S_N X, Y) = g(S_N Y, X), \qquad \forall X, Y \in \mathfrak{X}(M),\ N, N_1, N_2 \in \Gamma(\nu)$,

   $f, g \in C^{\infty}(M)$.

Also the symbol $\nabla^{\perp}$ appearing in the normal component of $\overline{\nabla}_X N$ defines the connection in the normal bundle ν over M called the normal connection.

Remark 1.3.2 The shape operator S_N and the second fundamental form of M are related by

$$g\left(S_N X, Y\right) = g\left(h\left(X, Y\right), N\right), \qquad X, Y \in \mathfrak{X}(M), N \in \Gamma\left(\nu\right).$$

Proposition 1.3.1 If $\overline{R}, R$ and $R^{\perp}$ denote the curvature tensor field of the connection $\overline{\nabla}, \nabla$ and $\nabla^{\perp}$ respectively, then we have the following equations:

(a) Gauss equation:

$$\begin{aligned} R\left(X, Y, Z, W\right) \;\; &= \;\; \overline{R}\left(X, Y, Z, W\right) + g\left(h\left(Y, Z\right), h\left(X, W\right)\right) \\ &\quad -g\left(h\left(X, Z\right), h\left(Y, W\right)\right), \quad X, Y, Z, W \in \mathfrak{X}(M) \end{aligned}$$

(b) Codazzi's equation:

$$\left[\overline{R}\left(X, Y\right) Z\right]^{\perp} = \left(\nabla'_x h\right)\left(Y, Z\right) - \left(\nabla'_Y h\right)\left(X, Z\right), \quad X, Y, Z \in \mathfrak{X}(M)$$

where $[\ ]^{\perp}$ denotes the normal component of $[\ ]$ and the term $\left(\nabla' h\right)\left(Y, Z\right)$ stands for

$$\nabla^{\perp}_x h\left(Y, Z\right) - h\left(\nabla_X Y, Z\right) - h\left(Y, \nabla_X Z\right)$$

(c) Maindary's equation:

$$\bar{R}\left(X, Y, N_1, N_2\right) = R^{\perp}\left(X, Y, N_1, N_2\right) - g\left(\left[S_{N_1}, S_{N_2}\right]\left(X\right), Y\right),$$

$X, Y \in \mathfrak{X}(M), N_1, N_2 \in \Gamma(\nu)$, where $[S_{N_1}, S_{N_2}](X) = S_{N_1}S_{N_2}X - S_{N_2}S_{N_1}X$.

Definition 1.3.1 Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $(\overline{M}, g)$, then

(1) M is called minimal if $H = 0$, where $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$ is the mean curvature vector field on M and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

(2) M is called totally umbilical if $h(X, Y) = g(X, Y)H$.

Definition 1.3.2 A submanifold M of a Riemannian manifold $(\overline{M}, g)$ is said to be totally geodesic if geodesics in M are carried in to geodesics in $\overline{M}$.

Remark 1.3.3 From above definition it follows that M is a totally geodesic submanifold of $\bar{M}$ if and only if $h(X, Y) = 0, X, Y \in \mathfrak{X}(M)$.

Definition 1.3.3 If M is an n-dimensional submanifold of an $n+1$-dimensional Riemannian manifold $(\overline{M}, g)$. Then M is called a hypersurface of $\overline{M}$.

In this case $T_p^{\perp}M$ is 1-dimensional. Then $\Gamma(\nu)$ is spanned by a single unit normal vector field N. Thus $X.g(N, N) = 0$ gives $g\left(\nabla_X^{\perp}N, N\right) = 0$, that is, $\nabla_X^{\perp}N = 0$. Since there is only one unit vector field, we will write S_N as S. The second fundamental form h takes the form $h(X, Y) = g(SX, Y)N$. Thus the Gauss and Weingarten formula for a hypersurface M of $\overline{M}$ take the form

$$\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y) N, \qquad X, Y \in \mathfrak{X}(M) \tag{1.3.3}$$

$$\bar{\nabla}_X N = -SX, \qquad X \in \mathfrak{X}(M) \tag{1.3.4}$$

and the equations of Gauss and Codazzi take the form

$$\begin{aligned} R(X, Y, Z, W) &= \overline{R}(X, Y, Z, W) + g(SY, Z) g(SX, W) \\ &\quad - g(SX, Z) g(SY, W) \end{aligned} \tag{1.3.5}$$

$$[\overline{R}(X, Y) Z]^{\perp} = g((\nabla S)(X, Y) - (\nabla S)(Y, X), Z) N, \tag{1.3.6}$$

$X, Y, Z \in \mathfrak{X}(M)$, where $(\nabla S)(X, Y) = \nabla_X SY - S\nabla_X Y$.

Definition 1.3.4 Let M be an n-dimensional hypersurface of a Riemannian manifold $\overline{M}$ and $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M. Then $\alpha = \frac{1}{n}\sum_{i=1}^{n} g(Se_i, e_i)$ defines a smooth function on M called the mean curvature of M.

Example 1.3.1 Consider $S^n(c) = \{(x_1, \ldots, x_{n+1}) \in R^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = a^2\}$ the $n-$sphere in R^{n+1}. Then $\psi : S^n(c) \to R^{n+1}$, the inclusion map satisfies $d\psi_p = I_p$ the identity map $\forall p \in S^n(c)$. Thus ψ is an immersion making $S^n(c)$ a hypersurface of R^{n+1}. The Riemannian connection $\bar{\nabla}$ on R^{n+1} defined in Example 1.2.1 by $\bar{\nabla}_X Y = \sum_{i=1}^{n+1} X(f^i) \frac{\partial}{\partial x^i}$, where $X \in \mathfrak{X}(M)$ and $Y = \sum_{i=1}^{n+1} f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(M), f^i \in C^\infty(M)$. Since at each point $p \in S^n(c)$, $\psi(p)$ is the position vector of p on $S^n(c)$, we have $\psi = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x^i}$ and thus $\bar{\nabla}_X \psi = X$, $X \in \mathfrak{X}(S^n(c))$. Also from $g(\psi, \psi) = a^2$, where g is the Euclidean metric on

R^{n+1}, we have $X \cdot g(\psi, \psi) = 0$, which gives $g(X, \psi) = 0$, $X \in \mathfrak{X}(S^n(c))$. Thus ψ is normal to $S^n(c)$. Hence the unit normal vector field N to $S^n(c)$ is given by $N = \frac{\psi}{\|\psi\|} = \frac{1}{a}\psi$, from which follows that $\bar{\nabla}_X N = \frac{1}{a}X$, comparing this with the formula (1.3.4), we get the Weingarten map S for $S^n(c)$ as $S = -\frac{1}{a}I$, I being the identity transformation of $\mathfrak{X}(M)$. Thus the Gauss formula (1.3.3) for $S^n(c)$ is

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{a^2} g(X, Y)\,\psi, \qquad X, Y \in \mathfrak{X}(S^n(c))$$

.

Also since the curvature tensor $\overline{R}$ of the connection $\overline{\nabla}$ is zero, from equation (1.3.5) we get the following expression for the curvature tensor R of $S^n(c)$

$$R(X, Y, Z, W) = \frac{1}{a^2}\{g(Y, Z)\, g(X, W) - g(X, Z)\, g(Y, W)\},$$

$X, Y, Z, W \in \mathfrak{X}(S^n(c))$, which shows that $S^n(c)$ has constant sectional curvature $\frac{1}{a^2}$.

CHAPTER 2

JACOBI FIELDS AND THEIR PROPERTIES

In this chapter we introduce Jacobi field and state their properties. As Jacobi field arise from variation of geodesic, we start with variation of a smooth curve.

2.1 VARIATION

Definition 2.1.1 Let (M, g) be a Riemannian manifold and $\sigma : [a, b] \to M$ be a continuous map. If there exists a partition of $[a, b]$ by points $a = t_1 < t_2 < \ldots < t_k = b$ such that $\sigma : (t_i, t_{i+1}) \to M$, $i = 1, ..., k-1$, is smooth. Then σ is called a piecewise smooth curve.

A vector field $Y(t)$ along a piecewise smooth curve $\sigma : [a, b] \to M$ is said to be piecewise smooth vector field if $Y : (t_i, t_{i+1}) \to TM$ is smooth.

Definition 2.1.2 Let (M, g) be a Riemannian manifold and $\sigma : [a, b] \to M$ be a piecewise smooth curve. A variation f of σ is a continuous map $f : [a, b] \times (-\epsilon, \epsilon) \to M$ such that $f(t, 0) = \sigma(t), t \in [a, b]$.

A variation f of σ is said to be proper variation if $f(a, s) = \sigma(a)$ and $f(b, s) = \sigma(b)$, for all $s \in (-\epsilon, \epsilon)$.

For the orthonormal frame $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\}$ on the subset $(a,b)\times(-\epsilon,\epsilon)$ of R^2. Define $T(t,s) = df_{(t,s)}\left[\left(\frac{\partial}{\partial t}\right)_{(t,s)}\right]$. Thus for fixed s, $T(t) = df(t,s)$ is tangential vector field to the horizontal curve $t \to f_s(t)$ also define $V(t) = df_{(t,0)}[\left(\frac{\partial}{\partial s}\right)_{(t,0)}]$, then $V(t)$ is a vector field defined along the curve σ. This vector field V is called variation vector field.

Remark 2.1.1 Note that

(i) $[V,T] = 0$.

(ii) $\dfrac{D^2X}{\partial t\partial s} - \dfrac{D^2X}{\partial s\partial t} = R(T,S)X$ where X is a vector field along f and $T = \frac{\partial f}{\partial t}$, $S = \frac{\partial f}{\partial s}$.

Examples 2.1.1

1. Take $\sigma : R \to R^2$ as $\sigma(t) = (t,t^2)$ and $f : R\times(-\epsilon,\epsilon) \to R^2$ as $f(t,s) = (t,(1-s)t^2)$. Then the variation field is given by $V(t) = -t^2\frac{\partial}{\partial s}$.

2. Consider a smooth curve $\sigma : \left[\frac{-\pi}{2},\frac{\pi}{2}\right] \to S^2$, $\sigma(t) = (\cos t, \sin t, 0)$ and $f : \left[\frac{-\pi}{2},\frac{\pi}{2}\right]\times\left(\frac{-\pi}{2},\frac{\pi}{2}\right) \to S^2$, $f(t,s) = (\cos t\cos s, \sin t\cos s, \sin s)$. Then choose a chart (U, i_d) around (t,s) with $U = \left(\frac{-\pi}{2},\frac{\pi}{2}\right)\times\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ and t,s are local coordinates on U, and a chart (V,ϕ) around $f(t,s)\in S^2$ with $V = \{(x,y,z)\in S^2 | x>0\}$ and $\phi(x,y,z) = (y,z)$, y,z are local coordinates on V. Then, it can be easily shown that the variation field is given by $V(t) = \frac{\partial}{\partial z}$.

Theorem 2.1.1 Let (M, g) be a Riemannian manifold. Given a piecewise smooth vector field $V(t)$ along a piecewise smooth curve $\sigma : [0, a] \to M$. Then there exists a variation $f : [0, a] \times (-\epsilon, \epsilon) \to M$ of σ, such that $V(t)$ is variation vector field of f. In addition, if $V(0) = V(a) = 0$, it is possible to choose f as proper variation.

Proof: Fix $t \in [0, a]$, then for $\sigma(t) = p \in M$, there is a $\delta_t > 0$ such that $\exp_{\sigma(t)} : B_{\delta_t}(o_p) \to U_t$ is a diffeomorphism. Since $\{U_t\}$ covers the compact set $\sigma([0, a]) \subset M$, there are finite numbers of $\delta_{t_1}, \ldots, \delta_{t_k}$ such that $\sigma([0, a]) \subset U_{t_1} \cup \ldots \cup U_{t_k}$. Choose $\delta = \min\{\delta_{t_1}, \ldots, \delta_{t_k}\}$ then $\exp_{\sigma(t)} : B_{\delta_t}(o_p) \to M$ is defined for all $t \in [0, a]$.

Define $f : [0, a] \times (-\epsilon, \epsilon) \to M$ by

$$f(t, s) = \exp_{\sigma(t)} sV(t)$$

where $0 < \epsilon < \frac{\delta}{\|V(t\|}$. Clearly f is a piecewise smooth map and $f(t, 0) = \sigma(t)$. Then, f is variation of σ.

Now, the variation vector field of f is given by

$$df_{(t,0)}\left(\frac{\partial}{\partial s}\right) = d\left(\exp_{\sigma(t)}\right)_{o_p}(V(t)) = V(t)$$

and it is clear that, from the definition of f, if $V(0) = V(a) = 0$ then f is proper.

2.2 JACOBI FIELDS

Definition 2.2.1 Let (M, g) be a Riemannian manifold, and $\gamma : [0, a] \to M$ be a geodesic. A vector field $X(t)$ along a geodesic γ is said to be a Jacobi field if it satisfies

$$X^{\cdot\cdot} + R\left(X, \dot{\gamma}\right)\dot{\gamma} = 0.$$

Examples 2.2.1

1. For a geodesic $\gamma : [0, a] \to M$, $\dot{\gamma}(t)$ and $t\dot{\gamma}(t)$ are Jacobi fields along γ. The first field has derivative zero and vanishes no where, the second field is zero if and only if $t = 0$.

2. Let $\gamma : [0, a] \to M$ be a geodesic and $f : [0, a] \times (-\epsilon, \epsilon) \to M$ be a geodesic variation of γ (i.e $f_s(t) = f(t, s)$ is a geodesic for all $s \in (-\epsilon, \epsilon)$). Then the variation vector field is a Jacobi field.

Theorem 2.2.1 Let (M, g) be a Riemannian manifold and $\gamma : [a, b] \to M$ be a geodesic. For $u, v \in T_{\gamma(a)}M$ there exists a unique Jacobi field $J(t)$ along γ such that $J(a) = u$ and $J^{\cdot}(a) = v$.

Remark 2.2.1 Let J_γ be the set of all Jacobi field along $\gamma : [a, b] \to M$ and define $\Phi : J_\gamma \to T_{\gamma(a)}M \times T_{\gamma(a)}M$ by

$$\Phi(J) = \left(J(a), J^{\cdot}(a)\right).$$

Clearly, Φ is linear map. Moreover, by above theorem it is an isomorphism. Thus, the space of Jacobi field along γ is $2n$-dimensional vector space over R where $n = \dim M$.

Theorem 2.2.2 (Jacobi fields on manifold of constant curvature).

Let (M, g) be a Riemannian manifold of constant curvature K, and let $\gamma : [0, \ell] \to M$ be a normalized geodesic on M and $w(t)$ be a parallel field along γ with $g(\dot{\gamma}(t), w(t)) = 0$ and $\| w(t) \| = 1$. Then, a Jacobi field J along γ is given by

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t) & \text{if } K > 0 \\ tw(t) & \text{if } K = 0 \\ \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} w(t) & \text{if } K < 0 \end{cases}$$

Theorem 2.2.3 Let $\gamma : [0, a] \to M$ be a geodesic. Then, a Jacobi field J along γ with $J(0) = 0$ is given by

$$J(t) = d\left(\exp_p\right)_{t\dot{\gamma}(0)} \left(tJ^{\cdot}(0)\right), \ t \in [0, a], \text{ where } \gamma(0) = p.$$

Proposition 2.2.1 Let (M, g) be a Riemannian manifold and $p \in M$. Let $\gamma : [0, a] \to M$ be a geodesic with $\gamma(0) = p$, $\dot{\gamma}(0) = v \in T_pM$ and $w \in T_v(T_pM)$ be such that $\| w \| = 1$ and J is a Jacobi field along γ given by $J(t) = d\left(\exp_p\right)_{tv}(tw)$, $t \in [0, a]$. Then, the Taylor series of $\| J(t) \|^2$ about $t = 0$ is given by

$$\| J(t) \|^2 = t^2 - \frac{1}{3} R(v, w, w, v) t^4 + R(t)$$

where $\lim\limits_{t \to 0} \frac{R(t)}{t^4} = 0$.

Corollary 2.2.1 For $v, w \in T_pM$, we suppose that v and w span a subspace $\pi \subset T_pM$. Then, $K(\pi) = R(v, w, w, v)$ and

$$\| J(t) \|^2 = t^2 - \frac{1}{3}K(\pi)\,t^4 + R(t)$$

where $\lim\limits_{t\to 0}\frac{R(t)}{t^4} = 0$, or

$$\| J(t) \| = t - \frac{1}{6}K(\pi)\,t^3 + \bar{R}(t) \tag{2.2.1}$$

where $\lim\limits_{t\to 0}\frac{\bar{R}(t)}{t^3} = 0$.

Remark 2.2.2 The expression (2.2.1) essentially contains the relation between geodesics and curvature. Indeed, considering the parametrized surface

$$f(t, s) = \exp_p tv(s), t \in [0, \delta], \quad s \in (-\epsilon, \epsilon)$$

where δ is chosen so small that $\exp_p tv(s)$ is defined, and $v(s)$ is a curve in T_pM with $\| v(s) \| = 1, v(0) = v, v^{\cdot}(0) = w$, we see that the rays $t \to tv(s), t \in [0, \delta]$, that start from the origin o of T_pM, are spreading a way from the ray $t \to tv$ with the velocity $\| \frac{\partial}{\partial s}tv(s)\,|_{s=0} \| = \| tw \| = t$ as $\| w \| = 1$.

