CONFORMAL DEFORMATION OF A Riemannian Metric

By

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To my wonderful parents and my husband who set me on my path

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PREFACE

For a given Riemannian manifold \((M, g)\), it is an interesting question to study the existence of a conformal diffeomorphism (also called as a conformal transformation) \(f : M \rightarrow M\) such that the metric \(g' = f^*g\) has one of the following properties:

(i) \((M, g')\) has constant scalar curvature.

(ii) \((M, g')\) is an Einstein manifold.

The question described in (i) above is called the Yamabe problem. If one starts with an Einstein manifold \((M, g)\) and seek the existence of a conformal transformation \(f : M \rightarrow M\) such that \((M, g')\), \(g' = f^*g\) is also an Einstein manifold, then the question reduces to the existence of the solution of a partial differential equation on the Riemannian manifold \((M, g)\) (cf. [5]). Also we know that there are special type of conformal transformation on the Complex plane of the type \(f(z) = \frac{az + b}{cz + d}\), \(ad - bc \neq 0\) called the Mobius transformation and they are characterized by \(S(f) = 0\) where \(S(f)\) is the Schwarzian derivative of \(f\) defined by

\[
S(f) = \left( \frac{f''}{f'''} \right) - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 = \frac{f'''}{f'} - 3 \left( \frac{f''}{f'} \right)^2
\]

There is a natural extension of the Schwarzian derivative to Riemannian manifold \((M, g)\), and thus a conformal transformation is said to be a Mobius transformation if \(S(f) = 0\). Thus existence of Mobius transformation is equivalent to the existence of solution of a Partial differential equation on the Riemannian manifold which is known as the Mobius equation (cf. [10]). This thesis is de-
voted to the study of the solutions of these partial differential equations and all the results are taken from the papers cited in [5], [6] and [10].

The thesis is divided into three chapters and each chapter is divided into subsections and the results in each section are numbered as \((a.b.c)\), for instance Theorem \(a.b.c\), means Theorem number \(c\) in the section \(b\) of chapter \(a\).

The first chapter is introductory and is basically intended to make the thesis as self-contained as possible. In this chapter we gave basic definitions and summarized the basic formulae and results on Riemannian manifolds, submanifolds, which are essential for the other two chapters.

The second chapter is devoted to the study of the conformal transformation between Einstein spaces. all the results of this chapter are taken form [5] & [6]. In the first section, we see that for an Einstein manifold \((M, g)\) the problem of finding a conformal transformation \(f : M \rightarrow M\), such that \((M, g')\), \(g' = f^*g\) is also an Einstein manifold is equivalent to finding a solution of the differential equation \(\nabla^2 \psi = \lambda g\), where \(\lambda : M \rightarrow \mathbb{R}\) and \(g' = \psi^{-2}g\). In the second section, we study the behavior of a non-constant local solution of the differential equation \(\nabla^2 \psi = \lambda g\). As a result of this study, we conclude that on the domain of these local solutions the line element corresponding to the Riemannian metric \(g\) may be expressed as a warped product of an interval with a unit sphere. In third section, we give the conditions for complete Einstein spaces to be mapped conformally onto another (possibly non-complete) Einstein space. It turns out that the conditions for complete Einstein spaces to be mapped conformally onto another complete Einstein space, are equiva-
lent to the conditions for complete Riemannian manifolds to admit a globally defined solution of the equation $\nabla^2 \psi = \frac{\Delta \psi}{n} g$. In the fourth section, we discuss some examples of global solutions of these differential equations and this section is very interesting. Here in this section we gave possible warped product expressions for the metrics on the Model spaces the sphere, Euclidean space and the Hyperbolic space. Also we gave examples of nontrivial Einstein manifolds that is those which do not have constant scalar curvature. Finally in this section we also see the special form of the this differential equation on the surfaces. In the last section we consider the question of classifying complete Riemannian manifolds $(M, g)$ on which there exist a smooth nonconstant function $\psi$ with $g' = \psi^{-2} g$ such that the difference $Ric_{g'} - Ric_g$ of the Ricci tensors is a constant multiple of the metric $g$ or $g'$. It turns out that such Riemannian manifolds are either standard spaces of constant curvature or is a warped product $R \times_{\exp} M_*$ of the Real line $R$ and a Ricci-flat manifold $M_*$. 

In third chapter, in first section we first extend the definition of Schwarzian derivative $S(f)$ to Riemannian manifolds and in the process we come across a tensor filed $B_\varphi$ associated to a smooth function $\varphi$. A diffeomorphism $f : (M, g) \to (M, \bar{g})$ is said to be a Mobuis transformation if $S(f) = 0$. The set of all Möbius transformation $\text{Mob}(M, g)$ forms a group which is a subgroup of the group $C(M, g)$ of the conformal transformation. In this section we study the properties of Schwarzian as well as the tensor filed $B_\varphi$ and also find the Mobius group for the spaces $R^n$ and $S^n$. One of related problem is the non-homogeneous Möbius equation $B_g(\varphi) = p$, where $p$ is symmetric tracefree tensor field of type $(0, 2)$ on the Riemannian manifold $(M, g)$. This
equation is studied in the second section of this chapter. In addition, in this section we illustrate the usefulness of Schwarizan tensor when computing the change in geometric quantities under a conformal change of metric. In the third section, we study the decomposition of Riemann curvature tensor and we discuss the natural components of it, and we see how one of the natural components of Riemann curvature tensor changes under a conformal change of metric. The substitution $u = e^{-\varphi}$ change the equation $B_g(\varphi) = 0$ into the linear equation $H_u = \frac{1}{n}(\Delta u)g$ which is the same differential equation considered in chapter-II. We let $U(M)$ denote the space of all solutions of this equation, moreover we single out a subspace $U_k(M)$ of $U(M)$ consisting of those for which $\frac{\Delta u}{n} + ku$ is a constant, where $k$ is a constant. On $U_k(M)$, we define the inner product $\langle \langle u, v \rangle \rangle_k = u(\frac{\Delta u}{n}) + v(\frac{\Delta u}{n}) + \langle \nabla u, \nabla v \rangle + kuv$, and then we conclude that $M$ has constant sectional curvature if and only if $U_k(M) = U(M)$. In last section of this chapter we show how the existence of non-constant solutions of $H_u = \frac{1}{n}(\Delta u)g$ lead to local warped product. All the results in this chapter are taken from [10].

The thesis ends with a list of references, which by no means is exhaustive on the subject, but lists only those references which have either been directly used in the thesis or have relevance to our work.
CHAPTER I

INTRODUCTION

In this introductory chapter, we will describe basic definitions, results and formulas which are related to our subsequent chapters. Throughout this thesis, we will denote by $T_pM$ the tangent space to $M$ at $p \in M$, by $\chi(M)$ the lie-algebra of the smooth vector field on $M$ and by $C^\infty(M)$ the ring of smooth functions on $M$. The differential of the map $f : M \to N$ at $p \in M$ is denoted by $df_p$ which is the linear map $df_p : T_pM \to T_{f(p)}N$.

1.1 RIEMANNIAN MANIFOLDS

In this section we will discuss the connection on a manifold, a Riemannian metric, Riemannian connection on a Riemannian manifold, a distribution on Riemannian manifold and the properties of the curvature tensor, Ricci tensor and scaler curvature. Moreover, we will discuss some types of spaces: space of constant curvature and an Einstein space, followed by some examples.

**Definition 1.1.1** Let $f : M \to N$ be a smooth map. Then

1. $f$ is called an immersion if $df_p : T_pM \to T_{f(p)}N$ is one-to-one map for all $p \in M$.

2. $f$ is called an imbedding if $f$ is one-to-one immersion.
(3) \( f \) is called a diffeomorphism if \( f \) is a bijection and \( f^{-1} \) is smooth.

Now we state the Implicit function theorem.

**Theorem 1.1.2** Suppose \( f : M \to N \) is a smooth map and that \( df_q : T_q M \to T_p N \) is onto for all \( q \in M \) with \( f(q) = p \). Then the set \( F = \{q \in M : f(q) = p\} \) is a smooth manifold and \( \text{dim}F = \text{dim}M - \text{dim}N \), moreover the inclusion \( i : F \to M \) is an imbedding.

**Definition 1.1.3** Let \( M \) be \( n \)-dimensional smooth manifold. A Riemannian metric on \( M \) is a tensor field \( g \) of type \((0,2)\) which satisfies:

(i) \( g(X,Y) = g(Y,X), \forall X,Y \in \chi(M) \).

(ii) for each \( p \in M \), \( g_p \) is positive definite non-degenerate bilinear form on \( T_p M \times T_p M \). that is: \( g_p(X_p,Y_p) = 0 \) implies \( X_p = 0 \), \( g_p(X_p,X_p) \geq 0 \) for all \( X_p \in T_p M \) and \( g_p(X_p,Y_p) = 0 \) implies \( X_p = 0 \).

A smooth manifold \( M \) together with a given a Riemannian metric \( g \) is called a Riemannian manifold, and is denoted by \((M,g)\).

Now, we define a distribution on a Riemannian manifold, and the condition under which the Distribution is called involutive.

**Definition 1.1.4** (1) Let \( M \) be a Riemannian manifold of dimension \( m = n + k \), and let us suppose that to each \( p \in M \) is assigned an \( n \)-dimensional
subspace $\triangle_p$ of $T_p(M)$. Suppose moreover that in a neighborhood $U$ of each $p \in M$ there are $n$ linearly independent $C^\infty$-vector fields $X_1, \ldots, X_n$, which form a basis of $\triangle_q$ for every $q \in U$. Then we shall say that $\triangle$ is a $C^\infty$ distribution of dimension $n$ on $M$, and $X_1, \ldots, X_n$ is a local basis of $\triangle$.

(2) We shall say that the distribution $\triangle$ is involutive if $[X, Y] \in \triangle, \forall X, Y \in \triangle$.

(3) Finally, if $\triangle$ is a $C^\infty$-distribution on $M$, we say that $\triangle$ is integrable, if for each $p \in M$, there exists a $n$-dimensional submanifold $N$ containing $p$ of $M$ such that

$$T_qN = \triangle_q, \forall q \in N$$

And then $N$ is called a leaf of $\triangle$.

Now we state theorem of Forbenius.

**Theorem 1.1.5** A distribution $\triangle$ on a manifold $M$ is integrable if and only if it is involutive.

We now introduce a connection on a smooth manifolds.

**Definition 1.1.6** A connection $\nabla$ on a smooth manifold $M$ is a map

$$\nabla : \chi(M) \times \chi(M) \to \chi(M), \quad (X, Y) \to \nabla_X Y$$

which satisfying the following:
Example 1.1.7 Let \( \{x_1, x_2, ..., x_n\} \) be the coordinate system on \( \mathbb{R}^n \). Then for \( X, Y \in \chi(\mathbb{R}^n) \), we have
\[
X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}, \quad f_i, g_i \in C^\infty(\mathbb{R}^n).
\]
Define \( \nabla : \chi(\mathbb{R}^n) \times \chi(\mathbb{R}^n) \to \chi(\mathbb{R}^n) \) by \( \nabla_X Y = \sum_{i=1}^{n} X(g^i) \frac{\partial}{\partial x_i} \), then it can be easily verified that \( \nabla \) satisfies the requirement for a connection. This connection \( \nabla \) on \( \mathbb{R}^n \) is also known as the Euclidean connection \( \nabla \).

Remark 1.1.8 On a smooth manifold \( M \) there could be several Riemannian metrics (once existence of one is known), for instance if \( g \) is a Riemannian metric on \( M \) and \( f : M \to M \) is a non-singular smooth map (that is, the matrix \( (df)_p \) is non-singular of each \( p \in M \)), then \( g_f(X,Y) = g(df(X),df(Y)) \) is also a Riemannian metric on \( M \).

The next interesting question is given a Riemannian metric \( g \) on \( M \), how to choose an appropriate connection \( \nabla \) on \( M \) among several connections on \( M \), in other words what should be the compatibility in \( \nabla \) and \( g ? \). This is answered in the following:

Theorem 1.1.9 On a Riemannian manifold \((M, g)\) there exists one and
only one connection $\nabla$ satisfying the following conditions

(i) $\nabla_X Y - \nabla_Y X = [X, Y]$

(ii) $X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad \forall X, Y, Z \in \chi(M)$.

The unique connection $\nabla$ on the Riemannian manifold $(M, g)$ satisfying (i) and (ii) in above theorem is called the Riemannian connection.

**Definition 1.1.10** For a connection $\nabla$ on a smooth manifold $M$, there is associated a tensor field $R$ of type $(1, 3)$ called the curvature tensor field of the connection $\nabla$, defined by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z, \quad X, Y, Z \in \chi(M)$$

where $[X, Y]$ is the Lie-bracket of the vector fields $X$ and $Y$.

The properties of the curvature tensor $R$ of the Riemannian connection $\nabla$ on $(M, g)$ are summarized in the following:

**Theorem 1.1.11** The curvature tensor $R$ of the Riemannian connection $\nabla$ on a Riemannian manifold $(M, g)$ satisfies

(i) $R(X, Y)Z + R(Y, X)Z = 0$.

(ii) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.

(iii) $R(X, Y; Z, W) = R(Z, W; X, Y), \quad X, Y, Z, W \in \chi(M)$,
where \( R(X,Y;Z,W) = g(R(X,Y)Z,W) \).

**Definition 1.1.12** Let \( P \) be a plane section on \((M,g)\). The sectional curvature of the plane section \( P \) is defined by

\[
K(P) = R(X,Y;Y,X)_{X \wedge Y}
\]

where \( X \) and \( Y \) are vector fields which span \( P \), and \( X \wedge Y = g(X,X)g(Y,Y) - g(X,Y)^2 \).

If \( K(P) \) is a constant \( c \) for all plane sections \( P \) on \( M \), then \( M \) is said to have constant sectional curvature \( c \). For constant sectional curvature manifolds, we will have a simple formula for \( R \) given in the following theorem:

**Theorem 1.1.13** If the Riemannian manifold \((M,g)\) is of constant sectional curvature \( c \), then its curvature tensor field is given by

\[
R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \quad X,Y \in \chi(M)
\]

The Riemannian manifold that has a constant sectional curvature is characterized by the following:

**Theorem 1.1.14** (*Schur's theorem*) Let \((M,g)\) be a connected Riemannian manifold \((M,g)\) of dimension \( n \geq 3 \), suppose that for each \( p \in M \),
the sectional curvature $K(p, \sigma)$ does not depend on $\sigma \subset T_p M$, then $M$ has a constant sectional curvature.

**Definition 1.1.15** Let $\{e_1, e_2, ..., e_n\}$ be a local orthonormal frame on a Riemannian manifold $(M, g)$. If $R$ is the tensor of type $(0, 4)$ described in Theorem 1.1.11, then the Ricci tensor field $Ric$ of $M$ is defined by

$$Ric(X, Y) = \sum_{i=1}^{n} R(e_i, X; Y, e_i), \quad X, Y \in \chi(M)$$

and the scaler curvature $S$ of $M$ is the trace of the Ricci tensor, that is $S$ is defined by:

$$S = \sum_{i=1}^{n} Ric(e_i, e_i)$$

**Example 1.1.16** Consider $(\mathbb{R}^n, g)$ where $g$ is the euclidean inner product on $\mathbb{R}^n$ defined by $g(X, Y) = \sum_{i=1}^{n} g^i f^i$, where $X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^{n} g^i \frac{\partial}{\partial x_i}$, $f^i, g^i \in C^\infty(\mathbb{R}^n)$. Then according to the euclidean connection defined in example 1.1.7 the curvature tensor field $R$ will be given by $R(X, Y) Z = \sum_{i=1}^{n} XY(h^i) \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} YX(h^i) \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} [X, Y](h^i) \frac{\partial}{\partial x_i} = 0$, where $Z = \sum_{i=1}^{n} h^i \frac{\partial}{\partial x_i} \Rightarrow g(R(X, Y) Z, W) = 0, \forall X, Y, Z, W \in \chi(\mathbb{R}^n)$, which means that $K(P) = 0$ for any plane section of $\mathbb{R}^n$, on the other world $\mathbb{R}^n$ is of constant sectional curvature zero.

**Definition 1.1.17** A Riemannian manifold $(M, g)$ of dimension $n \geq 3$, is called an Einstein manifold if there exist a constant $\lambda$ such that: $Ric = \lambda g$, and the constant $\lambda$ is called the Einstein constant.
**Remark 1.1.18** If \((M, g)\) is of constant sectional curvature \(c\) then \((M, g)\) is an Einstein manifold, indeed by Theorem 1.1.13 we have

\[
R(X, Y, Z, W) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}, \forall X, Y, Z, W \in \chi(M).
\]

\[
\Rightarrow Ric(Y, Z) = \sum_{i=1}^{n} R(e_i, Y, Z, e_i)
\]

\[
= c\sum_{i=1}^{n} \{g(e_i, e_i)g(Y, Z) - g(e_i Z)g(Y, e_i)\}
\]

\[
= c\{ng(Y, Z) - g(Y, Z)\}
\]

\[
= c(n-1)g(Y, Z), \quad \forall Y, Z \in \chi(R^n)
\]

that is, \((M, g)\) is an Einstein manifold.

The Theorem 1.1.14 and preceding remark gives the following

**Corollary 1.1.19** If \((M, g)\) is as in theorem 1.1.14 then \((M, g)\) is an Einstein manifold.

**Example 1.1.20** According to above remark \((R^n, g)\), where \(g\) is the euclidean inner product on \(R^n\) defined in example 1.1.16 is an Einstein manifold.

### 1.2 OPERATORS ON A RIEMANNIAN MANIFOLD

On a Riemannian manifold \((M, g)\) there are natural generalization of the well-known differential operators: gradient, divergence, Laplacian, and Hessian.
Definition 1.2.1 For a Riemannian manifold \((M, g)\) of dimension \(n\), with a local orthonormal frame \(\{e_1, e_2, \ldots, e_n\}\), we define the following:

1. The gradient \(\text{grad} f\) of a function \(f \in C^\infty (M)\) is a smooth vector field \(\text{grad} f\) (or \(\nabla f\)) defined by

\[
g(\text{grad} f, X) = df (X) = X (f), \quad \forall X \in \chi (M)
\]

For \(p \in M\), in terms of a coordinate system, \(\text{grad} f (p) = \sum_{i=1}^{n} (e_i (f)) e_i (p)\)

2. If \(X\) is a vector field then the divergence, \(\text{div} X\) of \(X\) is defined by

\[
\text{div} X = \sum_{i=1}^{n} g (\nabla_{e_i} X, e_i)
\]

3. The Laplacian, \(\Delta f\) of a function \(f \in C^\infty (M)\) is the divergence of its gradient, that is \(\Delta f : M \to \mathbb{R}\) is given by \(\Delta f = \text{div} (\nabla f)\)

4. The Hessian, \(H_f\) of a function \(f \in C^\infty (M)\) is the symmetric \((0, 2)\) tensor field such that

\[
H_f (X, Y) = XY (f) - (\nabla_X Y) (f) = g (\nabla_X \nabla f, Y), \quad X, Y \in \chi (M)
\]

We also use the notation \((\nabla^2 f) (X, Y) = g (H_f (X), Y)\) for the Hessian of \(f\).