On other hand, (2.2.1) tells us that the geodesics $t \to \exp_p tv(s)$ are spreading away from the geodesic $t \to \exp_p tv$ with the velocity that differs from t by a term of the third order in t, given by $-\frac{1}{6}\,K(\pi)\,t^3$. Thus we conclude that if $K(\pi) > 0$ the geodesics spread apart less than the rays in T_pM and if $K(\pi) < 0$ then the geodesics spread apart more than the rays in T_pM.

Actually, for t small, the value $K(\pi)\,t^3$ furnishes approximation for the extent of this spread with an error of order t^3.

For a geodesic $\gamma : [0, a] \to M$ on a Riemannian manifold the space of Jacobi fields along γ can be written as

$$J_\gamma = J_1 \oplus J_2 \oplus J_3$$

where $J_1 = \left\langle \dot{\gamma}(t), (t-a)\dot{\gamma}(t) \right\rangle$ is the space of Jacobi fields which is tangent to the geodesic and has dimension 2, $J_2 \oplus J_3$ is the space of Jacobi fields which is normal to the geodesic such that

$J_2 = \left\{ X \in J_\gamma \mid g\left(X(t), \dot{\gamma}(t)\right) = 0, X(0) = 0, X^{\cdot}(0) \neq 0 \right\}$ and has dimension $n-1$,

$J_3 = \left\{ X \in J_\gamma \mid g\left(X(t), \dot{\gamma}(t)\right) = 0, X(0) \neq 0, X^{\cdot}(0) = 0 \right\}$ and has dimension $n-1$.

Proposition 2.2.2 Let J be a Jacobi field along the geodesic $\gamma : [0, a] \to M$. Then,

$$g\left(J(t), \dot{\gamma}(t)\right) = g\left(J^{\cdot}(0), \dot{\gamma}(0)\right)t + g\left(J(0), \dot{\gamma}(0)\right), t \in [0, a].$$

Corollary 2.2.2 If $J(t_\circ) = J^{\cdot}(t_\circ) = 0$, then $J(t) \in J_2$.

Corollary 2.2.3 If $g\left(J, \dot{\gamma}\right)(t_1) = g\left(J, \dot{\gamma}\right)(t_2)$, $t_1, t_2 \in [0, a]$, $t_1 \neq t_2$, then $g\left(J, \dot{\gamma}\right)$ does not depend on t; in particular if $J(0) = J(a) = 0$, then $J(t) \in J_2$.

2.3 GAUSS LEMMA

In this section we are going to prove the following theorem which is known as Guass Lemma and is a very important tool for obtaining global results on a Riemannian manifold.

Theorem 2.3.1 Let (M,g) be a Riemannian manifold and $p \in M$, $v \in T_pM$ such that $\exp_p v$ is defined. Then for $w \in T_v(T_pM)$,

$$g\left(d\left(\exp_p\right)_v(v), d\left(\exp_p\right)_v(w)\right) = g(v,w).$$

Proof: Consider $h : R^2 \to T_pM$ defined by

$$h(t,s) = t(v+sw)$$

is a parametrized surface in T_pM satisfying $h_t = v + sw$, $h_s = tw$. Thus, $h_t(1,0) = v$ and $h_s(1,0) = w$.

Now, define $f : h^{-1}(N_\circ) \to M$ where $N_\circ$ is domain of $\exp_p$ by

$$f(t,s) = \exp_p(h(t,s)).$$

Then f is a parametrized surface in M. Now,

$$f_t(1,0) = \frac{\partial f}{\partial t}(1,0) = d\left(\exp_p\right)_{h(1,0)}\left(\frac{\partial h}{\partial t}(1,0)\right) = d\left(\exp_p\right)_v(v) \quad (2.3.1)$$

and

$$f_s(1,0) = \frac{\partial f}{\partial s}(1,0) = d\left(\exp_p\right)_{h(1,0)}\left(\frac{\partial h}{\partial s}(1,0)\right) = d\left(\exp_p\right)_v(w). \quad (2.3.2)$$

Observe that $\alpha(t) = \exp_p t(v + sw)$, for fixed s is a geodesic. Then, $\frac{D\alpha^{\cdot}}{dt} = 0$ and $\| \alpha^{\cdot}(t) \|$=constant, $\alpha^{\cdot}(0) = v + sw$. Since $\alpha(t) = f(t, s)$, s fixed, we get $\alpha^{\cdot}(t) = f_t$ and $f_{tt} = \frac{D\alpha^{\cdot}}{dt} = 0$. Also,

$$g(f_t, f_t) = \| \alpha^{\cdot}(t) \|^2 = \| \alpha^{\cdot}(0) \|^2 = g(v + sw, v + sw)$$

We compute,

$$\begin{aligned} \frac{\partial}{\partial t} g(f_t, f_s) &= g(f_{tt}, f_s) + g(f_t, f_{st}) \\ &= g(f_t, f_{ts}) \\ &= \frac{1}{2}\frac{\partial}{\partial s} g(f_t, f_t) \\ &= \frac{1}{2}\frac{\partial}{\partial s} g(v + sw, v + sw) \\ &= \frac{1}{2}\{g(w, v + sw) + g(v + sw, w)\} \\ &= g(v + sw, w). \end{aligned}$$

to conclude that $\left\{\frac{\partial}{\partial t} g(f_t, f_s)\right\}(t, 0) = g(v, w)$.

Integrating this last equation we get

$$g(f_t, f_s)(t, 0) = g(v, w)\, t + C \tag{2.3.3}$$

where C is a constant. Now,

$$\begin{aligned} g(f_t, f_s)(0, 0) &= g(f_t(0, 0), f_s(0, 0)) \\ &= g\left(d\left(\exp_p\right)_{o_p}(v), d\left(\exp_p\right)_{o_p}(o_p)\right) \\ &= g(v, o_p) = 0. \end{aligned}$$

Thus from (2.3.3), we have $C = 0$, that is $g(f_t, f_s)(t, 0) = g(v, w)\,t$ for all t, and for $t = 1$, $g(f_t(1,0), f_s(1,0)) = g(v, w)$.

Using (2.3.1) and (2.3.2) we get

$$g\left(d\left(\exp_p\right)_v(v), d\left(\exp_p\right)_v(w)\right) = g(v, w)$$

which proves the Guass Lemma.

2.4 INDEX FORM

For a smooth curve $\sigma : I \to M$, we denote $\mathfrak{X}_\sigma(M)$ the set of smooth vector field defined along σ.

Definition 2.4.1 Let (M, g) be a Riemannian manifold and $\gamma : [a, b] \to M$ be a geodesic. Define $I_\gamma : \mathfrak{X}_\gamma(M) \times \mathfrak{X}_\gamma(M) \to R$ by

$$I_\gamma(X, Y) = \int_a^b \left\{ g\left(\frac{DX}{dt}, \frac{DY}{dt}\right) - R\left(\dot{\gamma}, X, Y, \dot{\gamma}\right) \right\} dt.$$

It is easy to verify that I_γ is a symmetric bilinear form on $\mathfrak{X}_\gamma(M)$. This form I_γ is called index form of γ.

Notation. For a geodesic $\gamma : [0, a] \to M$, we put for any $t_\circ \in (0, a]$

$$I_{t_\circ}(V, W) = \int_0^{t_\circ} \left\{ g\left(\frac{DV}{dt}, \frac{DW}{dt}\right) - R\left(\dot{\gamma}, V, W, \dot{\gamma}\right) \right\} dt, \;\; V, W \in \mathfrak{X}_\gamma(M)$$

that is, $I_a(V, W) = I_\gamma(V, W)$.

Theorem 2.4.1 If X is a Jacobi field along a geodesic γ then for any vector field Y along γ vanishing at end points of γ, the index form satisfies $I_\gamma (X, Y) = 0$.

Definition 2.4.2 Let $\gamma : [0, a] \to M$ be a geodesic on a Riemannian manifold (M, g). The point $\gamma(t_\circ)$ is said to be conjugate to $\gamma(0)$ along γ, $t_\circ \in (0, a]$, if there exists a non-zero Jacobi field J along γ such that $J(0) = J(t_\circ) = 0$.

Theorem 2.4.2 (The Index Lemma). Let (M, g) be a Riemannian manifold and $\gamma : [0, a] \to M$ be a geodesic without conjugate points to $\gamma(0)$ in the interval $(0, a]$. Let J be a Jacobi field along γ, with $g(J(t), \gamma'(t)) = 0$, and let V be a smooth vector field along γ, with $g(V(t), \gamma'(t)) = 0$. Suppose that $J(0) = V(0) = 0$ and that $J(t_\circ) = V(t_\circ)$, for some $t_\circ \in (0, a]$. Then,

$$I_{t_\circ}(J, J) \leq I_{t_\circ}(V, V)$$

and the equality occurs if and only if $V = J$ on $[0, t_\circ]$.

Proof: We know that the space of Jacobi fields J along γ with $J(0) = 0$ and $g(J(t), \gamma'(t)) = 0$ has dimension $n - 1$, where $n = \dim M$.

Let $\{J_1(t), \ldots, J_{n-1}(t)\}$ be a basis for this space. Then,

$$J = \sum_{i=1}^{n-1} \alpha_i J_i, \qquad \alpha_i \in R$$

Also, since $\{J_1(t), \ldots, J_{n-1}(t), \gamma'(t)\}$ is basis for $T_{\gamma(t)}M$, with $g(V(t), \gamma'(t)) =$

0, we have

$$V(t) = \sum_{i=1}^{n-1} f_i(t) J_i(t)$$

where $f_i(t) = g(V(t), J_i(t)) : [0, a] \to R$ are smooth function.

We observe that, as $J(t_\circ) = V(t_\circ)$ we have

$$\alpha_i = f_i(t_\circ) \tag{2.4.1}$$

Also we have

$$\frac{DV}{dt} = \sum_{i=1}^{n-1} \left(f_i' J_i + f_i J_i' \right)$$

and

$$R(V, \gamma')\gamma' = R\left(\sum_{i=1}^{n-1} f_i J_i, \gamma'\right)\gamma' = \sum_{i=1}^{n-1} f_i R(J_i, \gamma')\gamma' = -\sum_{i=1}^{n-1} f_i J_i''$$

which implies

$$\begin{aligned} g\left(\frac{DV}{dt}, \frac{DV}{dt}\right) &= g\left(\sum_i \left(f_i' J_i + f_i J_i'\right), \sum_j \left(f_j' J_j + f_j J_j'\right)\right) \\ &= g\left(\sum_i f_i' J_i, \sum_j f_j' J_j\right) + g\left(\sum_i f_i' J_i, \sum_j f_j J_j'\right) \\ &\quad + g\left(\sum_i f_i J_i', \sum_j f_j' J_j\right) + g\left(\sum_i f_i J_i', \sum_j f_j J_j'\right). \end{aligned}$$

Consequently,

$$
\begin{aligned}
g\left(\frac{DV}{dt},\frac{DV}{dt}\right)-R\left(V,\gamma',\gamma',V\right) &= g\left(\sum_i f_i' J_i,\sum_j f_j' J_j\right)\\
&\quad +g\left(\sum_i f_i' J_i,\sum_j f_j J_j'\right)\\
&\quad +g\left(\sum_i f_i J_i',\sum_j f_j' J_j\right)\\
&\quad +g\left(\sum_i f_i J_i',\sum_j f_j J_j'\right)\\
&\quad +g\left(\sum_i f_i J_i'',\sum_j f_j J_j\right) \qquad (2.4.2)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{d}{dt}g\left(\sum_i f_i J_i',\sum_i f_j J_j\right) &= g\left(\sum_i \left(f_i' J_i' + f_i J_i''\right),\sum_j f_j J_j\right)\\
&\quad +g\left(\sum_i f_i J_i',\sum_j \left(f_j' J_j + f_j J_j'\right)\right)\\
&= g\left(\sum_i f_i' J_i',\sum_j f_j J_j\right)+g\left(\sum_i f_i J_i'',\sum_j f_j J_j\right)\\
&\quad +g\left(\sum_i f_i J_i',\sum_j f_j' J_j\right)+g\left(\sum_i f_i J_i',\sum_j f_j J_j'\right)
\end{aligned}
$$

Thus, (2.4.2) becomes

$$\begin{aligned} g\left(\frac{DV}{dt},\frac{DV}{dt}\right)-R\left(V,\gamma',\gamma',V\right) &= \frac{d}{dt}g\left(\sum_i f_i J_i', \sum_i f_j J_j\right) \\ &\quad -g\left(\sum_i f_i' J_i', \sum_j f_j J_j\right) \\ &\quad +g\left(\sum_i f_i' J_i, \sum_j f_j' J_j\right) \\ &\quad +g\left(\sum_i f_i' J_i, \sum_j f_j J_j'\right) \qquad (2.4.3) \end{aligned}$$

Now, write

$$h(t)=g\left(J_i'(t),J_j(t)\right)-g\left(J_i(t),J_j'(t)\right) \text{ for fixed } i,j.$$

Since $h(0)=0$ and

$$\begin{aligned} h'(t) &= g\left(J_i'',J_j\right)+g\left(J_i',J_j'\right)-g\left(J_i',J_j'\right)-g\left(J_i,J_j''\right) \\ &= -g\left(R\left(J_i,\gamma'\right)\gamma',J_j\right)+g\left(J_i,R\left(J_j,\gamma'\right)\gamma'\right) \\ &= -R\left(J_i,\gamma',\gamma',J_j\right)+R\left(J_i,\gamma',\gamma',J_j\right)=0, \end{aligned}$$

we conclude that $h(t)=0$. Which implies

$$g\left(J_i'(t),J_j(t)\right)=g\left(J_i(t),J_j'(t)\right).$$

In light of this, (2.4.3) becomes

$$g\left(\frac{DV}{dt},\frac{DV}{dt}\right)-R\left(V,\gamma',\gamma',V\right)=\frac{d}{dt}g\left(\sum_i f_i J_i', \sum_i f_j J_j\right)+g\left(\sum_i f_i' J_i, \sum_j f_j' J_j\right).$$

Thus,

$$I_{t_\circ}(V,V) = g\left(\sum_i f_i J_i', \sum_i f_j J_j\right)(t_\circ) + \int_0^{t_\circ} g\left(\sum_i f_i' J_i, \sum_j f_j' J_j\right) dt.$$

From (2.4.1) we have

$$I_{t_\circ}(V,V) = g\left(\sum_i \alpha_i J_i'(t_\circ), \sum_i \alpha_j J_j(t_\circ)\right) + \int_0^{t_\circ} \| \sum_i f_i' J_i \|^2 \, dt \qquad (2.4.4)$$

Now,

$$\begin{aligned} I_{t_\circ}(J,J) &= \int_0^{t_\circ} \left\{ g\left(\frac{DJ}{dt}, \frac{DJ}{dt}\right) - g\left(R\left(J,\gamma'\right)\gamma', J\right)\right\} dt \\ &= \int_0^{t_\circ} \left\{ g\left(J', J'\right) + g\left(J'', J\right)\right\} dt \\ &= \int_0^{t_\circ} \left\{ \frac{d}{dt} g\left(J, J'\right) - g\left(J, J''\right) + g\left(J'', J\right)\right\} dt \\ &= g\left(J(t_\circ), J'(t_\circ)\right) \\ &= g\left(\sum_i \alpha_i J_i(t_\circ), \sum_i \alpha_i J_i'(t_\circ)\right). \end{aligned}$$

Use this in (2.4.4) we get

$$I_{t_\circ}(V,V) = I_{t_\circ}(J,J) + \int_0^{t_\circ} \| \sum_i f_i' J_i \|^2 \, dt. \qquad (2.4.5)$$

It follows from (2.4.5) that $I_{t_\circ}(J,J) \leq I_{t_\circ}(V,V)$, which proves the first part of the lemma.

$$
\begin{aligned}
\text{Now, } I_{t_\circ}(J,J) = I_{t_\circ}(V,V) \text{ for } t_\circ \in (0,a] \quad &\Leftrightarrow \sum_i f_i'(t)\, J_i(t) = 0 \ \forall t \in (0,t_\circ] \\
&\Leftrightarrow \ f_i'(t) = 0,\ i = 1,\ldots,n-1. \\
&\Leftrightarrow \ f_i(t) = \text{constant} = \alpha_i \\
&\Leftrightarrow \ V = J \text{ on } [0,t_\circ].
\end{aligned}
$$

CHAPTER 3

VARIATION OF ARC-LENGTH

Most of deep results in geometry of Riemannian manifold use in their proofs the variational techniques. Thus calculus of variations is a vital and integral part of the study of Riemannian geometry. In this chapter we introduce this notion with study of variation of arc-length and energy.