Example 1.2.2 Let \((\mathbb{R}^n, g)\) where \(g\) is the euclidean inner product on \(\mathbb{R}^n\) defined in example 1.1.16, with coordinates \((x_1, x_2, \ldots, x_n)\), \(\frac{\partial}{\partial x_i} = (0, \ldots, 1, 0, \ldots, 0) = \)
\( e_i \), and \( f \in C^\infty (R^n) \) then using the definition of \( \text{grad} f \) we have \( \nabla f = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right) e_i \) and for \( X = \sum_{i=1}^{n} f_i e_i \) we have
\[
\text{div} X = \sum_{i=1}^{n} g \left( \nabla e_i, X, e_i \right) = \sum_{i=1}^{n} G \left( \frac{\partial f_i}{\partial x_i} \right) e_i, e_i = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}
\]
Finally, \( \triangle f = \text{div} (\nabla f) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \)

Moreover for \( X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x_i}, \ Y = \sum_{i=1}^{n} g^i \frac{\partial}{\partial x_i}, \ f^i, g^i \in C^\infty (R^n) \). The Hessian of \( f \) is given by:
\[
H_f (X, Y) = g \left( \nabla X \nabla f, Y \right) = g \left( \nabla X \left( \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right) e_i \right), Y \right) = g \left( \sum_{i=1}^{n} X \left( \frac{\partial f}{\partial x_i} \right) e_i, Y \right)
\]
\[
= \sum_{i=1}^{n} X \left( \frac{\partial f}{\partial x_i} \right) g^i.
\]

**Remark 1.2.3** For \( f, h \in C^\infty (M) \), using the definition of the Laplacian and gradient it can be shown that:

1. \( \triangle (fh) = f \triangle h + h \triangle f + 2 g (\nabla f, \nabla h) \).
2. \( \nabla (fh) = f \nabla h + h \nabla f \).
Definition 1.2.4 Let $f : M \to R$ be a smooth function on a Riemannian manifold $M$. A point $p \in M$ is said to be a critical point of $f$ if $df (p) = 0$, $f (p)$ is called a critical value of $f$. If $p$ is a critical point and $(x_1, x_2, \ldots, x_n)$ is a coordinate system on $M$ around $p$, the Hessian of $f$, given by the matrix 

$$
\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) (p).
$$

We say that the critical point is non-degenerate if the determinant of this matrix is different from zero.

Now we are in a position to state theorem of Reeb:

Theorem 1.2.5 If for a smooth compact $n$-dimensional manifold $M$ there is a smooth function $f : M \to R$ which has just two non-degenerate critical points then $M$ is homeomorphic to a sphere of dimension $n$.

1.3 CONFORMAL TRANSFORMATIONS

Conformal transformations on a Riemannian manifold are most important diffeomorphisms of a Riemannian manifold. In this section we define and discuss briefly important properties of conformal transformations.

Definition 1.3.1 Let $(M, g)$, $(\overline{M}, \overline{g})$ be two Riemannian manifolds, A
diffeomorphism \( f : M \rightarrow \overline{M} \) is said to be conformal transformation if:
\[ f^*(\bar{g}) = e^{\phi}g, \]
for some smooth function \( \phi \in C^\infty(M) \). If there exists such a diffeomorphism then the two Riemannian metrics \( g \) and \( \bar{g} \) are said to be conformally related.

**Remark 1.3.2** (1) If \( \phi \) is a constant in above definition, then \( f \) is called a homothety.

(2) If \( \phi = 0 \) then \( f \) is called an isometry.

For a Riemannian manifold \((M, g)\) we denote by \( C(M) \) the set of all conformal transformations from \( M \) to itself, which forms a group called the conformal group, and by \( I(M) \) the set of all isometries of \( M \), which forms a subgroup of \( C(M) \).

Consider the half space
\[ H^n = \{(y_1, y_2, ..., y_n) \in R^n : y_n > 0\} \]
with a Riemannian metric
\[ h = \frac{1}{y_n^2} (dy_1^2 + dy_2^2 + ... + dy_n^2) \]
and the open ball
\[ B^n = \{x \in R^n : \|x\| < 1\} \]
with a Riemannian metric

\[ g = \frac{4}{(1 - \|x\|^2)^2} \left( dx_1^2 + dx_2^2 + \ldots + dx_n^2 \right) \]

then the Riemannian manifolds \( (H^n, h) \) and \( (B^n, g) \) are called the \textit{poincare's} half space and \textit{poincare's} ball respectively.

**Example 1.3.3** Define \( \phi : B^2 \to H^2 \) by

\[ \phi(u, v) = \left( \frac{2u}{u^2 + (1 - v)^2}, \frac{1 - u^2 - v^2}{u^2 + (1 - v)^2} \right) \]

then it can be shown that \( \phi \) is a smooth. Indeed, for \((x, y) \in H^2,\)

\[ \phi^{-1}(x, y) = \left( \frac{2x}{x^2 + (y + 1)^2}, \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} \right) \]

which also is smooth.

We use coordinates \((u, v)\) on \(B^2\) and \((x, y)\) on \(H^2,\) such that \(\phi(u, v) = (x, y)\) then we have in local coordinates

\[ h = \frac{1}{y^2} (dx^2 + dy^2) \quad (1.3.1) \]

Now \( x = \frac{2u}{u^2 + (1 - v)^2} \) and \( y = \frac{1 - u^2 - v^2}{u^2 + (1 - v)^2} \) imply

\[ dx = \frac{2du}{u^2 + (1 - v)^2} - \frac{2u}{u^2 + (1 - v)^2} \left( \frac{2udu + 2(1 - v)dv}{u^2 + (1 - v)^2} \right) \]

and

\[ dy = -\frac{2udu - 2vdu}{u^2 + (1 - v)^2} - \left( \frac{1 - u^2 - v^2}{u^2 + (1 - v)^2} \right) \left( \frac{2udu + 2(1 - v)dv}{u^2 + (1 - v)^2} \right) \]
Squaring and substituting in (1.3.1) and simplifying we will get
\[ h = \frac{1}{y^2} (dx^2 + dy^2) = \frac{4}{1 - u^2 - v^2} (du^2 + dv^2) = g \]
this proves that \( \phi^* h = g \), that is \( \phi : B^2 \to H^2 \) is an isometry.

Using similar arguments we have the following

**Proposition 1.3.4** The poincare's half space\( (H^n, h) \) is isometric to poincare's ball \( (B^n, g) \).

Now we state the relations between the covariant derivatives and curvatures of two conformally related metrics.

**Lemma 1.3.5** For \( g \) and \( \bar{g} = \psi^{-2} g \), \( \psi = e^\phi \) we have:

(i) \( \bar{\nabla}_X Y = \nabla_X Y - \{ X(\phi) Y + Y(\phi) X - g(X, Y) \nabla \phi \} \)

(ii) \( \bar{R}(X, Y) Z = R(X, Y) Z - \{ g(X, Z) H(\phi) Y - g(Y, Z) H(X) \} \)
\[ + \{ \nabla^2 \phi (Y, Z) + Y(\phi) Z(\phi) - g(Y, Z) \| \nabla \phi \|^2 \} X \]
\[ - \{ \nabla^2 \phi (X, Z) + Z(\phi) X(\phi) - g(X, Z) \| \nabla \phi \|^2 \} Y \]
\[ + \{ X(\phi) g(Y, Z) - Y(\phi) g(X, Z) \} \nabla \phi \]

(iii) \( \bar{Ric}(X, Y) = Ric(X, Y) + \frac{(n-2)}{\psi^2} \psi^2 \psi(X, Y) + \left( \Delta \phi - (n - 2) \| \nabla \phi \|^2 \right) g(X, Y) \)

(iv) \( \psi^{-2} S = S + \frac{(n-2)}{\psi} (\Delta \psi) + n(\Delta \phi - (n - 2) \| \nabla \phi \|^2) \).
**Proof.** (i) We know from Koszul formula:

\[
2\bar{g}(\nabla_X Y, Z) = X\bar{g}(Y, Z) + Y\bar{g}(X, Z) - Z\bar{g}(X, Y) - \bar{g}([Y, Z], X) - \bar{g}([X, Z], Y) - \bar{g}([Y, X], Z)
\]

(1.3.2)

Now as \( \bar{g} = e^{-2\phi} g \) we have

\[
X\bar{g}(Y, Z) = X(e^{-2\phi} g(Y, Z)) = -2e^{-2\phi} X(\phi) g(Y, Z) + e^{-2\phi} Xg(Y, Z)
\]

Thus (1.3.2) takes the form:

\[
2\bar{g}(\nabla_X Y, Z) = -2e^{-2\phi} X(\phi) g(Y, Z) + e^{-2\phi} Xg(Y, Z) - 2e^{-2\phi} Y(\phi) g(X, Z)
\]

\[
+ e^{-2\phi} Y g(X, Z) + 2e^{-2\phi} Z(\phi) g(X, Y) - e^{-2\phi} Zg(Y, X)
\]

\[
- e^{-2\phi} g([Y, Z], X) - e^{-2\phi} g([X, Z], Y) - e^{-2\phi} g([Y, X], Z)
\]

\[
e^{-2\phi} \{-2X(\phi)g(Y, Z) - 2Y(\phi)g(X, Z) + 2Z(\phi)g(X, Y)\}
\]

\[
+ e^{-2\phi} \{Xg(Y, Z) + Yg(X, Z) - Zg(Y, X) - g([Y, Z], X)
\]

\[
- g([X, Z], Y) - g([Y, X], Z)\}
\]

\[
= -2e^{-2\phi} \{X(\phi)g(Y, Z) + Y(\phi)g(X, Z) - g(\nabla\phi, Z)g(X, Y)\}
\]

\[
+ 2e^{-2\phi} g(\nabla_X Y, Z)
\]

which implies

\[
2\bar{g}(\nabla_X Y, Z) = -2\{X(\phi)\bar{g}(Y, Z) + Y(\phi)\bar{g}(X, Z) - \bar{g}(\nabla\phi, Z)g(X, Y)\} + 2\bar{g}(\nabla_X Y, Z)
\]

Thus for any \( Z \in \chi(M) \), we get

\[
\bar{g}(\nabla_X Y, Z) = \bar{g}(\nabla_X Y, Z) - \{X(\phi)\bar{g}(Y, Z) + Y(\phi)\bar{g}(X, Z) - g(X, Y)\bar{g}(\nabla\phi, Z)\}.
\]
which implies

\[ \tilde{\nabla}_X Y = \nabla_X Y - \{X(\phi)Y + Y(\phi)X - g(X, Y)\nabla\phi\} \]

(ii): Using (i), we get

\[ \tilde{\nabla}_Y Z = \nabla_Y Z - \{Y(\phi)Z + Z(\phi)Y - g(Y, Z)\nabla\phi\} \]

which implies

\[
\begin{align*}
\tilde{\nabla}_X \tilde{\nabla}_Y Z &= \tilde{\nabla}_X (\nabla_Y Z) - \tilde{\nabla}_X (Y(\phi)Z) - \tilde{\nabla}_X (Z(\phi)Y) + \tilde{\nabla}_X (g(Y, Z)\nabla\phi) \\
&= \nabla_X \nabla_Y Z - X(\phi)\nabla_Y Z - (\nabla_Y Z)(\phi)X + g(X, \nabla_Y Z)\nabla\phi - XY(\phi)Z \\
&\quad - Y(\phi)\nabla_X Z - XZ(\phi)Y - Z(\phi)\nabla_X Y + Xg(Y, Z)\nabla\phi + g(Y, Z)\nabla_X (\nabla\phi)
\end{align*}
\]

that is

\[
\begin{align*}
\tilde{\nabla}_X \tilde{\nabla}_Y Z &= \nabla_X \nabla_Y Z - X(\phi)\nabla_Y Z - (\nabla_Y Z)(\phi)X + g(X, \nabla_Y Z)\nabla\phi - XY(\phi)Z \\
&\quad - Y(\phi)\nabla_X Z + Y(\phi)X(\phi)Z + 2Y(\phi)Z(\phi)X - Y(\phi)g(X, Z)\nabla\phi \\
&\quad - XZ(\phi)Y - Z(\phi)\nabla_X Y + Z(\phi)X(\phi)Y - Z(\phi)g(X, Y)\nabla\phi \\
&\quad + Xg(Y, Z)\nabla\phi + g(Y, Z)H_\phi(X) - g(Y, Z)\|\nabla\phi\|^2 X
\end{align*}
\]

Similarly,

\[
\begin{align*}
\tilde{\nabla}_Y \tilde{\nabla}_X Z &= \nabla_Y \nabla_X Z - Y(\phi)\nabla_X Z - (\nabla_X Z)(\phi)Y + g(Y, \nabla_X Z)\nabla\phi - YX(\phi)Z \\
&\quad - X(\phi)\nabla_Y Z + X(\phi)Y(\phi)Z + 2X(\phi)Z(\phi)Y - X(\phi)g(Y, Z)\nabla\phi \\
&\quad - YZ(\phi)X - Z(\phi)\nabla_Y X + Z(\phi)Y(\phi)X - Z(\phi)g(X, Y)\nabla\phi \\
&\quad + Yg(X, Z)\nabla\phi + g(X, Z)H_\phi(Y) - g(X, Z)\|\nabla\phi\|^2 Y
\end{align*}
\]
Finally,

$$\nabla_{[X,Y]}Z = \nabla_{[X,Y]}Z - [X,Y](\phi)Z - Z(\phi)[X,Y] + g([X,Y],Z)\nabla\phi$$

Thus we have

$$\tilde{R}(X,Y)Z = R(X,Y)Z - (\nabla_Y Z - YZ)(\phi)X + g(X, \nabla_Y Z)\nabla\phi - g(Y, \nabla_X Z)\nabla\phi$$

$$+ (\nabla_Y Z - XZ)(\phi)Y - Y(\phi)g(X, Z)\nabla\phi + X(\phi)g(Y, Z)\nabla\phi$$

$$+ Xg(Y, Z)\nabla\phi - Yg(X, Z)\nabla\phi + g(Y, Z)H_\phi(X)$$

$$- g(X, Z)H_\phi(Y) + g(X, Z)\|\nabla\phi\|^2 Y - g(Y, Z)\|\nabla\phi\|^2 X$$

$$- g([X,Y], Z)\nabla\phi + Y(\phi)Z(\phi)X - X(\phi)Z(\phi)Y$$

Note that

$$g(X, \nabla_Y Z)\nabla\phi - g(Y, \nabla_X Z)\nabla\phi + Xg(Y, Z)\nabla\phi - Yg(X, Z)\nabla\phi - g([X,Y], Z)\nabla\phi$$

$$= g(X, \nabla_Y Z)\nabla\phi + g(\nabla_Y X, Z)\nabla\phi - Yg(X, Z)\nabla\phi - g(\nabla_X Y, Z)\nabla\phi$$

$$- g(Y, \nabla_X Z)\nabla\phi + Xg(Y, Z)\nabla\phi = 0.$$
\begin{align*}
+ [\nabla^2 \phi(Y, Z) + Y(\phi)Z(\phi) - g(Y, Z) \| \nabla \phi \|^2]X \\
- [\nabla^2 \phi(X, Z) + Z(\phi)X(\phi) - g(X, Z) \| \nabla \phi \|^2]Y \\
+ [X(\phi)g(Y, Z) - Y(\phi)g(X, Z)]\nabla \phi.
\end{align*}

(iii) Let \( \{e_1, \ldots, e_n\} \) be a local orthonormal frame on \((M, g)\), then we have \( \tilde{g}(e^\phi e_i, e^\phi e_i) = e^{2\phi} \tilde{g}(e_i, e_i) = e^{2\phi}e^{-2\phi}g(e_i, e_i) = 1 \), which implies that \( \{e^\phi e_1, \ldots, e^\phi e_n\} \) is a local orthonormal frame on \((M, \tilde{g})\). Thus we have

\[
\overline{\text{Ric}}(Y, Z) = \sum_{i=1}^{n} \tilde{g}(\tilde{R}(e^\phi e_i, Y)Z, e^\phi e_i) = e^{2\phi} \sum_{i=1}^{n} e^{-2\phi}g(\tilde{R}(e_i, Y)Z, e_i)
\]

\[
= \sum_{i=1}^{n} g(\tilde{R}(e_i, Y)Z, e_i)
\]

\[
= \sum_{i=1}^{n} g(\{R(e_i, Y)Z - g(e_i, Z)H_\phi(Y) + g(Y, Z)H_\phi(e_i)
\]

\[
+ \nabla^2 \phi(Y, Z)e_i + Y(\phi)Z(\phi)e_i - g(Y, Z) \| \nabla \phi \|^2 e_i
\]

\[
- \nabla^2 \phi(e_i, Z)Y - Z(\phi)e_i(\phi)Y + g(e_i, Z) \| \nabla \phi \|^2 Y
\]

\[
+ e_i(\phi)g(Y, Z)\nabla \phi - Y(\phi)g(e_i, Z)\nabla \phi, e_i
\]

\[
= \text{Ric}(Y, Z) - \nabla^2 \phi(Y, Z) + g(Y, Z) \triangle \phi + n(\nabla^2 \phi(Y, Z))
\]

\[
+ n(Y(\phi)Z(\phi)) - n \left( g(Y, Z) \| \nabla \phi \|^2 \right) - (\nabla^2 \phi)(Y, Z) - Z(\phi)Y(\phi)
\]

\[
+ g(Y, Z) \| \nabla \phi \|^2 + g(Y, Z) \| \nabla \phi \|^2 - Y(\phi)Z(\phi)
\]

\[
\overline{\text{Ric}}(Y, Z) = \text{Ric}(Y, Z) + (n - 2)\nabla^2 \phi(Y, Z) + (n - 2)Y(\phi)Z(\phi)
\]

\[
-(n - 2)g(Y, Z) \| \nabla \phi \|^2 + g(Y, Z)\Delta \phi
\] (1.3.3)
Note that $\psi = e^\phi$, and consequently $Y(\psi) = e^\phi Y(\phi)$, that is $\nabla_Y Z(\psi) = e^\phi \nabla_Y Z(\phi)$ and

$$Y(Z(\psi)) = Y(e^\phi Z(\phi)) = e^\phi Y Z(\phi) + e^\phi Y(\phi) Z(\phi)$$

which gives

$$\nabla^2 \psi(Y, Z) = YZ(\psi) - \nabla_Y Z(\psi)$$

$$= e^\phi Y Z(\phi) + e^\phi Y(\phi) Z(\phi) - e^\phi \nabla_Y Z(\phi)$$

$$= e^\phi \{ YZ(\phi) - \nabla_Y Z(\phi) + Y(\phi) Z(\phi) \}$$

$$= \psi \{ \nabla^2 \phi(Y, Z) + Y(\phi) Z(\phi) \}.$$ 

Thus:

$$\frac{\nabla^2 \psi(Y, Z)}{\psi} = \nabla^2 \phi(Y, Z) + Y(\phi) Z(\phi)$$

Using this in (1.3.3) we get:

$$\overline{\text{Ric}}(Y, Z) = \text{Ric}(Y, Z) + (n - 2) \frac{\nabla^2 \psi(Y, Z)}{\psi} + (\Delta \phi - (n - 2) \| \nabla \phi \|^2) g(Y, Z)$$