3.1 VARIATION OF ARC-LENGTH

Definition 3.1.1 Let (M, g) be a Riemannian manifold and $\sigma : [a, b] \to M$ be a piecewise smooth curve. For a variation $f : [a, b] \times (-\epsilon, \epsilon) \to M$ of σ, we define $L : (-\epsilon, \epsilon) \to R$ and $E : (-\epsilon, \epsilon) \to R$ by $L(s) = \int_a^b \| T(t, s) \| \, dt$ and $E(s) = \int_a^b \| T(t, s) \|^2 \, dt$, where $T(t, s) = df_{(t,s)}\left(\frac{\partial}{\partial t}\right)$. Called the arc-length and energy functions.

Theorem 3.1.1 (First variation formula for energy) Let $\sigma : [0, a] \to M$ be a piecewise smooth curve on a Riemannian manifold (M, g) and $f : [0, a] \times (-\epsilon, \epsilon) \to M$ be a variation of σ. Then,

$$\frac{1}{2} E'(0) = -\int_0^a g\left(V(t), \frac{D\sigma^{\cdot}}{dt}\right) dt + \sum_{i=1}^{k} g\left(V(t_i), \sigma^{\cdot}\left(t_i^{+}\right) - \sigma^{\cdot}\left(t_i^{-}\right)\right)$$

$$-g\left(V(0), \sigma^{\cdot}(0)\right) + g\left(V(a), \sigma^{\cdot}(a)\right) \tag{3.1.1}$$

where $V(t)$ is variation field of f and $\sigma^{\cdot}\left(t_i^{+}\right) = \lim\limits_{t\to t_i^{+}} \sigma^{\cdot}(t)$, $\sigma^{\cdot}\left(t_i^{-}\right) = \lim\limits_{t\to t_i^{-}} \sigma^{\cdot}(t)$.

Proof: By defintion

$$E(s) = \int_0^a \left\| \frac{\partial f}{\partial t} \right\|^2 dt = \int_0^a g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) dt.$$

Differenting the equality we get

$$\begin{aligned} E^{'}(s) &= \frac{dE}{ds} = \frac{d}{ds}\int_0^a g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) dt \\ &= \int_0^a \frac{d}{ds} g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) dt \\ &= 2\int_0^a g\left(\frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2}E^{'}(s) &= \int_0^a g\left(\frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) dt \\ &= \int_0^a \left\{\frac{d}{dt} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) - g\left(\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right)\right\} dt \\ &= -\int_0^a g\left(\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right) dt + \int_0^a \frac{d}{dt} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) dt \\ &= -\int_0^a g\left(\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right) dt + \sum_{i=0}^{k} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\Bigg]_{t_i}^{t_{i+1}} \qquad (3.1.2) \end{aligned}$$

Note that

$$\sum_{i=0}^{k} [f(x)]_{t_i}^{t_{i+1}} = f(a) - f(0) + \sum_{i=1}^{k} f\left(t_i^{+}\right) - f\left(t_i^{-}\right),$$

where $\frac{\partial f}{\partial s}\mid_{s=0}= V(t)$, $\frac{\partial f}{\partial t}\mid_{s=0}= \sigma^{\cdot}(t)$. Putting $s=0$ in (3.1.2), we get (3.1.1).

Theorem 3.1.2 A piecewise smooth curve $\sigma : [0,a] \to M$ is a geodesic if and only if, for each proper variation f of σ, we have $E^{'}(0) = 0$.

Proof : If σ is a geodesic, then $\frac{D\sigma^{\cdot}}{dt} = 0$ and σ is smooth, that is, $\sigma^{\cdot}\left(t_i^{+}\right) = \sigma^{\cdot}\left(t_i^{-}\right) = \sigma^{\cdot}(t_i)$. Also variation f is proper then $V(0) = V(a) = 0$. Thus from first varition formula we get $E^{'}(0) = 0$.

Conversely, suppose that $E^{'}(0) = 0$ for each proper varition f of σ. Let $0 < t_1 < \ldots < t_k < t_{k+1} = a$ be the partion such that $\sigma : (t_i, t_{i+1}) \to M$ is smooth. Let $g(t)$ be smooth function such that $g(t) > 0$ on (t_i, t_{i+1}) and $g(t_i) = 0$, $i = 0, 1, \ldots, k+1$. Define $V(t) = g(t)\frac{D\sigma^{\cdot}}{dt}$, $t \in [0,a]$, then $V(t)$ is a piecewise smooth vector field along σ, also $V(0) = V(a) = 0$. Then there exists proper variation f of σ whose variation field is $V(t)$.

By first variation formula with $E'(0) = 0$, we have

$$\begin{aligned} 0 &= \int_0^a g\left(g(t)\frac{D\sigma^{\cdot}}{dt}, \frac{D\sigma^{\cdot}}{dt}\right) dt \\ &= \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} g(t)\, g\left(\frac{D\sigma^{\cdot}}{dt}, \frac{D\sigma^{\cdot}}{dt}\right) dt \\ &= \int_{t_i}^{t_{i+1}} g(t) \parallel \frac{D\sigma^{\cdot}}{dt} \parallel^2 dt \end{aligned}$$

It follows that

$$\frac{D\sigma^{\cdot}}{dt} = 0 \text{ on } (t_i, t_{i+1}).$$

Therefore, $\sigma : (t_i, t_{i+1}) \to M$ is geodesic for $i = 0, \ldots, k$.

Now, to check what happens at the points t_i, we consider the vector field $\overline{V}(t)$ along σ satisfying $\overline{V}(0) = \overline{V}(a) = 0$, $\overline{V}(t) = 0$ on (t_i, t_{i+1}), $1 \leq i \leq k$ and $\overline{V}(t_i) = \sigma^{\cdot}\left(t_i^+\right) - \sigma^{\cdot}\left(t_i^-\right)$. Then, there is a proper variation of σ with variation vector field $\overline{V}(t)$. The first variation formula gives

$$\sum_{i=1}^{k} \| \dot{\sigma}\left(t_i^+\right) - \dot{\sigma}\left(t_i^-\right) \|^2 = 0$$

$$\dot{\sigma}\left(t_i^+\right) = \dot{\sigma}\left(t_i^-\right) = \dot{\sigma}\left(t_i\right)$$

that is, $\sigma : [0, a] \to M$ of class C^1. Then, by uniqueness geodesic $\sigma : [0, a] \to M$ is smooth.

Theorem 3.1.3 (The second variation formula). Let $\gamma : [0, a] \to M$ be a geodesic and let $f : [0, a] \times (-\epsilon, \epsilon) \to M$ be a proper variation of γ. Then,

$$\begin{aligned} \frac{1}{2}E''(0) &= -\int_0^a g\left(V(t), \frac{D^2V}{dt^2} - R\left(\gamma'(t), V(t)\right)\gamma'(t)\right) dt \\ &\quad + \sum_{i=1}^{k} g\left(V(t_i), \frac{DV}{dt}\left(t_i^+\right) - \frac{DV}{dt}\left(t_i^-\right)\right) \qquad (3.1.3) \end{aligned}$$

where $V(t)$ is the variation vector field of f, R is the curvature tensor field of M and $\frac{DV}{dt}\left(t_i^+\right) = \lim_{t \to t_i^+} \frac{DV}{dt}(t)$, $\frac{DV}{dt}\left(t_i^-\right) = \lim_{t \to t_i^-} \frac{DV}{dt}(t)$.

Proof: From the proof of first variation formula. The equation (3.1.2) is

$$\frac{1}{2}E'(0) = -\int_0^a g\left(\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right) dt + \sum_{i=0}^{k} \left[g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\right]_{t_i}^{t_{i+1}}.$$

Thus,

$$\begin{aligned}\frac{1}{2}E''(0) &= -\int_0^a \frac{d}{ds} g\left(\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right) dt + \sum_{i=0}^{k} \frac{d}{ds}\left[g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\right]_{t_i}^{t_{i+1}} \\ &= -\int_0^a g\left(\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right) dt - \int_0^a g\left(\frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\right) dt \\ &\quad + \sum_{i=0}^{k}\left[g\left(\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\right]_{t_i}^{t_{i+1}} \\ &\quad + \sum_{i=0}^{k}\left[g\left(\frac{\partial f}{\partial s}, \frac{D}{ds}\frac{\partial f}{\partial t}\right)\right]_{t_i}^{t_{i+1}} \qquad (3.1.4)\end{aligned}$$

by putting $s = 0$.

Since γ is geodesic then $\frac{D}{dt}\frac{\partial f}{\partial t} = 0$. Also, as f is proper variation of γ,

$$\begin{aligned}\sum_{i=0}^{k}\left[g\left(\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\right]_{t_i}^{t_{i+1}} &= \left[g\left(\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\right]_0^a \\ &= g\left(\left(\nabla_V V\right)(a), \frac{\partial f}{\partial t}(a)\right) \\ &\quad - g\left(\left(\nabla_V V\right)(0), \frac{\partial f}{\partial t}(0)\right) \\ &= 0.\end{aligned}$$

Now,

$$\begin{aligned}\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} &= \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} \\ &= \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}\end{aligned}$$

Thus,

$$g\left(\frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t}\right) = g\left(\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}\right).$$

Finally,

$$g\left(\frac{\partial f}{\partial s}, \frac{D}{ds}\frac{\partial f}{\partial t}\right) = g\left(\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s}\right).$$

Putting $s = 0$ in (3.1.4) and using above equations we get (3.1.3).

Theorem 3.1.4 Let $\sigma : [0, a] \to M$ be a smooth curve and $f : [0, a] \times (-\epsilon, \epsilon) \to M$ be a variation of σ with variation vector field $V(t)$. Then

$$L'(0) = -\int_0^a g\left(V(t), \frac{D}{dt}\frac{\sigma^{\cdot}}{\|\sigma^{\cdot}\|}\right) dt + g\left(V(a), \frac{\sigma^{\cdot}(a)}{\|\sigma^{\cdot}(a)\|}\right) - g\left(V(0), \frac{\sigma^{\cdot}(0)}{\|\sigma^{\cdot}(0)\|}\right).$$

Theorem 3.1.5 A piecewise smooth curve $\sigma : [0, a] \to M$ is a geodesic if and only if for each proper variation f of σ, $L'(0) = 0$.

Theorem 3.1.6 Let $\gamma : [0, a] \to M$ be a geodesic and $f : [0, a] \times (-\epsilon, \epsilon) \to M$ be a proper variation of γ. Then

$$L''(0) = \int_0^a \frac{1}{\|\dot{\gamma}\|}\left\{g\left(\frac{DV^{\perp}}{dt}, \frac{DV^{\perp}}{dt}\right) - R\left(\dot{\gamma}, V, V, \dot{\gamma}\right)\right\} dt,$$

where $\frac{DV^{\perp}}{dt} \perp \frac{\dot{\gamma}}{\|\dot{\gamma}\|}$.

3.2 APPLICATION OF VARIATION OF ARC-LENGTH

In this section we use the variation formulas obtained in section 3.1 to prove the fundamental theorems of Myers, Synge and Rauch.

Theorem 3.2.1 (Myers Theorem). Let (M, g) be a complete Riemannian manifold. If the Ricci curvature of M satisfies

$$\mathrm{Ric}_p(v, v) \geq \frac{n-1}{r^2}\|v\|^2 > 0,$$

for all $p \in M$, $v \in T_pM$, $n = \dim M$, r a constant. Then M is compact and the diameter $diam(M) \leq \pi r$.

Proof: Let p and q be any two points in M. Since M is complete, there exists a minimizing geodesic $\gamma : [0, 1] \to M$ joining p to q. It is enough to show that the length $L(\gamma) = \ell \leq \pi r$, therefore compact, as then M is closed and bounded subset of complete space.

Suppose, to the contrary, that $\ell > \pi r$. Let $e_1(t), \cdots, e_{n-1}(t)$ be orthonormal parallel vector field along γ such that $g\left(e_i(t), \dot{\gamma}(t)\right) = 0, 1 \leq i \leq n-1$.

Define $V_j\ (t) = \sin \pi t e_j(t), 1 \leq j \leq n-1$, be a vector field along γ and $V_j(0) = V_j(1) = 0$.

Thus for each V_j there is a proper variation of γ whose variation vector field is V_j and the energy denoted by E_j.

Using the formula for the second variation of energy, we get

$$\begin{aligned}
\frac{1}{2}E_j''(0) &= -\int_0^1 g\left(V_j(t), \frac{D^2V_j}{dt^2} - R\left(\dot\gamma(t), V_j(t)\right)\dot\gamma(t)\right) dt \\
&= -\int_0^1 g\left(\sin\pi t e_j(t), -\pi^2\sin\pi t e_j(t) - R\left(\dot\gamma(t), \sin\pi t e_j(t)\right)\dot\gamma(t)\right) dt \\
&= \int_0^1 \left\{\pi^2\sin^2\pi t + \sin^2\pi t R\left(e_j, \dot\gamma, e_j, \dot\gamma\right)\right\} dt \\
&= \int_0^1 \left\{\pi^2\sin^2\pi t - \sin^2\pi t R\left(e_j, \dot\gamma, \dot\gamma, e_j\right)\right\} dt
\end{aligned}$$

Summing over j, we get

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^{n-1}E_j''(0) &= \int_0^1 \left\{(n-1)\pi^2\sin^2\pi t - \sin^2\pi t\sum_{j=1}^{n-1}R\left(e_j, \dot\gamma, \dot\gamma, e_j\right)\right\} dt \\
&= \int_0^1 \left\{(n-1)\pi^2\sin^2\pi t - \sin^2\pi t \operatorname{Ric}\left(\dot\gamma, \dot\gamma\right)\right\} dt
\end{aligned}$$

Since $\operatorname{Ric}\left(\dot\gamma, \dot\gamma\right) \geq \frac{n-1}{r^2}\left\|\dot\gamma\right\|^2$ and $\ell > \pi r$, we have

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^{n-1}E_j''(0) &\leq \int_0^1 \left\{\sin^2\pi t\left((n-1)\pi^2 - \frac{n-1}{r^2}\ell^2\right)\right\} dt \\
&< \int_0^1 \left\{\sin^2\pi t\left((n-1)\pi^2 - (n-1)\pi^2\right)\right\} dt = 0
\end{aligned}$$

Hence there is at least one j for which $E_j''(0) < 0$, which contradicts the fact that γ is minimizing geodesic. Therefore $\ell \leq \pi r$, that is, $d(p,q) \leq \pi r$ and since p and q are arbitrary, $diam(M) \leq \pi r$.

Remark 3.2.1 We note that the result requires that the Ricci curvatures are bounded below by a positive constant, the condition can not be relaxed to

$$\mathrm{Ric}\,(v, v) > 0.$$

For example, the paraboloid $M = \left\{(x, y, z) \in R^3 \mid z = x^2 + y^2\right\}$ has sectional curvature $K > 0$ and then $\mathrm{Ric}\,(v, v) > 0$, but M is complete and non-compact.

Theorem 3.2.2 (Synge's Theorem). Let (M, g) be an even dimensional compact connected Riemannian manifold of positive curvature. Then M is simply connected.

proof: Let $\dim M = 2n$, suppose $\pi^1(M, p) \neq \{0\}$ i.e M is not simply connected. Then there is a geodesic loop $\gamma : [0, \pi] \to M$ at p, $\gamma(0) = \gamma(\pi) = p$, which is not homotopic to a constant loop at p. Thus, $\dot{\gamma}(0) = \pm\dot{\gamma}(\pi) \neq 0$ and then $\dim \left\langle \dot{\gamma}(0) \right\rangle = 1$.

Also, $T_pM = \left\langle \dot{\gamma}(0) \right\rangle \oplus E$, where $\dim E = 2n - 1$.

Let $P_\gamma : T_pM \to T_pM$ be the parallel transport map. Since $\dot{\gamma}(t)$ is parallel along γ, $P_\gamma\left(\dot{\gamma}(0)\right) = \dot{\gamma}(\pi) = \pm\dot{\gamma}(0)$.

As P_γ is an isomorphism, $P_\gamma(E) = E$.

Since E has odd dimension, there is an egen vector $v \in E$ of P_γ with real egen value i.e. $P_\gamma(v) = \lambda v$, $\lambda \in R$. But $g(P_\gamma v, P_\gamma v) = g(v, v)$, then $\lambda^2 = 1$, take $\lambda = 1$,we get $P_\gamma(v) = v \in E \subset T_pM$.

Choose vector field $X(t)$ parallel along γ such that $X(0) = v$. Then for variation of γ where variation field is X,

$$\begin{aligned} \frac{1}{2}E''(0) &= \int_0^\pi \left\{ g\left(\frac{DX}{dt}, \frac{DX}{dt}\right) - R\left(X, \dot{\gamma}, \dot{\gamma}, X\right) \right\} dt \\ &= -\int_0^\pi R\left(X, \dot{\gamma}, \dot{\gamma}, X\right) dt. \end{aligned}$$

But the sectional curvature > 0 and then $R\left(X, \dot{\gamma}, \dot{\gamma}, X\right) > 0$, which gives $E''(0) < 0$ and then $E : (-\epsilon, \epsilon) \to R$ has maxima at $s = 0$ which is contradiction.

Theorem 3.2.3 (Rauch). Let (M^n, g) and $\left(\overline{M}^{n+m}, \overline{g}\right)$ be Riemannian manifolds, $m \geq 0$, and $\gamma : [0, a] \to M^n$, $\overline{\gamma} : [0, a] \to \overline{M}^{n+m}$ be geodesics with $\left\|\dot{\gamma}(t)\right\| = \|\dot{\overline{\gamma}}(t)\|$ and J, $\overline{J}$ be Jacobi fields along γ and $\overline{\gamma}$ respectively such that

$$J(0) = \overline{J}(0) = 0,\ g\left(J'(0), \gamma'(0)\right) = g\left(\overline{J}'(0), \overline{\gamma}'(0)\right),\ \left\|J'(0)\right\| = \left\|\overline{J}'(0)\right\|.$$

Assume that $\overline{\gamma}$ does not have conjugate points to $\gamma(0)$ on $(0, a]$ and that, for all t and all $v \in T_{\gamma(t)}M$, $\overline{v} \in T_{\overline{\gamma}(t)}\overline{M}$, we have

$$\overline{K}\left(\overline{v}, \overline{\gamma}'(t)\right) \geq K\left(v, \gamma'(t)\right).$$

Then,

$$\left\|\overline{J}(t)\right\| \leq \left\|J(t)\right\|.$$

In addition, if for some $t_0 \in (0,a]$, we have $\left\|\overline{J}(t_0)\right\| = \|J(t_0)\|$, then

$\overline{K}\left(\overline{J}(t), \overline{\gamma}'(t)\right) = K(J(t), \gamma'(t))$ for all $t \in [0, t_0]$.

Proof: Since $g(J(t), \gamma'(t)) = g(J'(0), \gamma'(0))\, t + g(J(0), \gamma'(0))$ and

$\overline{g}\left(\overline{J}(t), \overline{\gamma}'(t)\right) = \overline{g}\left(\overline{J}'(0), \overline{\gamma}'(0)\right) t + \overline{g}\left(\overline{J}(0), \overline{\gamma}'(0)\right)$ for $t \in [0,a]$ and given condition in statment, we get $g(J(t), \gamma'(t)) = \overline{g}\left(\overline{J}(t), \overline{\gamma}'(t)\right)$, $t \in [0,a]$. i.e. $J(t)$ and $\overline{J}(t)$ has the same tangential components. Therefore, we can suppose that

$$g\left(J(t), \gamma'(t)\right) = 0 = \overline{g}\left(\overline{J}(t), \overline{\gamma}'(t)\right).$$

Next, if $\|J'(0)\| = \left\|\overline{J}'(0)\right\| = 0$, then, as $J(t) = d\left(\exp_p\right)_{t\dot{\gamma}(0)}(tJ'(0))$, $\|J(t)\| = \left\|\overline{J}(t)\right\| = 0$ and then there is nothing to prove.

Therefore, assume $\|J'(0)\| = \left\|\overline{J}'(0)\right\| \neq 0$. Put $f(t) = \|J(t)\|^2$ and $\overline{f}(t) = \left\|\overline{J}(t)\right\|^2$.

Since $\overline{\gamma}$ does not have a conjugate point to $\gamma(0)$ on $(0,a]$, $\left(\frac{f}{\overline{f}}\right)(t)$ is well defined on $(0,a]$.

Also,

$$\lim_{t\to 0^+} \frac{f(t)}{\overline{f}(t)} = \lim_{t\to 0^+} \frac{f'(t)}{\overline{f}'(t)} = \lim_{t\to 0^+} \frac{f''(t)}{\overline{f}''(t)} = \frac{\|J'(0)\|^2}{\left\|\overline{J}'(0)\right\|^2} = 1.$$

Thus, to prove that $f(t) \geq \overline{f}(t)$, i.e. $\frac{f(t)}{\overline{f}(t)} \geq 1$, we have to prove $\left(\frac{f(t)}{\overline{f}(t)}\right)'(t) \geq 0$ on $(0,a]$. This is equivalent to proving that

$$\overline{f} f' \geq f \overline{f}' \text{ on } (0,a] \tag{3.2.1}$$

Take $t_0 \in (0, a]$. If $J(t_0) = 0$, then

$$f'(t_0) = 2g\left(J'(t_0), J(t_0)\right) = 0$$

and therefore inequality (3.2.1) is automatically satisfied at t_0. Therefore suppose $J(t_0) \neq 0$.

Define

$$U(t) = \frac{J(t)}{\|J(t_0)\|}, \qquad \overline{U}(t) = \frac{\overline{J}(t)}{\left\|\overline{J}(t_0)\right\|},$$

and observe that

$$\begin{aligned}
\frac{f'(t_o)}{f(t_0)} &= \frac{2g(J'(t_0), J(t_0))}{g(J(t_0), J(t_0))} = 2g\left(U'(t_0), U(t_0)\right) = g(U,U)'(t_0) \\
&= \int_0^{t_0} g(U,U)''(t)\,dt = 2\int_0^{t_0} \left\{g\left(U'', U\right) + g\left(U', U'\right)\right\} dt \\
&= 2\int_0^{t_0} \left\{g\left(U', U'\right) - R\left(U, \gamma', \gamma', U\right)\right\} dt \qquad \text{(as } U \text{ is a Jacobi field)} \\
&= 2I_{t_0}(U, U).
\end{aligned}$$

Similarly,

$$\frac{\overline{f'}(t_0)}{\overline{f}(t_0)} = 2I_{t_0}\left(\overline{U}, \overline{U}\right).$$

Thus to prove (3.2.1) is equivalent to prove

$$I_{t_0}\left(\overline{U}, \overline{U}\right) \leq I_{t_0}(U, U), t_0 \in (0, a]. \qquad (3.2.2)$$

For this, choose an orthonormal basis of parallel vector fields $e_1(t), \cdots, e_n(t)$ along γ such that $e_1 = \frac{\gamma'(t)}{\|\gamma'(t)\|}$ and $e_2(t_0) = U(t_0)$.

Similarly, we take orthonormal basis of parallel vector fields $\overline{e}_1(t), \cdots, \overline{e}_{n+m}(t)$ along $\overline{\gamma}$ such that $\overline{e}_1 = \frac{\overline{\gamma}'(t)}{\|\overline{\gamma}'(t)\|}$ and $\overline{e}_2(t_0) = U(t_0)$.

For any $V(t) \in \mathfrak{X}_{\gamma}(M)$, with $V(t) = \sum_i h_i(t) e_i(t), t \in (0,a]$, define $(\phi V)(t) \in \mathfrak{X}_{\overline{\gamma}}(\overline{M})$ by $\phi(V)(t) = \sum_i h_i(t) \overline{e}_i(t)$.

Thus we get a map $\phi : \mathfrak{X}_{\gamma}(M) \to \mathfrak{X}_{\overline{\gamma}}(\overline{M})$ satisfies the follawing properties:

(a) $\overline{g}((\phi V)(t), (\phi w)(t)) = g(v, w)$.

(b) $(\phi V)'(t) = \phi(V'(t))$.

Now, observe that the vector field $\phi(U)(t)$ and $\overline{U}(t)$ are both defined along $\overline{\gamma}$ of which $\overline{U}$ is a Jacobi field satisfying $\overline{U}(0) = 0$, $\overline{g}(\overline{U}(t), \overline{\gamma}'(t)) = 0$ and $\phi(U)$ is a vector field satisfying $(\phi U)(0) = 0$, $\overline{g}(\phi(U)(t), \overline{\gamma}'(t)) = 0$ and $(\phi U)(t_0) = \overline{U}(t_0)$.

Thus $\phi(U)$ and $\overline{U}$ satisfy the hypothesis of index theorem, and consequently

$$I_{t_0}\left(\overline{U}, \overline{U}\right) \leq I_{t_0}(\phi U, \phi U) \tag{3.2.3}$$

Now, we use the hypothesis $\overline{K} \geq K$ to get

$$\begin{aligned} I_{t_0}(\phi U, \phi U) &= \int_0^{t_0} \left\{ \overline{g}\left((\phi U)', (\phi U)'\right) - \overline{R}\left(\phi U, \overline{\gamma}', \overline{\gamma}', \phi U\right) \right\} dt \\ &= \int_0^{t_0} \left\{ \overline{g}\left(\phi U', \phi U'\right) - \overline{R}\left(\phi U, \overline{\gamma}', \overline{\gamma}', \phi U\right) \right\} dt \\ &\leq \int_0^{t_0} \left\{ g\left(U', U'\right) - R\left(U, \gamma', \gamma', U\right) \right\} dt \\ &= I_{t_0}(U, U) \end{aligned}$$

Then,

$$I_{t_0}(\phi U, \phi U) \leq I_{t_0}(U, U) \tag{3.2.4}$$

From (3.2.3) and (3.2.4) we get (3.2.2) and this proves $\left\|\overline{J}(t_0)\right\| \leq \|J(t_0)\|$ for $t_0 \in (0, a]$ which proves the inequality in the theorem.

Now, if $\left\|\overline{J}(t_0)\right\| = \|J(t_0)\|$, for some $t_0 \in (0, a]$. We have from (3.2.2)

$$I_{t_0}\left(\overline{U}, \overline{U}\right) \leq I_{t_0}(U, U), \; t_0 \in (0, a]$$

and also $\overline{f} f' \geq f \overline{f}'$ on $(0, a]$.

Since $\left\|\overline{J}(t_0)\right\| = \|J(t_0)\|$, it follows that

$$\overline{f} f'(t) = f \overline{f}'(t), \; t \in (0, t_0].$$

Now, we have

$$I_{t_0}\left(\overline{U},\overline{U}\right) = I_{t_0}\left(U,U\right) \text{ and } I_{t_0}\left(\overline{U},\overline{U}\right) \leq I_{t_0}\left(\phi U,\phi U\right) \leq I_{t_0}\left(U,U\right)$$

then

$$I_{t_0}\left(\overline{U},\overline{U}\right) = I_{t_0}\left(\phi U,\phi U\right) = I_{t_0}\left(U,U\right).$$

Now, as $\overline{U}$ and ϕU satisfying the hypothesis of index theorem and $I_{t_0}\left(\overline{U},\overline{U}\right) = I_{t_0}\left(\phi U,\phi U\right)$, then $\overline{U} = \phi U$ on $[0, t_0]$.

Next,

$$I_{t_0}\left(\phi U,\phi U\right) = I_{t_0}\left(U,U\right)$$

$$\int_0^{t_0} \left\{\overline{g}\left(\phi U',\phi U'\right) - \overline{R}\left(\phi U,\overline{\gamma}',\overline{\gamma}',\phi U\right)\right\} dt = \int_0^{t_0} \left\{g\left(U',U'\right) - R\left(U,\gamma',\gamma',U\right)\right\} dt$$

$$\int_0^{t_0} \left\{g\left(U',U'\right) - \overline{R}\left(\overline{U},\overline{\gamma}',\overline{\gamma}',\overline{U}\right)\right\} dt = \int_0^{t_0} \left\{g\left(U',U'\right) - R\left(U,\gamma',\gamma',U\right)\right\} dt$$

$$\overline{R}\left(\overline{U},\overline{\gamma}',\overline{\gamma}',\overline{U}\right) = R\left(U,\gamma',\gamma',U\right)$$

$$\frac{1}{\left\|\overline{J}\left(t_0\right)\right\|^2}\overline{R}\left(\overline{J},\overline{\gamma}',\overline{\gamma}',\overline{J}\right) = \frac{1}{\left\|J\left(t_0\right)\right\|^2}R\left(J,\gamma',\gamma',J\right)$$

$$\overline{R}\left(\overline{J},\overline{\gamma}',\overline{\gamma}',\overline{J}\right) = R\left(J,\gamma',\gamma',J\right)$$

$$\overline{K}\left(\overline{J},\overline{\gamma}'\right) = K\left(J,\gamma'\right) \text{ for } t \in (0,t_0]$$

which completes the proof of the Theorem.

3.3 A RICCI CURVATURE CRITERION FOR COMPACTNESS OF RIEMANNIAN MANIFOLDS

In this section we study the generalizations of Myers theorem and the results in this section are taken from [33].

First we review the index form suitable to present setting:

The index form Let $\gamma : [a,b] \to (M^n, g)$ be a geodesic segment with $\left\|\dot{\gamma}\right\| = 1$. Let $\mathfrak{X}_\gamma$ denote the space of piecewise smooth vector field along γ with $g\left(V,\dot{\gamma}\right) = 0$ and $V(a) = V(b) = 0$.

Recall that the index form $I : \mathfrak{X}_\gamma \times \mathfrak{X}_\gamma \to R$ is defined by

$$I(V,W) = \int_a^b \left\{ g\left(\frac{DV}{ds},\frac{DW}{ds}\right) - R\left(V,\dot{\gamma},\dot{\gamma},W\right)\right\} ds$$

If $V, W \in \mathfrak{X}_\gamma$ are smooth, we have

$$\frac{d}{ds} g\left(V', W\right) = g\left(V'', W\right) + g\left(V', W'\right)$$

$$I\left(V, W\right) = -\int_a^b g\left(V'' + R\left(V, \dot{\gamma}\right)\dot{\gamma}, W\right) ds$$

Now, let $X_1\left(s\right), \ldots, X_{n-1}\left(s\right)$, be parallel vector fields along γ, so that $\left\{X_1\left(s\right), \ldots, X_{n-1}\left(s\right), \dot{\gamma}\left(s\right)\right\}$ is an orthonormal basis for $T_{\gamma(s)}M$ for all $s \in [a, b]$.

If $\psi \in C^\infty\left([a, b]\right)$ with $\psi\left(a\right) = \psi\left(b\right) = 0$, then ψX_j is a smooth vector field along γ, which belongs to $\mathfrak{X}_\gamma$.

Hence we get

$$\begin{aligned} I\left(\psi X_j, \psi X_j\right) &= -\int_a^b g\left(\psi'' X_j + R\left(\psi X_j, \dot{\gamma}\right)\dot{\gamma}, \psi X_j\right) ds \\ &= -\int_a^b \psi\left\{g\left(\psi'' X_j, X_j\right) + g(R\left(\psi X_j, \dot{\gamma}\right)\dot{\gamma}, X_j)\right\} ds \\ &= -\int_a^b \psi\left\{\psi'' + \psi R\left(X_j, \dot{\gamma}, \dot{\gamma}, X_j\right)\right\} ds \end{aligned}$$

By summing over j,

$$\begin{aligned} \sum_{j=1}^{n-1} I\left(\psi X_j, \psi X_j\right) &= -\int_a^b \psi\left\{\left(n-1\right)\psi'' + \psi\sum_{j=1}^{n-1} R\left(X_j, \dot{\gamma}, \dot{\gamma}, X_j\right)\right\} ds \\ &= -\int_a^b \psi\left\{\left(n-1\right)\psi'' + \psi\,\mathrm{Ric}\left(\dot{\gamma}, \dot{\gamma}\right)\right\} ds \end{aligned}$$

which gives

$$\sum_{j-1}^{n-1} I\left(\psi X_j, \psi X_j\right) = -\left(n-1\right)\int_a^b \psi\left\{\psi'' + f\psi\right\} ds \text{ where } f = \frac{\operatorname{Ric}\left(\dot\gamma,\dot\gamma\right)}{n-1} \tag{3.3.1}$$

we are now in position to state the following

Lemma 3.3.1 If the differential equation

$$\psi''\left(s\right) + f\left(s\right)\psi\left(s\right) = 0 \tag{3.3.2}$$

has a non-trivial solution ψ with $\psi(a) = \psi(b) = 0$, then the geodesic segment γ from above contains a conjugate point to $\gamma(a)$.

Proof: From (3.3.1) we get $\sum_{j=1}^{n-1} I\left(\psi X_j, \psi X_j\right) = 0$, so for some j we must have $I\left(\psi X_j, \psi X_j\right) \leq 0$.

1. If $I\left(\psi X_j, \psi X_j\right) < 0$. We have $V_j = \psi X_j$ is a piecewise smooth vector field along γ with $V_j(a) = V_j(b) = 0$, then there exists a proper variation $f : [a,b] \times (-\epsilon, \epsilon) \to M$ of γ such that V_j is variation field of f.

Then second variation formula for f is

$$\begin{aligned} \frac{1}{2}E_j''(0) &= \int_a^b \left\{ g\left(\frac{DV_j}{ds}, \frac{DV_j}{ds}\right) - R\left(V_j, \dot\gamma, \dot\gamma, V_j\right)\right\} ds \\ &= I\left(\psi X_j, \psi X_j\right) < 0. \end{aligned}$$

So $E''_j(0) < 0$. Then E_j has maxima at $s = 0$ which is contradiction to γ is minimizing geodesic.

2. If $I(\psi X_j, \psi X_j) = 0$. Suppose that γ has no conjugate point to $\gamma(a)$, then we can find a Jacobi field J along γ with $J(a) = J(b) = 0$ and $g\left(J, \dot{\gamma}\right) = 0$.

Using Index Lemma we have $I(J, J) < I(\psi X_j, \psi X_j) = 0$, then $I(J, J) < 0$, which is contradiction to $I(J, J) = 0$.