(iv) We have

$$\tilde{S} = \sum_{n=1}^{n=\psi} \overline{\text{Ric}}(e^\phi e_i, e^\phi e_i) = e^{2\phi} \sum_{n=1}^{n=\psi} \overline{\text{Ric}}(e_i, e_i)$$

$$= e^{2\phi} \{ \sum_{n=1}^{n=\psi} \text{Ric}(e_i, e_i) + \frac{(n-2)}{\psi} \sum_{n=1}^{n=\psi} (\nabla^2 \psi)(e_i, e_i)$$

$$+(\Delta \phi - (n - 2) \| \nabla \phi \|^2) \sum_{n=1}^{n=\psi} g(e_i, e_i) \}$$

$$= e^{2\phi} \{ S + \frac{(n-2)}{\psi} [\Delta \psi] + n(\Delta \phi - (n - 2) \| \nabla \phi \|^2) \}. $$
Thus:
\[ e^{-2\phi} \tilde{S} = S + \frac{(n-2)}{\psi} [\Delta \psi] + n(\Delta \phi - (n-2) \|\nabla \phi\|^2) \]

**Remark 1.3.6** Using the same arguments we can prove that for \( \tilde{g} = e^{2\phi} g \), if \( \{e_1, e_2, \ldots, e_n\} \) is a local frame on \((M, g)\) then \( \{e^{-\phi} e_1, e^{-\phi} e_2, \ldots, e^{-\phi} e_n\} \) is a local frame on \((M, \tilde{g})\), furthermore, we have the following:

(i) \( \tilde{\nabla}_x Y = \nabla_x Y + X(\phi)Y + Y(\phi)X - g(X, Y)\nabla \phi. \)

(ii) \( \tilde{R}(X, Y)Z = R(X, Y)Z + \{g(X, Z)H_\phi(Y) - g(Y, Z)H_\phi(X)\} \]
\[-\{\nabla^2 \phi(Y, Z) + g(Y, Z) \|\nabla \phi\|^2 - Y(\phi)Z(\phi)\}X \]
\[+\{\nabla^2 \phi(X, Z) + g(X, Z) \|\nabla \phi\|^2 - X(\phi)Z(\phi)\}Y \]
\[+\{X(\phi)g(Y, Z) - Y(\phi)g(X, Z)\}\nabla \phi. \]

(iii) \( \overline{\text{Ric}}(X, Y) = Ric(X, Y) - (n-2) \nabla^2 \phi(X, Y) + (n-2) X(\phi)Y(\phi) \]
\[-\{\Delta \phi + (n-2) \|\nabla \phi\|^2\}g(X, Y). \]

(iv) \( \tilde{S} = e^{-2\phi}[S - 2(n-1) \Delta \phi - (n-2) (n-1) \|\nabla \phi\|^2]. \)

1.4 **SUBMANIFOLDS**

Given two smooth manifolds \( M \) and \( \overline{M} \), if there exists a smooth immersion
If $f : M \to \overline{M}$, then we say that $M$ is a submanifold of $\overline{M}$. If $f$ an imbedding, then $M$ is said to be an imbedded submanifold of $\overline{M}$.

If $\tilde{M}$ is a Riemannian manifold with a Riemannian metric $\tilde{g}$ then the immersion induces a Riemannian metric $g$ on $M$ defined by

$$g(X,Y) = \tilde{g}(df(X), df(Y)), \ X,Y \in \chi(M)$$

With respect to this induced metric $g = f^* (\tilde{g})$ the submanifold $M$ becomes a Riemannian manifold $(M, g)$.

**Remark 1.4.1** (1) Since an immersion $f : M \to \overline{M}$ (by an implicit function theorem) $f$ is a local imbedding, therefore when we are dealing with local expressions on a submanifold $M$ of $\overline{M}$, we shall identify $df(X)$ with $X \in \chi(M)$.

(2) Let $(\overline{M}, \tilde{g})$ be $m$-dimensional Riemannian manifold, and $M$ be its $n$-dimensional submanifold, the difference $n - m$ is called the co-dimension of the immersion $f$.

For a $p \in M$, let $T^\perp_p M = \{N_p \in \overline{M} / \tilde{g}(X_p, N_p) = 0, \forall X_p \in T_p M\}$, where $T_p M$ is the tangent space of $M$ then $T^\perp_p M$ is a subspace of $T_p \overline{M}$, precisely it is the orthogonal complement of $T_p M$ in $T_p \overline{M}$, in fact we have $T_p \overline{M} = T_p M \oplus T^\perp_p M$, where dim $T^\perp_p M = m - n$. The subspace $T^\perp_p M$ is called the
normal subspace of $M$ at $p$. Moreover we call $\nu = \bigcup_{p \in M} T^\perp_p M$ the normal bundle of $M$, and so $T^M_M = TM \oplus \nu$ and $\chi(M) = \chi(M) \oplus \Gamma(\nu)$, where $\Gamma(\nu)$ is the space of normal vector field to $M$. Thus if $\nabla$ be the Riemannian connection on $\tilde{M}$, then for $X, Y \in \chi(M)$ (as $X, Y$ are also in $\chi(\tilde{M})$, $\nabla_X Y \in \chi(\tilde{M})$), we can express $\nabla_X Y$ as follows

$$\nabla_X Y = \nabla_X Y + h(X, Y) \quad (1.4.1)$$

Where $\nabla_X Y \in \chi(M)$ is the tangential component of $\nabla_X Y$ to $M$ and $h(X, Y)$ is the normal component of $\nabla_X Y$ to $M$. Using the properties of $\nabla$ we can prove that $\nabla$ defines the Riemannian connection on $M$ with respect to the induced Riemannian metric $g$ on $M$.

The normal component $h(X, Y)$ defines a map $h : \chi(M) \times \chi(M) \rightarrow \Gamma(\nu)$ which satisfies:

(i) $h((X + Y), Z) = h(X, Z) + h(Y, Z)$.

(ii) $h(X, Y) = h(Y, X)$, $\forall X, Y, Z \in \chi(M)$.

This map $h$ is called the second fundamental form of the submanifold.

For $X \in \chi(M)$ and $N \in \Gamma(\nu)$, we have $\nabla_X N \in \chi(M)$ and thus we can express $\nabla_X N$ as

$$\nabla_X N = -A_N X + \nabla^\perp_X N \quad (1.4.2)$$

The map $A_N : \chi(M) \rightarrow \chi(M)$ is called the Weingarten map with respect to $N \in \Gamma(\nu)$ and formula (1.4.2) is called the Weingarten formula.
Remark 1.4.2 The Weingarten map $A_N$ and the second fundamental form of $M$ are related by

$$g(A_NX,Y) = g(h(X,Y),N), \quad X,Y \in \chi(M), N \in \Gamma(\nu)$$

Now we will define a totally umbilical submanifolds:

**Definition 1.4.3** A submanifold $M$ of dimension $n$ of a Riemannian manifold $(M,g)$ is said to be totally umbilical if $h(X,Y) = g(X,Y)H$, where $H = \frac{1}{n} \sum_{i=1}^{n} h(e_i,e_i)$, is said to be the mean curvature vector field and $\{e_1, ..., e_n\}$ is a local orthonormal frame on $M$.

**Definition 1.4.4** A submanifold $M$ of dimension $n$ of a Riemannian manifold $(\overline{M},g)$ is said to be totally geodesic if geodesics of $M$ are the restrictions of the geodesics of $\overline{M}$.

### 1.5 Warped Products

**Definition 1.5.1** Consider two Riemannian manifolds $(B,g_1)$ and $(F,g_2)$ and a smooth function $f : B \to R$, $f > 0$. The warped product $M = B \times_f F$ with warping function $f$, is the product $B \times F$ together with the metric

$$g = (\pi_1^*g_1) + (f \circ \pi_1)^2(\pi_2^*g_2)$$

**Remark 1.5.2** (1) Let $M = B \times_f F$ be the warped product, then for $(p,q) \in M$ we call $B \times \{q\}$ leaves of $M$ and $\{p\} \times F$ are called fibers of $M$. 
(2) For each $Z \in \chi(M)$ we will express it as $Z = X + V$, $X \in \chi(B)$, $V \in \chi(F)$, $X$ is called horizontal vector field and $V$ is called vertical vector field.

(3) Consider the projections $\pi_1 : M \to B$ and $\pi_2 : M \to F$ then a vector field $Z \in \chi(M)$ is called horizontal lift of $X \in \chi(B)$ if

(i) $d \pi_2 (Z) = 0$. (ii) $Z$ is $\pi_1$-related to $X$, i.e. $d \pi_1 \circ Z = X \circ \pi_1$.

Similarly, a vector field $W \in \chi(M)$ is called vertical lift of $V \in \chi(F)$ if

(i) $d \pi_1 (W) = 0$. (ii) $W$ is $\pi_2$-related to $V$, i.e. $d \pi_2 \circ W = V \circ \pi_2$.

**Remark 1.5.3** (i) If $f = 1$, then $B \times_f F$ reduces to a Riemannian product manifold $B \times F$.

(ii) As in the case of Riemannian product it is easy to see that the fibers $\{p\} \times F = \pi^{-1}(p)$ and the leaves $B \times \{q\} = \sigma^{-1}(q)$ are Riemannian submanifold of $M$. And the warped metric is characterized by

(1) For each $q \in F$, the map $\pi|(B \times \{q\})$ is an isometry onto $B$.

(2) For each $p \in B$, the map $\sigma|\{(p) \times F\}$ is a positive homothety onto $F$, with scale factor $1/f(p)$.

(3) For each $(p, q) \in M$, the leaf $B \times \{q\}$ and the fiber $\{p\} \times F$ are orthogonal at $(p, q)$. 
In next proposition we will see the relation between the covariant derivatives of the warped product and the covariant derivatives of the leave and the fiber of this warped product.

**Proposition 1.5.4** Let \( M = B \times f F \), if \( X, Y \in \chi(B) \) and \( V, W \in \chi(F) \), then

\[
\begin{align*}
(1) \quad \nabla_X Y & \in \chi(B) \text{ is the lift of } \nabla_X Y \text{ on } B. \\
(2) \quad \nabla_X V & = \nabla_Y X = \left( \frac{X(f)}{f} \right) V. \\
(3) \quad \nabla_V W & = h(V, W) = -(V, W) / f) grad f. \\
(4) \quad tan \ \nabla_V W & \in \chi(F) \text{ is the lift of } \nabla_Y W \text{ on } F.
\end{align*}
\]

**Corollary 1.5.5** The leaves \( B \times \{q\} \) of a warped product are totally geodesic; the fibers \( \{p\} \times F \) are totally umbilic.

**Lemma 1.5.6** If \( B \) and \( F \) are complete Riemannian manifolds, then \( M = B \times f F \) is complete for every warping function \( f \).

In the next proposition we will get the relation between the curvature of the warped product and the curvature of the leaves and the fibers of this warped product.
Proposition 1.5.7 Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor $R$. If $X, Y, Z \in \chi(B)$ and $U, V, W \in \chi(F)$, then

(1) $R_{XY}Z \in \chi(B)$ is the lift of $R^B_{XY}Z$ on $B$.

(2) $R_{XY}Y = -(H_f(X,Y)/f)V$, where $H_f$ is the Hessian of $f$.

(3) $R_{XY}V = R_{VW}X = 0$.

(4) $R_{XY}W = -\langle (V,W) / f \rangle \nabla_X (\text{grad } f)$.

(5) $R_{VW}U = R^F_{VW}U - \langle (\text{grad } f, \text{grad } f) / f^2 \rangle \{ (W,U)V - (V,U)W \}$.

Corollary 1.5.8 On a warped product $M = B \times_f F$, with $d = \dim F > 1$, let $X, Y$ be horizontal and $V, W$ be vertical. Then

(1) $\text{Ric}(X,Y) = \text{Ric}^B(X,Y) - (d/f)H_f(X,Y)$.

(2) $\text{Ric}(X,V) = 0$.

(3) $\text{Ric}(V,W) = \text{Ric}^F(V,W) - \langle V,W \rangle f^\#$, where $f^\# = \frac{\Delta f}{f} + (d-1)\frac{(\text{grad } f, \text{grad } f)}{f^2}$, and $\Delta f$ is the Laplacian of $f$ on $B$.

Example 1.5.9 $R^3 - 0$ as a warped product: If $(x,y,z)$ are the (rectangular) coordinates on $R^3 - 0$, then the spherical coordinates $(\rho, \phi, \varphi)$ are given by

$$x = \rho \sin \phi \cos \varphi, \quad y = \rho \sin \phi \sin \varphi, \quad z = \rho \cos \phi$$
\[
\begin{align*}
\alpha_{x} &= (\sin \phi \cos \theta) \, d\rho, \\
\alpha_{\theta} &= (\cos \phi \cos \theta) \, d\phi, \\
\alpha_{\phi} &= (-\rho \sin \phi \sin \theta) \, d\varphi.
\end{align*}
\]
\[
\begin{align*}
\alpha_{y} &= (\sin \phi \sin \theta) \, d\rho, \\
\alpha_{\phi} &= (\rho \cos \phi \sin \theta) \, d\phi, \\
\alpha_{\theta} &= (\rho \sin \phi \cos \theta) \, d\varphi.
\end{align*}
\]
Also \[
\begin{align*}
\alpha_{z} &= (\cos \phi) \, d\rho, \\
\alpha_{\rho} &= (-\rho \sin \phi) \, d\phi, \\
\alpha_{\phi} &= 0,
\end{align*}
\]
which imply
\[
\begin{align*}
\gamma_{x} &= (\sin \phi \cos \theta) \, d\rho + (\rho \cos \phi \cos \theta) \, d\phi + (-\rho \sin \phi \sin \theta) \, d\varphi, \\
\gamma_{y} &= (\sin \phi \sin \theta) \, d\rho + (\rho \cos \phi \sin \theta) \, d\phi + (\rho \sin \phi \cos \theta) \, d\varphi, \\
\gamma_{z} &= (\cos \phi) \, d\rho + (-\rho \sin \phi) \, d\phi.
\end{align*}
\]
Thus if \( ds^2 \) is the line element of \( R^3 - 0 \) (with respect to the rectangular coordinates) then
\[
\begin{align*}
\gamma_{x} &= d\rho^2 + (\rho^2 \sin^2 \phi) \, d\phi^2 + (\rho^2 \sin^2 \theta) \, d\varphi^2
\end{align*}
\]
But as \( d\varphi^2 + (\sin^2 \phi) \, d\varphi^2 \) is a line element of \( S^2 \) (this it has been proven in example 2.4.1 of this thesis).

Evidently \( R^3 - 0 \) is diffeomorphic to \( R^+ \times \rho S^2 \) under the natural map \((x, y, z) \rightarrow (\rho, (\phi, \varphi))\). Thus the formula for \( ds^2 \), shows that \( R^3 - 0 \) can be identified with the warped product \( R^+ \times \rho S^2 \). In general, \( R^n - 0 \) is naturally isometric to \( R^+ \times \rho S^{n-1} \).
Chapter 2

Conformal Transformations on Einstein Spaces

In this chapter we are interested with the following question:

For a given Riemannian manifold \((\mathcal{M}, g)\), to find a conformal diffeomorphism \(f : (\mathcal{M}, g) \rightarrow (\mathcal{M}, \tilde{g})\) such that the metric \(\tilde{g} = f^*g\) has one of the following properties:

(i) \((\mathcal{M}, \tilde{g})\) has constant scalar curvature, (ii) \((\mathcal{M}, \tilde{g})\) is an Einstein manifold.

2.1 A Problem and the Associated Differential Equation

In this section, we will discuss the following problem:

“Given an Einstein manifold \((\mathcal{M}, g)\), to find the conditions under which there exist a \(\psi \in C^\infty(\mathcal{M})\), such that \((\mathcal{M}, \tilde{g})\) where \(\tilde{g} = \psi^{-2}g\) is again an Einstein manifold”. To understand this problem, we need the following lemma.

Lemma 2.1.1 Let \((\mathcal{M}, g)\) be Einstein, \(\tilde{g} = \psi^{-2}g\). Then the following conditions (i), (ii), (iii) and (iv) are equivalent:

(i) \((\mathcal{M}, \tilde{g})\) is Einstein.
(ii) there exist a function $\lambda : M \to R$ with $\nabla^2 \psi = \lambda g$.

(iii) $\nabla^2 \psi = \frac{\Delta \psi}{n} g$.

(iv) there exists a constant $B$ such that $\nabla^2 \psi = (-\rho \psi + B)g$.

Each of these conditions implies:

(v) there are constants $B, C$ such that $\| \nabla \psi \|^2 = -\rho \psi^2 + 2B\psi + C$.

**Proof.** (i) $\Rightarrow$ (ii): If $(M, \bar{g})$ is Einstein, then there exist a constant $\mu_1$ such that $\bar{Ric} = \mu_1 \bar{g}$. But $(M, g)$ is Einstein also, so there exist a constant $\mu_2$ such that $Ric = \mu_2 g$. Now using

$$\bar{Ric} = Ric + \frac{(n-2)}{\psi} \nabla^2 \psi + (\Delta \psi - (n-2) \| \nabla \psi \|^2)g$$

and $\bar{g} = \psi^{-2} g$ we arrive at

$$\mu_1 \psi^{-2} g = \mu_2 g + \frac{(n-2)}{\psi} \nabla^2 \psi + (\Delta \psi - (n-2) \| \nabla \psi \|^2)g$$

or

$$\nabla^2 \psi = \frac{\psi}{n-2} \{\mu_1 \psi^{-2} - \mu_2 - \Delta \psi + (n-2) \| \nabla \psi \|^2\}g$$

Thus $\nabla^2 \psi = \lambda g$ where

$$\lambda = \frac{\psi}{n-2} \{\mu_1 \psi^{-2} - \mu_2 - \Delta \psi + (n-2) \| \nabla \psi \|^2\}$$

(ii) $\Rightarrow$ (i): If $\nabla^2 \psi = \frac{\Delta \psi}{n} g$, then by using $\bar{Ric} = Ric + \frac{(n-2)}{\psi} \nabla^2 \psi + (\Delta \psi - (n-2) \| \nabla \psi \|^2)g$ & $Ric = \mu_2 g$, we get $\bar{Ric} = hg$, where $h : M \to R$ and then by Schurts theorem $h$ is constant, and so $(M, \bar{g})$ is an Einstein manifold.
(ii) ⇒ (iii): By taking trace in (ii) we get $\Delta \psi = n \lambda$, so $\lambda = \frac{\Delta \psi}{n}$, or
\[ \nabla^2 \psi = \frac{\Delta \psi}{n} g. \]

(iii) ⇒ (ii) and (iv) ⇒ (ii) are trivial.