Definition 3.3.1 Consider the differential equation

$$\psi''(s) + f(s)\psi(s) = 0 \qquad (*)$$

for some fixed $f \in C^\infty(R)$.

Let a and b be real numbers with $b > a$ for which a nontrivial solution to $(*)$ exists with $\psi(a) = \psi(b) = 0$.

Then b is called a conjugate point to a with respect to $(*)$. For a given a we denote by $\eta(a)$ the first conjugate point to a, if it exists.

Lemma 3.3.2 If $a_1 \leq a_2$ and if $\eta(a_2)$ exists, then $\eta(a_1)$ exists, and $a_1 < \eta(a_1) \leq \eta(a_2)$.

Proof: Let $\psi(s)$ be a nontrivial solution to $(*)$ with $\psi(a_1) = 0$. Since $\eta(a_2)$ exists, then there exists ψ_1 be nontrivial solution to $(*)$ with $\psi_1(a_2) = \psi_1(\eta(a_2)) = 0$.

There is zero of $\psi(s)$ between any two zeros of $\psi_1(s)$ because :

if $\begin{cases} \psi_1'' + f\psi_1 = 0 \\ \psi'' + f\psi = 0 \end{cases}$ then

$$\psi\psi_1'' - \psi_1\psi'' = 0$$

$$\left[\psi\psi_1' - \psi_1\psi'\right]' = 0$$

$$\psi\psi_1' - \psi_1\psi' = \text{constant} = \alpha$$

Now, if $\alpha \neq 0$, let s_1, s_2 be two successive zero of ψ_1. Then $\alpha = \left(\psi\psi_1'\right)(s_1) = \left(\psi\psi_1'\right)(s_2)$.

As $\alpha \neq 0$, $\psi(s)$ and $\psi_1'(s)$ are both non zero at s_1, s_2. Also, as $\psi_1(s_1) = \psi_1(s_2) = 0$, $\psi_1'(s_1)$ and $\psi_1'(s_2)$ must have opposite signs.

Then $\psi(s_1)$ and $\psi(s_2)$ must have opposite signs and ψ must have zero on (s_1, s_2).

If $\alpha = 0$ then $\psi\psi_1' = \psi_1\psi'$ and $\frac{\psi'}{\psi} = \frac{\psi_1'}{\psi_1}$.

Integrating last equation we get $\psi = c\psi_1$, then $\psi(s_2) = c\psi_1(s_2) = 0$.

From above ψ must have zero on $[a_2, \eta(a_2)]$, which proves the Lemma.

Proposition 3.3.1 Suppose the function f in definition (3.3.1) satisfies the following conditions, where $r > 0$, $b > a \geq a_1$ are constants.

(i) $f(s) \leq \frac{1}{r^2}$ for all $s \in [a, b]$.

(ii) $\int_a^b f(s)\, ds \geq \frac{\pi}{r}$.

Then $\eta(a_1) \leq b$.

Proof: Since $a_1 \leq a$, and if $\eta(a)$ exists, then by lemma (3.3.2), $\eta(a_1)$ exists and $a_1 < \eta(a_1) < \eta(a)$. So we only have to show that $\eta(a)$ exists and $\eta(a) \leq b$.

Define $F : [a, b] \to R$ by $F(t) = \int\limits_a^t f(s)\, ds$.

Note that F is continuous and $F(a) = 0$ and $F(b) = \int\limits_a^b f(s)\, ds \geq \frac{\pi}{r}$. Since $\frac{\pi}{2r} \in \left[0, \frac{\pi}{r}\right]$ then by intermadiet value theorem there is a smallest $c \in [a, b]$ such that $F(c) = \int\limits_a^c f(s)\, ds = \frac{\pi}{2r}$.

Let $\psi(s)$ be a nontrivial solution to the differential equation $(*)$ satisfying $\psi'(a) > 0$ and $\psi(a) = 0$. We shall prove that there is a value $\widetilde{c} \in (a, c]$ such that $\psi'(\widetilde{c}) = 0$. Suppose not.

Define $x(s) = \tan^{-1}\left(\frac{\psi(s)}{r\psi'(s)}\right)$.Then $x(s)$ is well defined in $[a, c]$ with $x(s) < \frac{\pi}{2}$ and $x(a) = 0$ as $\psi(a) = 0$.

Now,

$$\tan x = \frac{\psi}{r\psi'} \tag{3.3.3}$$

or

$$\sin x = \frac{\psi}{r\psi'} \cos x. \tag{3.3.4}$$

Differentiating (3.3.3), we get $\sec^2 x.x' = \frac{r\psi'^2 - r\psi\psi''}{r^2\psi'^2}$ but equation $(*)$ gives $x' = \cos^2 x \left\{\frac{r\psi'^2 + rf\psi^2}{r^2\psi'^2}\right\}$. Using (3.3.4) we have $x' = \frac{1}{r}\cos^2 x + rf \sin^2 x$ and from condition (i), $rf \leq \frac{1}{r}$, then $x' \geq rf \cos^2 x + rf \sin^2 x = rf$.

Integrating last equation we have

$$\int_a^c x'(s)\,ds \geq \int_a^c rf(s)\,ds$$

$$x(c) - x(a) \geq r\frac{\pi}{2r}$$

$$x(c) \geq \frac{\pi}{2}$$

which is a contradiction to $x < \frac{\pi}{2}$ for all $s \in [a, c]$. Hence there must be some $\widetilde{c} \leq c$ where $\psi'(\widetilde{c}) = 0$.

Now, assume that $\psi(s) > 0$ for all $s \in [\widetilde{c}, b]$, so that $y(s) = \tan^{-1}\left(\frac{-r\psi'}{\psi}\right)$ is well defined with $y(s) < \frac{\pi}{2}$ in $[\widetilde{c}, b]$, and $y(\widetilde{c}) = 0$.

Next,

$$\tan y = \frac{-r\psi'}{\psi}$$

Differentiating this equation and using $(*)$, we have

$$y' \sec^2 y = -\left\{\frac{-r\psi'^2 + r\psi\psi''}{\psi^2}\right\} = \frac{r\psi'^2}{\psi^2} + rf \tag{3.3.5}$$

but $\sin y = \frac{-r\psi'}{\psi}\cos y$ substitute in (3.3.5) to get $y' = \frac{1}{r}\sin^2 y + rf\cos^2 y \geq rf$.

Integrating last equation on $[\tilde{c}, b]$, we get

$$\int_{\widetilde{c}}^b y'(s)\,ds \geq \int_{\widetilde{c}}^b rf ds$$

$$y(b) - y(\widetilde{c}) \geq \int_a^b rf ds - \int_a^{\widetilde{c}} rf ds.$$

But

$$\int_a^{\widetilde{c}} rf ds = \int_a^{c} rf ds - \int_{\widetilde{c}}^{c} rf ds = \frac{\pi}{2} - \int_{\widetilde{c}}^{c} rf ds \leq \frac{\pi}{2}.$$

Thus,

$$y(b) \geq \int_a^{b} rf ds - \frac{\pi}{2} \geq \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

So $y(b) \geq \frac{\pi}{2}$ and this contradicts that $y(s) < \frac{\pi}{2}$ for all $s \in [\widetilde{c}, b]$.

Hence there must be some $\widetilde{b} \leq b$ where $\psi\left(\widetilde{b}\right) = 0$. Then we have $a < b$ and $\psi(a) = \psi\left(\widetilde{b}\right) = 0$, gives $\eta(a)$ exists and $\eta(a) \leq \widetilde{b} \leq b$.

This complete the proof.

Theorem 3.3.1 Let $\psi(s)$ be a solution to the differential equation $(*)$ satisfying $\psi'(\widetilde{c}) = 0$ and $\psi(\widetilde{c}) > 0$. If $\lim\limits_{\alpha\to\infty} \int_{\widetilde{c}}^{\alpha} f(s)\, ds > 0$, then there is a value $\widetilde{b} < \infty$ for which $\psi\left(\widetilde{b}\right) = 0$.

Corollary 3.3.1 Suppose f satisfies the following conditions, where $r > 0$, $c > a > a_1$ are constants.

(i) $f(s) \leq \frac{1}{r^2}$ for all $s \in [a, c]$,

(ii) $\int_a^c f(s)\, ds \geq \frac{\pi}{2r}$,

(iii) $\lim\limits_{\alpha\to\infty} \int_t^{\alpha} f(s)\, ds > 0$ for all $t \in (a, c]$.

Then, $\eta(a_1)$ exists.

Proof: From the proof of last proposition there is a value $\widetilde{c} \in (a, c]$ such that $\psi'(\widetilde{c}) = 0$ where ψ is non trivial solution to $*$ with $\psi'(a) > 0$ and $\psi(a) = 0$ and then $\psi(\widetilde{c}) > 0$.

Putting $t = \tilde{c}$ and using theorem (3.3.1), there is a value $\widetilde{b} < \infty$ for which $\psi\left(\widetilde{b}\right) = 0$.

Thus $\eta(a_1)$ exists and $\eta(a_1) \leq \widetilde{b}$.

Condition R (The compactness condition) A Riemannian manifold (M^n, g) is said to satisfy condition R if there is a point $p \in M^n$, and for any γ in the set $G_p(M^n)$ of normal geodesics starting from p we can find real numbers $r_\gamma > 0$, $b_\gamma > a_\gamma \geq 0$ which satisfy

(i) $\mathrm{Ric}\left(\dot{\gamma}(s), \dot{\gamma}(s)\right) \leq \frac{n-1}{r_\gamma^2}$ for all $s \in [a_\gamma, b_\gamma]$,

and

(ii) $\int\limits_{a_\gamma}^{b_\gamma} \mathrm{Ric}\left(\dot{\gamma}(s), \dot{\gamma}(s)\right) ds \geq \frac{\pi(n-1)}{r_\gamma}$.

Theorem 3.3.2 Let (M^n, g) be a complete, connected Riemannian manifold, and assume that there is a point $p \in M^n$ with the property, that any geodesic starting from p contains a conjugate point to p. Then M^n is compact.

A very rough estimate of the diameter of a compact manifold (M^n, g) is obtained in the following way. Consider two points q_1, q_2 at maximal distance in M^n. For any $p \in M^n$ we then have

$$diam\,(M^n, g) \leq dist\,(p, q_1) + dist\,(p, q_2)\,.$$

Thus we have proved the following theorem

Theorem 3.3.3 If (M^n, g), $n \geq 2$, is a complete, connected Riemannian manifold, which satisfy condition R, then M^n is compact.

Furthermore, if $\{b_\gamma\}$ is the set of constants in the definition of condition R, then we get the estimate

$$diam\,(M^n, g) \leq \sup_{\gamma \in G_p} \{2b_\gamma\}\,.$$

CHAPTER 4

NON-IMMERSIBILITY THEOREM

One of the interesting questions in geometry of a Riemannian manifolds is to investigate whether given compact Riemannian manifold can be immersed into another Riemannian manifold. Very well known result in this direction are those of Whitney [34], Nash [27]. Chern and Kuiper [12] has considered opposite problem under what circumstances a compact Riemannian manifold can not be immersed into same known Riemannian manifold. Since then Tamking [46], Jacowibz [26], and Deshmukh [13] have obtained different conditions on a compact Riemannian manifolds which do not admit isometric immersions in certain Euclidean spaces. Recently Moore[35] using applications of second variation formula obtained most generalized non-immersibility result. The aim of this chapter is to study in detail the Moore's theorem.

Theorem 4.1.1 Let $\overline{M}$ be a complete simply connected Riemannian manifold whose sectional curvature $\overline{K}(\sigma)$ satisfy the inqualities

$$a \leq \overline{K}(\sigma) \leq b \leq 0, \tag{4.1.1}$$

M a compact Riemannian manifold whose sectional curvature $K(\sigma)$ satisfy $K(\sigma) \leq a - b$. If $\dim \overline{M} < 2 \dim M$, then M possesses no isometric immersion in $\overline{M}$.

Proof: We assume the existence of an isometric immersion $f : M \to \overline{M}$ which satisfies the hypotheses of the theorem, and derive a contradiction.

Choose a point $\overline{p} \in \overline{M}$. Let Ω be the set of pairs (q, γ), where $q \in M$ and $\gamma : [0,1] \to \overline{M}$ is a smooth path satisfying $\gamma(0) = \overline{p}$, $\gamma(1) = f(q)$. (To avoid clumsy notation, we will write γ for the pair (q, γ), and identify $q \in M$ with its image $f(q)$). We have a function

$$J : \Omega \to R$$

defined by

$$J(\gamma) = \frac{1}{2}\int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt,$$

where $\langle\ ,\ \rangle$ is the Riemannian metric of $\overline{M}$. A critical point for J is a constant speed geodesic which hits M orthogonally.

If $\gamma \in \Omega$, let $T_\gamma \Omega$ denote the space of smooth vector field X along γ such that $X(0) = 0$ and $X(1)$ is tangent to M.

At a critical point γ, we can define the Hessian d^2J on $T_\gamma \Omega$. To compute d^2J, take $X \in \Gamma_\gamma\left(\overline{M}\right)$ as a variation vector field. Then

$$\begin{aligned} XJ(\gamma) &= \int_0^1 \langle \overline{\nabla}_X \dot{\gamma}, \dot{\gamma} \rangle dt \\ &= \int_0^1 \langle \overline{\nabla}_{\dot{\gamma}} X, \dot{\gamma} \rangle dt \qquad \text{(As } X \text{ is a variation vector field)} \end{aligned}$$

and

$$\begin{aligned} XXJ(\gamma) &= \int_0^1 \langle \overline{\nabla}_X \overline{\nabla}_{\dot{\gamma}} X, \dot{\gamma}\rangle + \langle \overline{\nabla}_{\dot{\gamma}} X, \overline{\nabla}_{\dot{\gamma}} X\rangle dt \\ &= \int_0^1 \langle \overline{R}\left(X, \dot{\gamma}\right) X + \overline{\nabla}_{\dot{\gamma}} \overline{\nabla}_X X, \dot{\gamma}\rangle + \langle \overline{\nabla}_{\dot{\gamma}} X, \overline{\nabla}_{\dot{\gamma}} X\rangle dt \\ &= I(X, X) + \int_0^1 \langle \overline{\nabla}_{\dot{\gamma}} \overline{\nabla}_X X, \dot{\gamma}\rangle dt \\ &= I(X, X) + \langle \overline{\nabla}_X X, \dot{\gamma}\rangle\Big|_0^1 \\ &= I(X, X) + \langle h\left(X(1), X(1)\right), \dot{\gamma}(1)\rangle \qquad (4.1.2) \end{aligned}$$

Note that if γ is a critical point of J, then

$$XJ(\gamma) = 0 \qquad \forall X \in \Gamma_\gamma\left(\overline{M}\right)$$

which gives

$$\overline{\nabla}_X XJ(\gamma) = 0 \qquad (4.1.3)$$

Using $(4.1.2), (4.1.3)$ we get

$$\begin{aligned} d^2 J(X, X) &= XXJ - \overline{\nabla}_X XJ \\ &= I(X, X) + \langle h\left(X(1), X(1)\right), \dot{\gamma}(1)\rangle. \end{aligned}$$

Consequently,

$$d^2 J(X+Y, X+Y) = I(X+Y, X+Y) + \langle h\left(X+Y(1), X+Y(1)\right), \dot{\gamma}(1)\rangle$$

The linear and symmetric property gives

$$\begin{aligned} d^2J(X,X)+2d^2J(X,Y)+d^2J(Y,Y) &= I(X,X)+2I(X,Y)+I(Y,Y) \\ &\quad +\langle h(X(1),X(1)),\dot\gamma(1)\rangle \\ &\quad +2\langle h(X(1),Y(1)),\dot\gamma(1)\rangle \\ &\quad +\langle h(Y(1),Y(1)),\dot\gamma(1)\rangle \end{aligned}$$

and hence

$$d^2J(X,Y)=I(X,Y)+\langle h(X(1),Y(1)),\dot\gamma(1)\rangle.$$

Note that if X is a Jacobi field, (cf. chapter III, the proof of Rauch's Theorem).

$$I(X,X)=\frac{1}{2}\frac{d}{dt}\langle X(t),X(t)\rangle|_{t=1} \tag{4.1.4}$$

We have index form for Riemannian manifold of constant curvature c:

$$I_c(X,Y)=\int_0^1\left\{\langle\overline{\nabla}_{\dot\gamma}X,\overline{\nabla}_{\dot\gamma}Y\rangle-c\left[\langle X,Y\rangle\langle\dot\gamma,\dot\gamma\rangle-\langle X,\dot\gamma\rangle\langle Y,\dot\gamma\rangle\right]\right\}dt.$$

In our situation, it follows from (4.1.1) that

$$I_b(X,X)\le I(X,X)\le I_a(X,X).$$

Let v be a unit vector in $\overline{M}$ perpendicular to $\dot\gamma(1)$, E a parallel vector field along γ such that $E(1)=v$. Let d be the length of γ and consider the vector field

$$X(t)=\begin{cases} tE(t), & \text{if } b=0 \\ \frac{\sinh\left(d\sqrt{-b}t\right)}{\sinh\left(d\sqrt{-b}\right)}E(t), & \text{if } b<0. \end{cases}$$

X is a Jacobi field for I_b satisfying $X(0) = 0$, $X(1) = v$ and from (4.1.4),

$$\begin{aligned} I_b(X,X) &= \frac{1}{2}\frac{d}{dt}\langle X(t), X(t)\rangle|_{t=1} \\ &= \left\{ \begin{array}{ll} 1 & \text{if } b = 0 \\ \left(d\sqrt{-b}\right)\coth\left(d\sqrt{-b}\right) & \text{if } b < 0. \end{array} \right\} > d\sqrt{-b}. \end{aligned}$$

Since γ has no I_b-conjugate points (b is non positive sectional curvature) it follows from Index lemma that if Y is any vector field along γ satisfying $Y(0) = 0$, $Y(1) = v$, then

$$I_b(Y,Y) \geq I_b(X,X) > d\sqrt{-b}.$$

Now let q be a point in M which has maximal distance d from $\overline{p}$. Let γ be the minimal geodesic from $\overline{p}$ to q. For each $v \in T_qM$, there is a unique Jacobi field V along γ such that $V(0) = 0$, $V(1) = v$.