(i) ⇒ (iv): If (i) holds then (ii) holds or equivalently $H \psi = \lambda I_d$. Now, for $X, Y \in \chi(M)$ we have:
\[
R(X,Y)\nabla \psi = \nabla_X \nabla_Y \nabla \psi - \nabla_Y \nabla_X \nabla \psi - \nabla_{[X,Y]} \nabla \psi
\]
\[ = \nabla_X H \psi (Y) - \nabla_Y H \psi (X) - H \psi [X,Y]
\]
\[ = \nabla_X (\lambda Y) - \nabla_Y (\lambda X) - \lambda [X,Y]
\]
\[ = \lambda \nabla_X Y + X(\lambda) Y - \lambda \nabla_Y X - Y(\lambda) X - \lambda \nabla_X Y + \lambda \nabla_Y X.
\]

or
\[ R(X,Y)\nabla \psi = X(\lambda) Y - Y(\lambda) X \quad (2.1.1) \]

Now, let $\{e_1, e_2, ..., e_n\}$ be a local orthonormal frame on $(M, g)$ then using that $Ric = \mu_2 g$, we get
\[
S = \sum_{i=1}^{n} Ric(e_i, e_i) = \sum_{i=1}^{n} \mu_2 g(e_i, e_i) = n \mu_2
\]
that is, $S$ is a constant, or $\mu_2 = \frac{S}{n}$ and hence, $Ric = \frac{S}{n} g$ from which we get
\[
S(Y(\psi)) = S (g(Y, \nabla \psi)) = n Ric(Y, \nabla \psi) = n \sum_{i=1}^{n} R (e_i, Y, \nabla \psi, e_i)
\]
Now using (2.1.1) we arrive at

\[ S(Y(\psi)) = n \sum_{i=1}^{n} \{ e_i(\lambda) g(Y, e_i) - Y(\lambda) g(e_i, e_i) \} \]

\[ = n \{ g(\nabla \lambda, Y) - n Y(\lambda) \} \]

\[ = -n(n - 1) Y(\lambda) \]

or  \( Y(S\psi + n(n - 1)\lambda) = 0, \quad \forall Y \in \chi(M). \)

And so \( S\psi + n(n - 1)\lambda \) is a constant say \( A \), that is \( \lambda = \frac{-S\psi}{n(n-1)} + \frac{A}{n(n-1)} \), but \( \nabla^2 \psi = \lambda g \) implies \( \nabla^2 \psi = (\frac{-S\psi}{n(n-1)} + B)g \), where \( B \) is a constant.

\[(iv) \Rightarrow (v): \text{ For any } X \in \chi(M), \text{ we have} \]

\[ X\{ \| \nabla \psi \|^2 + \frac{S}{n(n - 1)} \psi^2 - 2B \psi \} = 2g(\nabla X \nabla \psi, \nabla \psi) + 2\frac{S}{n(n - 1)} \psi X(\psi) - 2B X(\psi) \]

\[ = 2\nabla^2 \psi(X, \nabla \psi) + \frac{2S}{n(n-1)} \psi X(\psi) - 2B X(\psi) \]

Which by \( (iv) \) gives:

\[ X\{ \| \nabla \psi \|^2 + \frac{S}{n(n - 1)} \psi^2 - 2B \psi \} = 2(\frac{-S\psi}{n(n - 1)} + B)g(X, \nabla \psi) \]

\[ + \frac{2S}{n(n - 1)} \psi X(\psi) - 2B X(\psi) \]

\[ = \frac{-2S\psi}{n(n-1)} X(\psi) + 2B X(\psi) \]

\[ + \frac{2S}{n(n-1)} \psi X(\psi) - 2B X(\psi) \]

\[ = 0 \]
Thus $\|\nabla \psi\|^2 + \frac{S}{n(n-1)} \psi^2 - 2B\psi = C$, for some constant $C$, or $\|\nabla \psi\|^2 = \frac{-S}{n(n-1)} \psi^2 + 2B\psi + C$. Thus the proof of lemma 2.1.1 is complete.

This lemma suggests that the above problem is equivalent to find a solution of the differential equation

$$\nabla^2 \psi = \frac{\triangle \psi}{n} g$$

since the Riemannian manifolds of constant curvature are also Einstein manifold, we look into the following form of the above problem, namely:

“If $(M, g)$ is a Riemannian manifold of constant curvature, to find conditions under which there exists a $\psi \in C^\infty(M)$ such that $(M, g)$, where $\bar{g} = \psi^{-2} g$ is also a Riemannian manifold of constant curvature”. For this we have

**Proposition 2.1.2** Let $(M, g)$ be a space of constant sectional curvature $K$ and dimension $n \geq 3$ and $\psi : M \to R$ a function, $\bar{g} = \psi^{-2} g$. Then the following conditions are equivalent:

(i) $(M, \bar{g})$ is a space of constant sectional curvature $\bar{K}$.

(ii) $(M, \bar{g})$ is Einstein.

(iii) $\nabla^2 \psi = \frac{\triangle \psi}{n} \cdot g$.

(iv) $\nabla^2 \psi = (-K \psi + B) \cdot g$ for a constant $B$. 

Each condition implies that $K, \overline{K}$ satisfy

$$\overline{K} = \psi^2 K + \frac{2}{n} \psi \Delta \psi - \|\nabla \psi\|^2$$

**Proof.** $(i) \Rightarrow (ii)$ is trivial, and $(ii) \iff (iii) \iff (iv)$ hold by lemma 2.1.1, thus it only remains to show that $(iii) \iff (i)$. Let $\sigma$ be the 2-plane spanned by two vectors $X, Y$ which are orthonormal with respect to $g$. Let $\overline{K}_\sigma$ denote the sectional curvature of $\sigma$ in $(M, g)$. Then using lemma 1.3.5 $(ii)$ and $\psi = e^{\varphi}$, we get:

$$\psi^{-2} \overline{K}_\sigma = \psi^{-2} \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y)} = \psi^{-4} \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y)}$$

$$= g([R(X, Y)Y - \{g(X, Y)H_\varphi(Y) - g(Y, Y)H_\varphi(X)\}$$

$$+\{(\nabla^2 \varphi)(Y, Y) + Y(\varphi)Y(\varphi) - g(Y, Y)\|\nabla \varphi\|^2\}X$$

$$-\{(\nabla^2 \varphi)(X, Y) + Y(\varphi)X(\varphi) - g(X, Y)\|\nabla \varphi\|^2\}Y$$

$$+\{X(\varphi)g(Y, Y) - Y(\varphi)g(X, Y)\} \nabla \varphi, X)$$

$$= K_\sigma + g(H_\varphi(X), X) + \nabla^2 \varphi(Y, Y) - \|\nabla \varphi\|^2$$

$$+X(\varphi)X(\varphi) + Y(\varphi)Y(\varphi)$$

$$= K_\sigma + \nabla^2 \varphi(X, X) + X(\varphi)X(\varphi) + \nabla^2 \varphi(Y, Y)$$

$$+Y(\varphi)Y(\varphi) - \|\nabla \varphi\|^2$$

Now using: $\nabla^2 \varphi(X, X) + X(\varphi)X(\varphi) = \frac{\nabla^2 \psi(X, X)}{\psi}$, we get

$$\psi^{-2} \overline{K}_\sigma = K_\sigma + \frac{\nabla^2 \psi(X, X)}{\psi} + \frac{\nabla^2 \psi(Y, Y)}{\psi} - \|\nabla \varphi\|^2$$
Which by (iii) gives
\[ \psi^{-2}K_{\sigma} = K_{\sigma} + 2 \frac{\Delta \psi}{n \psi} - \|\nabla \varphi\|^2 \]
therefore $K_{\sigma}$ is a function on $M$ that does not depend on $\sigma$. By Schur's theorem $K$ is constant.

Thus above proposition shows that this problem is also equivalent to solving the differential equation
\[ \nabla^2 \psi = \frac{\Delta \psi}{n} g \]

Finally, we have the following proposition

**Proposition 2.1.5** Let $(M, g)$ be of constant scalar curvature $S$, and $\psi : M \to \mathbb{R}$ be a function satisfying: $\nabla^2 \psi = \frac{\Delta \psi}{n} g$. Then (i) and (ii) are equivalent:

(i) $(M, g)$ has constant scalar curvature

(ii) there exist a constant $B$ such that $\Delta \psi = \frac{S}{n-1} \psi + B$.

**Proof.** Taking trace in the equation
\[ \nabla^2 \psi(X, Y) = \nabla^2 \varphi(X, Y) + X(\varphi)Y(\varphi) \]
we get: $\frac{\Delta \psi}{\psi} = \Delta \varphi + \|\nabla \varphi\|^2$. Using this and that $\|\nabla \varphi\|^2 = \frac{\|\nabla \psi\|^2}{\psi^2}$ in 1.3.5 (iv), we get
\[ \psi^{-2} \tilde{S} = S + 2(n-1) \frac{\Delta \psi}{\psi} - n(n-1) \frac{\|\nabla \psi\|^2}{\psi^2} \]
or

\[ \overline{S} = \psi^2 S + 2(n-1)\psi \Delta \psi - n(n-1) \| \nabla \psi \|^2 \]

Thus, for any \( X \in \chi(M) \) we have:

\[
X(\overline{S}) = 2\psi SX(\psi) + 2(n-1)X(\psi)\Delta \psi + 2(n-1)\psi X(\Delta \psi) \\
-2n(n-1)g(\nabla X \nabla \psi, \nabla \psi)
\]

But, \( \nabla^2 \psi = \frac{\Delta \psi}{n} g \), and so

\[
X(\overline{S}) = 2\psi SX(\psi) + 2(n-1)X(\psi)\Delta \psi + 2(n-1)\psi X(\Delta \psi) - 2(n-1)\Delta \psi g(X, \nabla \psi) \\
= 2\psi SX(\psi) + 2(n-1)X(\psi)\Delta \psi + 2(n-1)\psi X(\Delta \psi) - 2(n-1)\Delta \psi X(\psi) \\
= 2\psi X \{ S\psi + (n-1)\Delta \psi \} \\
= 2(n-1)\psi X \{ \frac{S\psi}{n-1} + \Delta \psi \}
\]

And hence, \( \overline{S} \) is a constant \( \iff \frac{S}{(n-1)} \psi + \Delta \psi = B \), for some constant \( B \)

\( \iff \Delta \psi = \frac{-S}{(n-1)} \psi + B \).