Corresponding to V, we have a one-parameter family of geodesics from $\overline{p}$ to M, each of which is minimal and hence no longer than γ. Therefore,

$$0 \geq d^2J(V,V) = I(V,V) + \langle h(v,v), \dot{\gamma}(1)\rangle.$$

If v has unit-length, then

$$I(V,V) \geq I_b(V,V) > d\sqrt{-b}$$

and hence

$$\langle h(v,v), \dot{\gamma}(1)\rangle < -d\sqrt{-b} \leq 0.$$

Since $\left\|\dot{\gamma}(1)\right\| = d$, it follows that

$$\|h(v,v)\| > \sqrt{-b}, \text{ for all unit length } v \in T_qM. \tag{4.1.5}$$

On the other hand, if v and w are orthogonal unit vectors in T_qM which span a two-plane σ, the Gauss equation yields

$$\langle h(v,v), h(w,w)\rangle - \|h(v,w)\|^2 = K(\sigma) - \overline{K}(\sigma) \leq -b. \tag{4.1.6}$$

We shall show that (4.1.5) and (4.1.6) are incompatible with the condition $\dim \overline{M} < 2 \dim M$:

Let S denote the unit sphere in T_qM, and consider the function

$$f : S \to R \text{ defined by } f(v) = \langle h(v,v), h(v,v)\rangle.$$

If $v \in S$ where f assumes its minimum, then for all $w \in S$ such that $w \perp v$ we have $v^{\cdot} = w$, $v^{\cdot\cdot} = -v$ and

$$\begin{aligned} 0 &= df(v) = 2\langle 2h(v, v^{\cdot}), h(v,v)\rangle \\ &= 4\langle h(v,w), h(v,v)\rangle \end{aligned} \tag{4.1.7}$$

Also,

$$\begin{aligned} 0 &\leq d^2f(v) = 4\{\langle h(w,w) + h(v,-v), h(v,v)\rangle + \langle h(v,w), 2h(v,w)\rangle\} \\ &= 4\langle h(w,w), h(v,v)\rangle - 4\langle h(v,v), h(v,v)\rangle + 8\langle h(v,w), h(v,w)\rangle \end{aligned}$$

Now look at the linear map $w \to h(v,w)$. Equation (4.1.7) and the fact that $\dim \overline{M} < 2 \dim M$ imply that there is a $w \in S$, $w \perp v$, for which $h(v,w) = 0$. For this choice of w, the Hessian d^2f on S satisfies

$$\begin{aligned} 0 &\leq d^2f(v) \\ &= 4\langle h(w,w), h(v,v)\rangle - 4\langle h(v,v), h(v,v)\rangle. \end{aligned}$$

Now apply (4.1.5) and (4.1.6) to obtain $0 < -4b - 4\left(\sqrt{-b}\right)^2 = 0$.

This contradiction establishes the theorem.

CHAPTER 5

CHARACTERIZATION OF A REAL SPACE FORM

In this chapter as an appliation of Jacobi fields, we discuss a basic formula for the shape operator of geodesic spheres and obtain a characterization of a real space form (a Riemannian manifold of constant sectional curvature) in terms of geodesic spheres.

5.1 JACOBI FIELDS AND THE SHAPE OPERATOR

Let (M, g) be a Riemannian manifold of dimension n. Let $m \in M$ and let ξ be a unit vector of the tangent space T_mM at m. Further, let γ be a geodesic with initial velocity vector ξ. We always suppose that $|r|$ is sufficiently small, so that $\gamma(r)$ belongs to a fixed normal neighborhood of m . Next, let $\{e_i, i = 1, ..., n\}$ be an orthonormal basis at m with $e_1 = \xi$ and denote by $\{E_i, i = 1, \ldots, n\}$ be an orthonormal basis along γ obtained by the parallel transport of $\{e_i\}$ along γ. Then, the Jacobi vector fields X along γ are given by the Jacobi equation

$$\ddot{X} + R\left(X, \dot{\gamma}\right)\dot{\gamma} = 0. \tag{5.1.1}$$

Next, let $X_a(s)$, $a = 2, ..., n$ be the Jacobi vector field satisfying the initial conditions

$$X_a(0) = 0, \qquad \dot{X}_a(0) = e_a.$$

and put

$$X_a(r) = (AE_a)(r), \qquad a = 2, \ldots, n \tag{5.1.2}$$

From (5.1.1) and (5.1.2) we see that the endomorphism-valued function $r \to A(r) : \{\gamma'(r)\}^{\perp} \to \{\gamma'(r)\}^{\perp}$ satisfies the differential equation

$$A'' + R \circ A = 0 \tag{5.1.3}$$

and the initial conditions

$$A(0) = 0, \qquad A'(0) = I. \tag{5.1.4}$$

Now, let $(x^1, \ldots, x^n)$ be a system of normal coordinates centered at m and such that

$$\frac{\partial}{\partial x^i}(m) = e_i, \qquad i = 1, \ldots, n.$$

Then we have

Lemma 5.1.1 The vector fields $\frac{\partial}{\partial x^a}$, $a = 2, \ldots, n$, and the Jacobi vector fields X_a along γ are related by

$$X_a(r) = r \frac{\partial}{\partial x^a}(\gamma(r)), \qquad a = 2, \ldots, n. \tag{5.1.5}$$

Proof: Consider the map $\Gamma(r, \alpha) = \exp_m(r \cos\alpha \xi + r \sin\alpha e_a)$. This is a variation of the geodesic $\gamma(r) = \Gamma(r, 0)$ by a one-parameter family of geodesics and the corresponding transverse vector field

$$\begin{aligned} \frac{\partial}{\partial \alpha}\Gamma(r, \alpha)\bigg|_{\alpha=0} &= d(\exp_m)_{r\xi}(re_a) \\ &= r\frac{\partial}{\partial x^a}(\gamma(r)) \end{aligned}$$

must be a Jacobi vector field. It satisfies the same initial conditions as $X_a(r)$ and hence our Lemma follows.

Further, we put

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad i, j = 1, \dots, n.$$

Then we have

Lemma 5.1.2 At $p = \exp_m(r\xi)$ it holds

$$\begin{cases} g_{11}(p) = 1, & g_{1a}(p) = 0, \\ g_{ab}(p) = \frac{1}{r^2} g(AE_a, AE_b)(r), & a, b = 2, \dots, n \end{cases} \tag{5.1.6}$$

or in matrix notation

$$(g_{ij})(p) = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2}\ {}^t[A][A](p) \end{bmatrix} \tag{5.1.7}$$

where $[A]$ denotes the matrix of the endomorphism A with respect to the canonical frame $\{E_2, \dots, E_n\}$ and ${}^t[A]$ is the transpose of $[A]$.

Proof: The first two relations are clear. The third relation comes from

$$g_{ab}(p) = g_p\left(\left(\frac{\partial}{\partial x^a}\right)_p, \left(\frac{\partial}{\partial x^b}\right)_p\right)$$

Using Lemma (5.1.1) and equation (5.1.2) we get

$$\begin{aligned} g_{ab}(p) &= \frac{1}{r^2} g_p(X_a(r), X_b(r)) \\ &= \frac{1}{r^2} g_p((AE_a)(r), (AE_b)(r)) \\ &= \frac{1}{r^2} g_p\left({}^tAAE_a, E_b\right) \end{aligned}$$

and hence our Lemma follows.

Finally, let $S_m(p)$ denote the shape operator of the geodesic sphere $G(m, \overline{r})$ at $p = \exp_m(\overline{r}\xi) = \gamma(\overline{r})$. This operator is defined by

$$S_m(p)X = -\nabla_X \frac{\partial}{\partial r}, \qquad X \in T_pG(m, \overline{r}),$$

where $\frac{\partial}{\partial r}$ means the field of unit normal vectors along the sphere $G(m, \overline{r})$. As in the proof of Lemma (5.1.1) consider the map

$$\Gamma(r, \alpha) = \exp_m (r \cos \alpha \xi + r \sin \alpha e_a),$$

which is a diffeomorphism of a neighborhood of the point $(\overline{r}, 0)$ in R^2 onto a piece of a surface through p in M. On this surface we have

$$\left[\frac{\partial \Gamma}{\partial r}, \frac{\partial \Gamma}{\partial \alpha}\right] = 0, \qquad \text{i.e. } \nabla_{\frac{\partial \Gamma}{\partial \alpha}} \frac{\partial \Gamma}{\partial r} = \nabla_{\frac{\partial \Gamma}{\partial r}} \frac{\partial \Gamma}{\partial \alpha}.$$

On the other hand,

$$\left(\frac{\partial \Gamma}{\partial r}\right)(\overline{r}, \alpha) = \frac{\partial}{\partial r}(\Gamma(\overline{r}, \alpha)), \ \left(\frac{\partial \Gamma}{\partial r}\right)(\overline{r}, 0) = \gamma'(\overline{r}), \ \left(\frac{\partial \Gamma}{\partial \alpha}\right)(r, 0) = X_a(r).$$

For $r = \overline{r}$, $\alpha = 0$ and the unit normal vector field $-\xi$ of $G(m, \overline{\gamma})$ at m we get

$$\begin{aligned} S_m(p) X_a(\overline{r}) &= -\nabla_{X_a(\overline{r})} \frac{\partial}{\partial r} \\ &= -\nabla_{-\gamma'(\overline{r})} X_a(r) \\ &= X_a'(\overline{r}). \end{aligned} \tag{5.1.8}$$

Then, using (5.1.2) and the fact that $A(r)$ is non-singular for small $r \neq 0$, we obtain

Lemma 5.1.3 At $p = \exp_m(r\xi)$ the shape operator $S_m(p)$ is given by

$$S_m(p) = \left(A' A^{-1}\right)(r). \tag{5.1.9}$$

Put $C(r) = r S_m(p)$. Then we have $(CA)(r) = r A'(r)$. Differentiating this twice and using (5.1.3) and (5.1.4) we get for $r = 0$:

$$C(0) = I, \qquad C'(0) = 0.$$

5.2 WRONSKIAN AND SOME APPLICATIONS

The Jacobi equation for the endomorphism-valued functions along geodesics

$$L'' + R \circ L = 0 \tag{5.2.1}$$

is fundamental in the geometry of geodesic spheres. Let F, G be two solutions of (5.2.1) along the geodesic γ. Then

$$W(F,G) = {}^tF'G - {}^tFG' \tag{5.2.2}$$

is called the Wronskian of F and G.

Lemma 5.2.1 Under the identification of endomorphisms by parallel transport, the Wronskian $W(F,G)$ is constant along γ.

Proof: We have to show that

$$W'(F,G) = 0.$$

Differentiating the Wronskian covariantly along our geodesic

$$\begin{aligned} W'(F,G) &= {}^tF''G + {}^tF'G' - {}^tF'G' - {}^tFG'' \\ &= {}^tF''G - {}^tFG'' \end{aligned}$$

Substituting from (5.2.1) and the symmetry of R give us

$$\begin{aligned} W'(F,G) &= -{}^tF \circ {}^tR \circ G + {}^tF \circ R \circ G \\ &= 0. \end{aligned}$$

This simple property has some remarkable consequences. In what follows we shall give some applications related to the geodesic spheres.

First, from (5.1.4) and Lemma (5.2.1) we obtain

Lemma 5.2.2 $W(A, A) = 0$, i.e. ${}^tA'A = {}^tAA'$.

Note that that this is equivalent to the property that the shape operator $S_m(p)$ from (5.1.9) is symmetric (self-adjoint).

Next, we consider the geodesic sphere $G(p, r)$ with center $p = \exp_m(r\xi)$ and radius r. Our next aim is to express the shape operator $S_p(m)$ of this sphere at the point m in a form which will be convenient for further mainpulations in section 3. Therefore, we consider first the solution B of the Jacobi equation (5.2.1) with initial conditions

$$B(r) = 0, \qquad B'(r) = -I. \tag{5.2.3}$$

Then we get, analogously to formula (5.1.9),

$$S_p(m) = -B'(0)\, B^{-1}(0) \tag{5.2.4}$$

where the shape operator is computed with respect to the unit normal vector ξ of $G(p, r)$ at m.

We shall now rewrite (5.2.4), using other endomorphism-valued Jacobi functions and Lemma (5.2.1). Substituting the initial conditions (5.1.4) and (5.2.3) into the equality $W(A, B)(r) = W(A, B)(0)$, we obtain first

$$B(0) = {}^tA(r).$$

Further, let D be the solution of (5.2.1) satisfying the initial conditions

$$D(0) = I, \qquad D'(0) = 0. \tag{5.2.5}$$

Using (5.2.3) and (5.2.5) in the equality $W(B, D)(r) = W(B, D)(0)$, we obtain

$$B'(0) = -{}^{t}D(r).$$

Substituting for $B(0)$ and $B'(0)$ into (5.2.4) we get

$$\begin{aligned} S_p(m) &= {}^{t}D(r)\,{}^{t}A^{-i}(r) \\ &= {}^{t}\left(A^{-1}(r)\,D(r)\right). \end{aligned} \tag{5.2.6}$$

Now, because $S_p(m)$ is a symmetric endomorphism, we get finally

$$S_p(m) = A^{-1}(r)\,D(r). \tag{5.2.7}$$

Let us notice we were using currently the identification of endomorphisms via parallel transport along γ.

5.3 NEW FORMULA FOR THE SHAPE OPERATOR

We start with the following

Proposition 5.3.1 Let $S_p(m)$ denote the shape operator of the geodesic sphere $G(p, \overline{r})$ at m, where $p = \exp_m(\overline{r}\xi) = \gamma(\overline{r})$. Then we have (via the parallel transport along γ)

$$\dot{S}_p(m) = -A^{-1}(\overline{r})\,{}^{t}A^{-1}(\overline{r}) \tag{5.3.1}$$

where

$$\dot{S}_p(m) = \frac{d}{dr} S_{\gamma(r)}(m)\Big|_{r=\overline{r}} \tag{5.3.2}$$

denotes the derivative with respect to the variable p along γ, keeping m fixed, and $A(r)$ is the Jacobi function given by (5.1.4).

proof: We use Lemma (5.2.1) for A and D at the points $r = 0$ and $r = \overline{r}$:

$$W(A, D)(\overline{r}) = W(A, D)(0).$$

With the initial conditions (5.1.4) and (5.2.5) we obtain

$$^{t}A'(\overline{r})\, D(\overline{r}) - {}^{t}A(\overline{r})\, D'(\overline{r}) = I. \tag{5.3.3}$$

Next, we write (5.2.7) in the form

$$D(r) = A(r)\, S_{\gamma(r)}(m), \qquad r \in (\overline{r} - \delta, \overline{r} + \delta)$$

and differentiate this covariantly for $r = \overline{r}$. We get

$$D'(\overline{r}) = A(\overline{r})\, \dot{S}_p(m) + A'(\overline{r})\, S_p\ (m).$$

Substituting both in (5.3.3), we obtain

$$\left({}^{t}A'A - {}^{t}AA'\right)(\overline{r})\, S_p(m) - \left({}^{t}AA\right)(\overline{r})\, \dot{S}_p(m) = I.$$

Now, the first term vanishes according to Lemma (5.2.2) and hence our formula follows.

Now, we are able to prove the following

Theorem 5.3.1 Let $\left(x^1, \ldots, x^n\right)$ be the system of normal coordinates centered at $m \in M$ and such that $\frac{\partial}{\partial x^1}(m) = \xi$, $\frac{\partial}{\partial x^a}(m) = e_a$, $a = 2, \ldots, n$.

Let $\left(g^{ij}\right)$ denote the matrix of contravariant components of the metric tensor g calculated in these coordinates. Then we have at $p = \exp_m(r\xi)$

$$\begin{cases} g^{11}(p) = 1, \qquad g^{1a}(p) = 0, \qquad a = 2, \ldots, n \\ g^{ab}(p) = -r^2 g\left(\dot{S}_p(m)\, e_a, e_b\right), \qquad a, b = 2, \ldots, n \end{cases} \tag{5.3.4}$$

Or in matrix form,

$$\left(g^{ij}\right)(p) = \begin{bmatrix} 1 & 0 \\ 0 & -r^2\left[\dot{S}_p(m)\right] \end{bmatrix} \tag{5.3.5}$$

where $\left[\dot{S}_p(m)\right]$ denotes the matrix of the endomophism $\dot{S}_p(m)$ with respect to the canonical frame $\{e_2, \ldots, e_n\}$ of $\{\xi\}^{\perp} \subset T_mM$.

Proof: According to (5.1.7) in Lemma (5.1.2) we have first

$$\left(g^{ij}\right)(p) = \begin{bmatrix} 1 & 0 \\ 0 & r^2 [A]^{-1\,t} [A]^{-1} \end{bmatrix}$$

and according to (5.3.1) we have

$$\left[\dot{S}_p(m)\right] = -[A]^{-1\,t}[A]^{-1}(p)$$

where the matrix on the right-hand side is calculated with respect to the frame $\{E_2, \ldots, E_n\}(p)$ and the matrix on the left-hand side is calculated with respect to the frame $\{e_2, \ldots, e_n\}$ at m. Hence our theorem follows.

5.4 CHARACTERIZATION OF REAL SPACE FORM

Let M be a connected n-dimensional Riemannian manifold. Let γ be a fixed geodesic in M, mapping some open interval which contain the real interval $[a, b]$ into M, parametrized by arc length, $\left\|\dot{\gamma}\right\| = 1$. We assume that $\gamma(a) = p$.

Choose an orthonormal frame at p so that e_n is along the tangent to γ, and e_a, $a = 1, 2, ..., n-1$, complete the frame. We denote by $E_i(s)$, $i = 1, ..., n$, the frame at $\gamma(s)$ obtained by parallel transport at p along the geodesic γ.

Now, let X be a Jacobi vector field along γ, then

$$\ddot{X} + R\left(X, \dot{\gamma}\right)\dot{\gamma} = 0. \tag{5.4.1}$$

We denote by $X_\alpha(s)$, $\alpha = 1, ..., n$ the Jacobi vector field uniquely determined along γ by the initial conditions

$$X_\alpha(a) = 0, \qquad \dot{X}_\alpha(a) = e_\alpha. \tag{5.4.2}$$

We form an $n \times n$ matrix F from the components of $X_\alpha(s)$ with respect to the frame $E_i(s)$ previously constructed by arranging that the first column of F shall be the components of X_1, the second column the components of X_2, etc. Thus we have

$$FE_\alpha = X_\alpha,$$

and hence

$$\dot{F}E_\alpha = \dot{X}_\alpha.$$

Now, from (5.1.9) the shape operator S associated with geodesic sphere center p and radius s satisfies

$$S = \dot{F}F^{-1}. \tag{5.4.3}$$

We now prove

Theorem 5.4.1 Let M^n be a connected n-dimensional Riemannian manifold $(n \geq 3)$ with the property that every sufficiently small geodesic sphere is totally umbilical. Then M^n is of constant sectional curvature.

Proof: Recall that a submanifold for which $h(X,Y) = \langle X,Y \rangle H$ is called totally umbilical, where H is the mean curvature vector of the submanifold. For a hyper surface $h(X,Y) = \langle SX,Y \rangle N$, and $H = \langle H,N \rangle N = \lambda N$, $\lambda \in C^{\infty}(M)$ and consequently the shape operator $S = \lambda I$.

Now, our hypothesis of total umbilicity gives

$$\dot{F}F^{-1} = \lambda I, \qquad \lambda \in C^{\infty}(M).$$

that is

$$\dot{F} = \lambda F \tag{5.4.4}$$

Differentiate

$$\ddot{F} = \dot{\lambda}F + \lambda\dot{F}$$

Use (5.4.4) to get

$$\ddot{F} = \left(\dot{\lambda} + \lambda^2\right)F. \tag{5.4.5}$$

Substitute in (5.4.1) to get

$$\left(\dot{\lambda} + \lambda^2\right)F + RF = 0 \tag{5.4.6}$$

Now it is known that the matrix F has maximal rank for $a < s < b$ provided that b occurs before the first conjugate point to a along γ. As $\det F \neq 0$, F^{-1} exist. Hence (5.4.6) gives

$$\left(\dot{\lambda} + \lambda^2\right)I + R = 0 \tag{5.4.7}$$

and

$$\left(\dot{\lambda}+\lambda^{2}\right)e_{a}+R\left(e_{a},\dot{\gamma}\right)\dot{\gamma}=0$$

Taking the inner product with e_b,we obtain

$$\left(\dot{\lambda}+\lambda^{2}\right)\langle e_{a},e_{b}\rangle+R\left(e_{a},e_{n},e_{n}e_{b}\right)=0$$

which gives

$$R\left(e_{a},e_{n},e_{n},e_{b}\right)=0\text{ if }a\neq b,\ a,b=1,...,n-1.$$

However, the geodesic γ at p may be chosen in an arbitrary direction. Thus we have proved that for any three orthogonal vectors at p we have

$$R\left(X,Y,Y,Z\right)=0.$$

From which it follows that the space is of constant curvature and the theorem is proved.

We now prove the converse,

Theorem 5.4.2 Let M^n be a connected n-dimensional Riemannian manifold ($n\geq 3$) of constant sectional curvature. Then every sufficiently small geodesic sphere is totally umbilical.

Proof: Since the space has constant sectional curvature K it follows from

$$R\left(X,Y,Z,W\right)=K\left\{\langle Y,Z\rangle\langle X,W\rangle-\langle X,Z\rangle\langle Y,W\rangle\right\}$$

that

$$\begin{aligned} R\left(e_a, e_n, e_n, e_b\right) &= K\left\{\left\langle e_n, e_n\right\rangle \left\langle e_a, e_b\right\rangle - \left\langle e_a, e_n\right\rangle \left\langle e_b, e_n\right\rangle\right\} \\ &= 0 \qquad \text{for } a \neq b, \quad a, b = 1, ..., n-1. \end{aligned}$$

Since

$$\begin{aligned} R\left(X_\alpha, \dot{\gamma}\right)\dot{\gamma} &= K\left\{X_\alpha - \left\langle X_\alpha, \dot{\gamma}\right\rangle \dot{\gamma}\right\} \\ &= KX_\alpha \qquad \text{for } \alpha = 1, ..., n-1. \end{aligned}$$

Thus equation (5.4.1) reduces to

$$\ddot{F} + KF = 0,$$

with initial conditions $F(0) = 0$, $\dot{F}(0) = I$.

This equation can be solved explicitly to give

$$F = I\lambda(s)$$

where

$$\lambda(s) = \begin{cases} \frac{\sin s\sqrt{K}}{\sqrt{K}} & \text{if } K > 0 \\ s & \text{if } K = 0 \\ \frac{\sinh s\sqrt{-K}}{\sqrt{-K}} & \text{if } K < 0. \end{cases}$$

Thus

$$\dot{F}(s) = I\dot{\lambda}(s)$$

and

$$\dot{F}(s)F^{-1}(s) = \frac{\dot{\lambda}(s)}{\lambda(s)}I,$$

from which we deduce that the geodeisc sphere is totally umbilical.

CHAPTER 6

JACOBI FIELDS ON SASAKIAN MANIFOLDS

In this chapter, we study the Jocobi fields on a Sasakian manifold and in particular on a Sasakian space form we completely solve the differential equation involved for a Jacobi field. First we start with preliminaries on Sasakian manifold.

6.1 PRILIMINARIES ON SASAKIAN MANIFOLDS

Definition 6.1.1 A smooth manifold M^{2n+1} is said to have an almost contact structure if the structural group of its tangent bundle is reducible to $U(n) \times 1$.

Definition 6.1.2 A smooth manifold M^{2n+1} is said to have a (ϕ, ξ, η)-structure if it admits a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 \tag{6.1.1}$$

and

$$\phi^2 = -I + \eta \otimes \xi \tag{6.1.2}$$

ξ is called a characteristic vector field and η is a contact form.

Proposition 6.1.1 Suppose M^{2n+1} has a (ϕ, ξ, η)-structure. Then

$$\phi\xi = 0 \tag{6.1.3}$$

and

$$\eta \circ \xi = 0. \tag{6.1.4}$$

Proof:

First, note that (6.1.1) and (6.1.2) give $\phi^2 \xi = -\xi + \eta(\xi)\xi = 0$ and hence either $\phi\xi = 0$ or $\phi\xi$ is a non-trivial eigenvector of ϕ corresponding to eigenvalue 0. So by (6.1.2) again $0 = \phi^2\phi\xi = -\phi\xi + \eta(\phi\xi)\xi$ or $\phi\xi = \eta(\phi\xi)\xi$. Now, if $\phi\xi$ is a non-trivial eigenvector of the eigenvalue 0, $\eta(\phi\xi) \neq 0$ and therefore $0 = \phi^2\xi = \eta(\phi\xi)\phi\xi = [\eta(\phi\xi)]^2 \xi \neq 0$, a contradiction. Thus, $\phi\xi = 0$.

Now, since $\phi\xi = 0$, we also have from (6.1.2) that $\eta(\phi X)\xi = \phi^3 X + \phi X = -\phi X + \phi[\eta(X)\xi] + \phi X = \eta(X)\phi\xi = 0$ for any vector field X and hence $\eta \circ \phi = 0$.

Moreover, a manifold M^{2n+1} with (ϕ, ξ, η)-structure admits a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{6.1.5}$$

for any vector field X, Y. The manifold M^{2n+1} together with (ϕ, ξ, η, g)-structure is said to be an almost contact metric manifold and g is called a compatible metric. Setting $Y = \xi$ in (6.1.5) we get

$$\eta(X) = g(X, \xi) \tag{6.1.6}$$

Definition 6.1.3 An almost contact metric manifold is said to be a

Sasakian manifold if

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y) X \tag{6.1.7}$$

$X, Y \in \mathfrak{X}(M)$ and where ∇ denotes the Riemannian connection of g.

It is easy to see from (6.1.7) that

$$\nabla_X \xi = -\phi X, \tag{6.1.8}$$

and using (6.1.6) to see that

$$d\eta(X, Y) = g(X, \phi Y).$$

Example 6.1.1 (Odd-dimensional sphere)

Let S^{2n+1} be a sphere in $R^{2n+2} = C^{n+1}$ with an almost complex structure J, defined by $J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$, $J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$, which is Hermitian with respect to the Riemannian connection $\overline{\nabla}$ of the Euclidean metric on R^{2n+2} i.e $J^2 = -I$ and

$$\langle JX, JY \rangle = \langle X, Y \rangle, X, Y \in \mathfrak{X}(R^{2n+2}) \tag{6.1.9}$$

Let ∇ and g denote the induced connection and metric on S^{2n+1} and N denote the unit normal vector field.

Now, it is easy to see from (6.1.9) that JN is a unit vector field tangent to S^{2n+1}, call $JN = -\xi$.

Now,

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi), \qquad X \in \mathfrak{X}(S^{2n+1})$$

or

$$-\overline{\nabla}_X JN = \nabla_X \xi + g\left(AX, \xi\right) N$$

For a sphere $A = -I$, and where $\left(\overline{\nabla}_X J\right) Y = 0$ we get

$$-J\overline{\nabla}_X N = \nabla_X \xi - g\left(X, \xi\right) N$$

Since $\overline{\nabla}_X N = -AX = X$, we have

$$\nabla_X \xi - g\left(X, \xi\right) N = -JX \tag{6.1.10}$$

Now, for $X \in \mathfrak{X}(S^{2n+1})$ define $JX = \phi\left(X\right) + \eta\left(X\right) N$ where $\phi\left(X\right)$ is the tangential component and $\eta\left(X\right)$ is 1-form the normal component of JX.

Put $X = \xi$, $N = \phi\left(\xi\right) + \eta\left(\xi\right) N$ and hence

$$\phi\left(\xi\right) = 0 \text{ and } \eta\left(\xi\right) = 1$$

Also, operate J on JX, we get

$$-X = J\left(\phi X\right) + \eta\left(X\right) JN$$

$$-X = \phi^2 X + \eta\left(\phi X\right) N - \eta\left(X\right) \xi$$

and then equating the tangential and normal components we get

$$\phi^2 X = -X + \eta\left(X\right) \xi \text{ and } \eta \circ \phi = 0.$$

Also, we can see that

$$g\left(\phi X, \phi Y\right) = g\left(X, Y\right) - \eta\left(X\right) \eta\left(Y\right) \text{ and } g\left(X, \xi\right) = \eta\left(X\right).$$

Then, the structure induced above (ϕ, ξ, η, g) is an almost contact metric structure.

Now, for $X, Y \in \mathfrak{X}(S^{2n+1})$ we have

$$\left(\overline{\nabla}_X\phi\right)Y = \overline{\nabla}_X\phi Y - \phi\overline{\nabla}_X Y$$

but $\overline{\nabla}_X Y = \nabla_X Y - g\left(X, Y\right)N$, so

$$\begin{aligned}\left(\overline{\nabla}_X\phi\right)Y &= \nabla_X\phi Y - g\left(X, \phi Y\right)N - \phi\nabla_X Y + g\left(X, Y\right)\phi N \\ &= \left(\nabla_X\phi\right)Y - g\left(X, Y\right)\xi - g\left(X, \phi Y\right)N, \qquad (6.1.11)\end{aligned}$$

where $\phi N = JN = -\xi$

On the other hand, since $JX = \phi X + \eta\left(X\right)N$,

$$\begin{aligned}\left(\overline{\nabla}_X\phi\right)Y &= \overline{\nabla}_X\phi Y - \phi\overline{\nabla}_X Y \\ &= \overline{\nabla}_X JY - \overline{\nabla}_X\left(\eta\left(Y\right)N\right) - J\left(\overline{\nabla}_X Y\right) + \eta\left(\overline{\nabla}_X Y\right)N \\ &= J\overline{\nabla}_X Y - X\left(\eta\left(Y\right)\right)N - \eta\left(Y\right)\overline{\nabla}_X N - J\overline{\nabla}_X Y + \eta\left(\overline{\nabla}_X Y\right)N \\ &= -\eta\left(Y\right)X - \left\{X\eta\left(Y\right) + \eta\left(\overline{\nabla}_X Y\right)\right\}N, \qquad (6.1.12)\end{aligned}$$

where $\overline{\nabla}_X N = X$. By equating the tangential component of (6.1.11) and (6.1.12) we get

$$\left(\nabla_X\phi\right)Y - g\left(X, Y\right)\xi = -\eta\left(Y\right)X$$

or

$$\left(\nabla_X\phi\right)Y = g\left(X, Y\right)\xi - \eta\left(Y\right)X$$

and hence S^{2n+1} with (ϕ, ξ, η, g) is Sasakian manifold.

Definition 6.1.4 A plan section of the tangent space at a point of M is called a ϕ-section if it is spanned by vectors X and ϕX orthogonal to ξ. The sectional curvature of a ϕ-section is called ϕ-sectional curvature.

A Sasakian manifold of constant ϕ-sectional curvature c is called a Sasakian space form and its curvature tensor is given by (cf. [5])

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\quad +g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\} \quad (6.1.13) \end{aligned}$$

We now concentrate on some further properties of Sasakian manifolds which will be used in the next section.

Let $p \in M$ and let u be a unit tangent vector at p. Further, let γ be the geodesic parametrized by arc length and such that $\gamma(0) = p$ and $\dot{\gamma}(0) = u$. We also denote by u the unit tangent field to γ. Now, for $u \neq \xi$, let E denote the field of planes spanned by ϕu and $\xi - \eta(u)u$ along γ and let $E^{\perp}$ denote the orthogonal complement of $E \oplus [u]$. Then, we have

Lemma 6.1.1 E and $E^{\perp}$ are parallel along γ.

Proof: First, (6.1.7) implies

$$\nabla_u(\phi u) = \xi - \eta(u)u. \quad (6.1.14)$$

Next, since $\eta(u) = g(u,\xi)$, (6.1.8) implies

$$u(\eta(u)) = g(u, \nabla_u \xi) = -g(u, \phi u) = 0.$$

Hence, $\eta(u)$ is constant along γ. Finally, this implies

$$\begin{aligned}\nabla_u(\xi-\eta(u)u) &= \nabla_u\xi-\nabla_u(\eta(u)u)\\ &= -\phi u-u(\eta(u))u-\eta(u)\nabla_u u\\ &= -\phi u \qquad (6.1.15)\end{aligned}$$

Equations (6.1.14), (6.1.15) give that for $X\in E$, $\nabla_u X\in E$ and hence we see that E is parallel along γ.

Similarly, take $v\in E^{\perp}$ then $v\perp u,\phi u,\xi-\eta(u)u$. Using (6.1.14) we have

$$\begin{aligned}g(\nabla_u v,\phi u) &= ug(v,\phi u)-g(v,\nabla_u\phi u)\\ &= -g(v,\xi-\eta(u)u)\end{aligned}$$

and using (6.1.15), we have

$$\begin{aligned}g(\nabla_u v,\xi-\eta(u)u) &= ug(v,\xi-\eta(u)u)-g(v,\nabla_u(\xi-\eta(u)u))\\ &= g(v,\phi u)=0\end{aligned}$$

Finally,

$$g(\nabla_u v,u)=ug(v,u)-g(v,\nabla_u u)=0$$

Thus, we conclude that $\nabla_u v\in E^{\perp}$ for every $v\in E^{\perp}$ and therefore $E^{\perp}$ is parallel along γ.

Now, when $u\neq\xi$, note that

$$\begin{aligned}g(\xi-\eta(u)u,\xi-\eta(u)u) &= g(\xi,\xi)-2g(\eta(u)u,\xi)+g(\eta(u)u,\eta(u)u)\\ &= 1-2\eta(u)^2+\eta(u)^2=1-\eta(u)^2.\end{aligned}$$

and

$$g(\phi u, \phi u) = g(u, u) - \eta(u)\eta(u) = 1 - \eta(u)^2.$$

Also,

$$\begin{aligned} g(\xi - \eta(u)u, \phi u) &= g(\xi, \phi u) - \eta(u) g(u, \phi u) \\ &= -g(\phi\xi, u) = 0. \end{aligned}$$

Therefore, the two vectors

$$\left(1 - \eta(u)^2\right)^{-\frac{1}{2}} (\xi - \eta(u)u), \ \left(1 - \eta(u)^2\right)^{-\frac{1}{2}} \phi u$$

are orthonormal vectors which span E at each point of γ.

6.2 JACOBI FIELD EQUATION

In this section we study Jacobi field on a Sasakian manifold M.

Let p be a point in the Sasakian manifold and let γ be a geodesic parametrized by arc length and such that $\gamma(0) = p$ and $\dot{\gamma}(0) = u$. Then, a Jacobi field X along γ is a vector field that satisfies the equation

$$\nabla_u \nabla_u X - R(u, X)u = 0 \tag{6.2.1}$$

First, we consider the case where $u = \xi$, i.e. the geodesic is an integral curve of the characteristic vector field. Since on a Sasakian manifold we have

$$\begin{aligned} R(Y,X)\xi &= \nabla_Y \nabla_X \xi - \nabla_X \nabla_Y \xi - \nabla_{[Y,X]}\xi \\ &= -\nabla_Y \phi X + \nabla_X \phi Y + \phi[Y,X] \\ &= -\{g(Y,X)\xi - \eta(X)Y + \phi(\nabla_Y X)\} + \{g(X,Y)\xi \\ &\quad -\eta(Y)X + \phi(\nabla_X Y)\} + \phi(\nabla_Y X) - \phi(\nabla_X Y) \\ &= \eta(X)Y - \eta(Y)X \end{aligned}$$

Putting $Y = \xi$, we obtain

$$R(\xi,X)\xi = \eta(X)\xi - X.$$

For a Jacobi vector field orthogonal to ξ, (6.2.1) then becomes

$$\nabla_\xi \nabla_\xi X - R(\xi,X)\xi = 0$$

or

$$\nabla_\xi \nabla_\xi X - \eta(X)\xi + X = 0$$

Since X orthogonal to ξ, $\eta(X) = g(X,\xi) = 0$ and then

$$\nabla_\xi \nabla_\xi X + X = 0 \qquad (6.2.2)$$

Now, let $\{u, E_a; a = 2, \ldots, 2n+1\}$ be a parallel basis and put

$$X(s) = \sum_{a=2}^{2n+1} f_a(s) E_a$$

then

$$\ddot{X}(s) = \sum_{a=2}^{2n+1} f_a''(s) E_a$$

and (6.2.2) becomes

$$f_a''(s) + f_a(s) = 0,$$

which gives

$$f_a(s) = A_a \sin s + B_a \cos s$$

where A_a and B_a are constant along γ. Therefore,

$$X = \sum_{a=2}^{2n+1} (A_a \sin s + B_a \cos s) E_a$$

and hence we have

Proposition 6.2.1 Let γ be an integral curve of the characteristic vector field ξ on a Sasakian manifold and let X be a Jacobi vector field orthogonal to ξ along γ. With respect to a parallel basis $\{u, E_a; a = 2, \ldots, 2n+1\}$ we have

$$X = \sum_{a=2}^{2n+1} (A_a \sin s + B_a \cos s) E_a$$

where A_a and B_a are constant along γ.

Now, let M be a Sasakian space form and let γ be a geodesic through $p \in M$, tangent to $u \neq \xi$. Our aim is to solve (6.2.1) completely. First, from (6.1.13)

$$\begin{aligned} R(u, X)u &= \frac{c+3}{4}\{g(X,u)u - g(u,u)X\} + \frac{c-1}{4}\{\eta(u)\eta(u)X \\ &\quad -\eta(X)\eta(u)u + g(u,u)\eta(X)\xi - g(X,u)\eta(u)\xi \\ &\quad +g(u,\phi X)\phi u - g(u,\phi u)\phi X + 2g(u,\phi X)\phi u\} \\ &= \frac{c+3}{4}\{g(X,u)u - X\} + \frac{c-1}{4}\{\alpha^2 X - \alpha\eta(X)u \\ &\quad +\eta(X)\xi - \alpha g(X,u)\xi + 3g(u,\phi X)\phi u\} \end{aligned} \tag{6.2.3}$$

where $\alpha = \eta(u)$. We recall that α is constant along γ. Next, let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis at $p \in M$ with $u = e_1$ and

$$e_{2n} = \left(1-\alpha^2\right)^{-\frac{1}{2}}(\xi - \alpha u), \qquad e_{2n+1} = \left(1-\alpha^2\right)^{-\frac{1}{2}}\phi u.$$

Further, let $\{E_1, \ldots, E_{2n+1}\}$ be the basis obtained by parallel translation of the basis $\{e_1, ..., e_{2n+1}\}$ along γ. Then, we have

$$\begin{cases} E_{2n} = \dfrac{\xi - \alpha u}{\sqrt{1-\alpha^2}} \cos s + \dfrac{\phi u}{\sqrt{1-\alpha^2}} \sin s, \\ E_{2n+1} = -\dfrac{\xi - \alpha u}{\sqrt{1-\alpha^2}} \sin s + \dfrac{\phi u}{\sqrt{1-\alpha^2}} \cos s, \end{cases} \tag{6.2.4}$$

where s denotes the arc length from p along γ. Further, let X be a Jacobi vector field orthogonal to γ and put

$$X = \sum_{a=2}^{2n-1} f_a E_a + f_{2n} E_{2n} + f_{2n+1} E_{2n+1}. \tag{6.2.5}$$

Using (6.2.3) and (6.2.4) we compute

$$\begin{aligned} R(u, E_b)\, u &= \frac{c+3}{4} \{g(u, E_b)\, u - E_b\} + \frac{c-1}{4} \{\alpha^2 E_b - \alpha \eta(E_b)\, u \\ &\quad + \eta(E_b)\, \xi - \alpha g(E_b, u)\, \xi + 3g(u, \phi E_b)\, \phi u\}, \qquad b = 2, ..., 2n-1. \end{aligned}$$

Since $E_b \in E^{\perp}$ and $\xi - \eta(u)\, u \in E$,

$$g(E_b, \xi - \eta(u)\, u) = 0,$$

which gives

$$g(E_b, \xi) = \eta(u)\, g(E_b, u) = 0,$$

or

$$\eta(E_b) = 0$$

and hence

$$R(u, E_b)\, u = -\frac{1}{4} \left\{c + 3 - \alpha^2 (c-1)\right\} E_b.$$

Therefore,

$$R(u, E_b, u, E_{2n}) = 0, \qquad R(u, E_b, u, E_{2n+1}) = 0$$

and

$$R\left(u,E_b,u,E_a\right)=-\frac{1}{4}\left\{c+3-\alpha^2\left(c-1\right)\right\}\delta_{ba}.$$

Similarly,

$$\begin{aligned}R\left(u,\xi\right)u &= \frac{c+3}{4}\left\{g\left(u,\xi\right)u-\xi\right\}+\frac{c-1}{4}\{\alpha^2\xi-\alpha\eta\left(\xi\right)u\\ &\quad +\eta\left(\xi\right)\xi-\alpha g\left(\xi,u\right)\xi+3g\left(u,\phi\xi\right)\phi u\}\\ &= \frac{c+3}{4}\left\{\alpha u-\xi\right\}+\frac{c-1}{4}\left\{\xi-\alpha u\right\}\\ &= \alpha u-\xi\end{aligned}$$

hence

$$R\left(u,\xi,u,\xi\right)=\alpha^2-1,\qquad R\left(u,\xi,u,\phi u\right)=0. \tag{6.2.6}$$

Also,

$$\begin{aligned}R\left(u,\phi u\right)u &= \frac{c+3}{4}\left\{g\left(u,\phi u\right)u-\phi u\right\}+\frac{c-1}{4}\{\alpha^2\phi u-\alpha\eta\left(\phi u\right)u\\ &\quad +\eta\left(\phi u\right)\xi-\alpha g\left(\phi u,u\right)\xi+3g\left(u,\phi^2u\right)\phi u\}\\ &= \frac{c+3}{4}\left\{-\phi u\right\}+\frac{c-1}{4}\{\alpha^2\phi u-3\phi u+3\alpha^2\phi u\}\\ &= -\frac{1}{4}\left\{c+3-4\alpha^2\left(c-1\right)+3\left(c-1\right)\right\}\phi u\\ &= -\frac{1}{4}\left\{4c-4\alpha^2\left(c-1\right)\right\}\phi u\\ &= -\left\{c-\left(c-1\right)\alpha^2\right\}\phi u,\end{aligned}$$

using (6.1.5), we get

$$\begin{aligned}R\left(u,\phi u,u,\phi u\right) &= -\left\{c-\left(c-1\right)\alpha^2\right\}g\left(\phi u,\phi u\right)\\ &= -\left\{c-\left(c-1\right)\alpha^2\right\}\left(1-\alpha^2\right)\end{aligned} \tag{6.2.7}$$

Next, using (6.2.6), (6.2.7) we see that

$$\begin{aligned} R(u,E_{2n},u,E_{2n}) &= \frac{\cos s}{\sqrt{1-\alpha^2}}R(u,\xi,u,E_{2n})+\frac{\sin s}{\sqrt{1-\alpha^2}}R(u,\phi u,u,E_{2n}) \\ &= \frac{\cos^2 s}{1-\alpha^2}R(u,\xi,u,\xi)+\frac{\sin^2 s}{1-\alpha^2}R(u,\phi u,u,\phi u) \\ &= -\cos^2 s-\sin^2 s\left\{c-(c-1)\alpha^2\right\}, \end{aligned}$$

$$\begin{aligned} R(u,E_{2n},u,E_{2n+1}) &= \frac{\cos s}{\sqrt{1-\alpha^2}}R(u,\xi,u,E_{2n+1}) \\ &\quad +\frac{\sin s}{\sqrt{1-\alpha^2}}R(u,\phi u,u,E_{2n+1}) \\ &= -\frac{\cos s\sin s}{1-\alpha^2}R(u,\xi,u,\xi)+\frac{\sin s\cos s}{1-\alpha^2}R(u,\phi u,u,\phi u) \\ &= \cos s\sin s-\cos s\sin s\left\{c-(c-1)\alpha^2\right\} \\ &= -(c-1)\left(1-\alpha^2\right)\cos s\sin s \end{aligned}$$

and finally,

$$\begin{aligned} R(u,E_{2n+1},u,E_{2n+1}) &= -\frac{\sin s}{\sqrt{1-\alpha^2}}R(u,\xi,u,E_{2n+1}) \\ &\quad +\frac{\cos s}{\sqrt{1-\alpha^2}}R(u,\phi u,u,E_{2n+1}) \\ &= \frac{\sin^2 s}{1-\alpha^2}R(u,\xi,u,\xi)+\frac{\cos^2 s}{1-\alpha^2}R(u,\phi u,u,\phi u) \\ &= -\sin^2 s-\cos^2 s\left\{c-(c-1)\alpha^2\right\}. \end{aligned}$$

Now, using (6.2.5), (6.2.1) is equivalent to

$$\begin{aligned} 0 &= \sum_{a=2}^{2n-1} f_a'' E_a + f_{2n}'' E_{2n} + f_{2n+1}'' E_{2n+1} \\ &\quad -R\left(u,\sum_{a=2}^{2n-1} f_a E_a + f_{2n}E_{2n} + f_{2n+1}E_{2n+1}\right)u. \qquad (6.2.8) \end{aligned}$$

Taking the inner product of (6.2.8) with E_a and using above equations, we get

$$f_a''+\frac{1}{4}\left\{c+3-(c-1)\alpha^2\right\}f_a=0, \qquad a=2,...,2n-1.$$

With E_{2n}, we obtain

$$\begin{aligned}
0 &= f_{2n}'' - f_{2n}\left[-\cos^2 s - \sin^2 s\left\{c-(c-1)\,\alpha^2\right\}\right] \\
&\quad -f_{2n+1}\left[-(c-1)\left(1-\alpha^2\right)\cos s\sin s\right] \\
&= f_{2n}'' + f_{2n} + \left\{(c-1)\left(1-\alpha^2\right)\right\}\sin^2 s f_{2n} \\
&\quad +\left\{(c-1)\left(1-\alpha^2\right)\cos s\sin s\right\} f_{2n+1} \\
&= f_{2n}'' + f_{2n} + (c-1)\left(1-\alpha^2\right)\sin s\left\{f_{2n}\sin s + f_{2n+1}\cos s\right\}
\end{aligned}$$

With E_{2n+1}, we obtain

$$\begin{aligned}
0 &= f_{2n+1}'' + f_{2n}\,(c-1)\left(1-\alpha^2\right)\cos s\sin s \\
&\quad +f_{2n+1}\left[\sin^2 s + \cos^2 s\left\{c-(c-1)\,\alpha^2\right\}\right] \\
&= f_{2n+1}'' + f_{2n}\,(c-1)\left(1-\alpha^2\right)\cos s\sin s + f_{2n+1} \\
&\quad +f_{2n+1}\left\{(c-1)\left(1-\alpha^2\right)\cos^2 s\right\} \\
&= f_{2n+1}'' + f_{2n+1} + (c-1)\left(1-\alpha^2\right)\cos s\left\{f_{2n}\sin s + f_{2n+1}\cos s\right\}.
\end{aligned}$$

Therefore, (6.2.1) is equivalent to the following system of differential equations

$$f_a'' + \frac{1}{4}\left\{c+3-\alpha^2\,(c-1)\right\} f_a = 0, \qquad a = 2, ..., 2n-1 \tag{6.2.9}$$

$$\left\{\begin{array}{l} f_{2n}'' + f_{2n} + (c-1)\left(1-\alpha^2\right)\sin s\left\{f_{2n}\sin s + f_{2n+1}\cos s\right\} = 0 \\ f_{2n+1}'' + f_{2n+1} + (c-1)\left(1-\alpha^2\right)\cos s\left\{f_{2n}\sin s + f_{2n+1}\cos s\right\} = 0. \end{array}\right. \tag{6.2.10}$$

The solutions of the $2n-2$ equations (6.2.9) are standard. Next, we show that the solution of the equations (6.2.10) leads to two other equations where one is of the same form as (6.2.9) and the other is still more elementary.

Therefore, put

$$\left\{\begin{array}{l} Z_{2n} = f_{2n}\sin s + f_{2n+1}\cos s \\ Z_{2n+1} = f_{2n}\cos s - f_{2n+1}\sin s \end{array}\right. \tag{6.2.11}$$

Then the equations (6.2.11) become

$$\begin{cases} f''_{2n} + f_{2n} + (c-1)\left(1-\alpha^2\right)\sin s Z_{2n} = 0, \\ f''_{2n+1} + f_{2n+1} + (c-1)\left(1-\alpha^2\right)\cos s Z_{2n} = 0. \end{cases}$$

These equations are equivalent to

$$\begin{cases} f''_{2n}\cos s + f_{2n}\cos s - f''_{2n+1}\sin s - f_{2n+1}\sin s = 0, \\ f''_{2n}\sin s + f_{2n}\sin s + f''_{2n+1}\cos s + f_{2n+1}\cos s + (c-1)\left(1-\alpha^2\right)Z_{2n} = 0, \end{cases}$$

or

$$\begin{cases} Z''_{2n+1} + 2Z'_{2n} = 0, \\ Z''_{2n} - 2Z'_{2n+1} + (c-1)\left(1-\alpha^2\right)Z_{2n} = 0. \end{cases}$$

Further, put $w = Z'_{2n}$. Then,

$$\begin{cases} Z''_{2n+1} = -2w, \\ w' - 2Z'_{2n+1} + (c-1)\left(1-\alpha^2\right)Z_{2n} = 0. \end{cases}$$

Differentiation of the second equation gives

$$\begin{aligned} w'' - 2Z''_{2n+1} + (c-1)\left(1-\alpha^2\right)Z'_{2n} &= 0 \\ w'' + 4w + (c-1)\,w - (c-1)\,\alpha^2 w &= 0 \\ w'' + \left\{(c+3) - (c-1)\,\alpha^2\right\} w &= 0. \end{aligned}$$

Integration of last equation gives the remaining parts for the complete solution.

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