### 2.2 Local Solutions of the Differential Equation

In the last section we have seen that the problem of finding \( \psi \in C^\infty(M) \) on an Einstein manifold \((M, g)\) (or a Riemannian manifold of constant curvature) so that \((M, \tilde{g})\), where \( \tilde{g} = \psi^{-2} g \) is also an Einstein manifold (or a Riemannian
manifold of constant curvature) is equivalent to finding a solution of the differential equation

\[ \nabla^2 \psi = \frac{\Delta \psi}{n} g \]  

\((*)\)

Since \( \psi = \text{constant} \) is also a solution of this equation in which case the problem becomes trivial, therefore in this section, we are interested in studying the non-constant local solutions of the differential equation \((*)\). First we prove:

**Lemma 2.2.1** Let \( U \subseteq M \) be an open set which does not contain any critical points of \( \psi \) of the differential equation \((*)\), that is \( \nabla \psi(p) \neq 0 \) for all \( p \in U \). Then the following holds:

(i) the integral curves of \( N \) are geodesics (up to parametrization).

(ii) the level hypersurfaces of \( \psi \) are totally umbilical.

(iii) \( \| \nabla \psi \| \) is constant along the levels of \( \psi \).

(iv) there exists a function \( \alpha \) of \( \psi \) such that \( \nabla \Delta \psi = \alpha \nabla \psi \).

(v) the sectional curvature of every plane containing \( \nabla \psi \) is \( K = -\frac{\alpha}{n} \).

**Proof.** The unit normal to the level hypersurface \( M_c = \{ x \in M : \psi(x) = c \} \) is \( N = \frac{\nabla \psi}{\| \nabla \psi \|} \). Then using the differential equation \( \nabla^2 \psi = \lambda g \) we have the following:
\[ \nabla_X(\nabla\psi) = \lambda X, \] so we calculate

\[ \nabla_X(N) = \nabla X\left(\frac{\nabla\psi}{\|\nabla\psi\|}\right) = \frac{1}{\|\nabla\psi\|} \left( \nabla X(\nabla\psi) + X\left(\frac{1}{\|\nabla\psi\|}\right) \nabla\psi \right) \]

\[ = \frac{1}{\|\nabla\psi\|} \nabla X(\nabla\psi) - \frac{X(\|\nabla\psi\|)}{(\|\nabla\psi\|)^2} \nabla\psi \]

\[ = \frac{1}{\|\nabla\psi\|} \nabla X(\nabla\psi) - \frac{1}{(\|\nabla\psi\|)^2} \left\{ X \sqrt{g(\nabla\psi, \nabla\psi)} \right\} \nabla\psi \]

\[ = \frac{1}{\|\nabla\psi\|} \left\{ \nabla X(\nabla\psi) - \frac{g(\nabla X(\nabla\psi), \nabla\psi)}{\|\nabla\psi\|^2} \nabla\psi \right\} \]

However, as \[ \nabla_X(\nabla\psi) = \lambda X \] the last equation takes the form

\[ \nabla_X(N) = \frac{\lambda}{\|\nabla\psi\|} \left\{ X - \frac{g(X, \nabla\psi)}{\|\nabla\psi\|^2} \nabla\psi \right\} = \frac{\lambda}{\|\nabla\psi\|} \left\{ X - g(X, N)N \right\} \] for an arbitrary tangent vector \( X \) on \( M \).

(i) For \( X = N \), we have

\[ \nabla_N N = \frac{\lambda}{\|\nabla\psi\|} \{ N - g(N, N)N \}, \]

but as \( g(N, N) = 1 \) we have

\[ \nabla_N N = \frac{\lambda}{\|\nabla\psi\|} \{ N - N \} = 0 \]

(ii) For \( X \) orthogonal to \( N \), the weingarten map will be

\[ A(X) = -\nabla_X N = -\frac{\lambda}{\|\nabla\psi\|} X \]

Thus \( A \) is a multiple of the identity.
(iii) for $X$ orthogonal to $N$ we calculate
\[
X(\|\nabla \psi\|) = X \sqrt{g(\nabla \psi, \nabla \psi)} = g(\nabla_x(\nabla \psi, \nabla \psi) / \|\nabla \psi\|) = g(\nabla_x(\nabla \psi), N) \\
= \nabla^2 \psi(X, N) = \frac{\Delta \psi}{n} g(X, N) = 0
\]

(iv) The Ricci identity (2.1.1) is
\[
R(X, Y) \nabla \psi = (X\lambda)Y - (Y\lambda)X
\]
and thus
\[
R(X, \nabla \psi) \nabla \psi = (X\lambda) \nabla \psi - ((\nabla \psi)\lambda)X
\]
consequently
\[
R(X, \nabla \psi, \nabla \psi, \nabla \psi) = (X\lambda) \|\nabla \psi\|^2 - ((\nabla \psi)\lambda) g(X, \nabla \psi)
\]

Now,
\[
0 = R(\nabla \psi, \nabla \psi, X, \nabla \psi) = R(X, \nabla \psi, \nabla \psi, \nabla \psi) = (X\lambda) \|\nabla \psi\|^2 - ((\nabla \psi)\lambda) g(X, \nabla \psi)
\]
or
\[
(X\lambda) \|\nabla \psi\|^2 = ((\nabla \psi)\lambda) g(X, \nabla \psi) = \|\nabla \psi\|^2 ((\nabla \psi)\lambda) g(X, N)
\]
Thus for $X$ orthogonal to $N$ we have $(X\lambda) \|\nabla \psi\|^2 = 0$, that is $X\lambda = 0$, which gives: $X(\frac{\Delta \psi}{n}) = \frac{1}{n} X(\Delta \psi) = 0$. Hence, $\Delta \psi$ is a constant along the levels of $\psi$, and thus $g(\nabla(\Delta \psi), X) = 0$, for all $X \in \chi(M_c)$, or equivalently $\nabla \Delta \psi$.
CONFORMAL DEFORMATION OF A RIEMANNIAN METRIC

is normal to $M_c$, but the only normal of $M_c$ is $N = \frac{\nabla \psi}{\|\nabla \psi\|}$, therefore $\nabla \nabla \psi$ is proportional to $N$. This proves that there exist a scalar factor $\alpha$ such that $\nabla \nabla \psi = \alpha \nabla \psi$, because $\alpha$ is constant along the levels, $\alpha$ may be regarded as a function of $\psi$.

(v) Let $X$ be a vector field satisfying $g(X, \nabla \psi) = 0$ and $\|X\| = 1$, then the Ricci identity implies

$$K = \frac{R(X, \nabla \psi, \nabla \psi, X)}{g(X, X)g(\nabla \psi, \nabla \psi)} = \frac{(X\lambda)g(\nabla \psi, X) - (\nabla \psi \lambda)g(X, X)}{g(\nabla \psi, \nabla \psi)} = -\frac{(\nabla \psi \lambda)}{g(\nabla \psi, \nabla \psi)}$$

But; $\nabla \lambda = \nabla \left( \frac{\Delta \psi}{n} \right) = \frac{1}{n} \nabla \Delta \psi = \frac{n}{n} \nabla \psi$ and thus $K = -\frac{\alpha g(\nabla \psi, \nabla \psi)}{n g(\nabla \psi, \nabla \psi)} = -\frac{\alpha}{n}$.

The following lemma show that on the domain of the local solution of the differential equation $(\ast)$; the metric $g$ becomes warped product metric.

**Lemma 2.2.2** The following conditions are equivalent:

(i) there exist a non-constant solution $\psi$ of $\nabla^2 \psi = \Delta \frac{\psi}{n} g$ in a neighborhood of $p \in M$ with $\nabla \psi (p) \neq 0$.

(ii) there exist local coordinates $(u, u_1, u_2, \ldots, u_{n-1})$ in a neighborhood of $p$ and a function $\psi = \psi(u)$ with $\psi'(p) \neq 0$ and a $(n-1)$-dimensional Riemannian metric $g_* = g_*(u_1, u_2, \ldots, u_{n-1})$ such that $g(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = 1$, $g(\frac{\partial}{\partial u}, \frac{\partial}{\partial u_i}) = 0$ and $g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}) = (\psi'(u))^2 \cdot g_*(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j})$, for $i, j = 1, 2, \ldots, n - 1$.

The condition (ii) implies that the line element of $g$ may be written as $ds^2 = du^2 + (\psi'(u))^2 ds_*^2$, which is the warped product metric, and $g_*$ is a metric on an ideal level of $\psi$. 
Proof. \((ii) \implies (i)\): Let \(u\) be the arc length parameter on the integral curves of \(\nabla \psi = \psi' \frac{\partial}{\partial u}\), where \(\psi' = \frac{d\psi}{du}, \psi'' = \frac{d^2\psi}{du^2}\). Then we shall show that: \(\nabla^2 \psi = \psi'' g\). Now, \(N = \frac{\nabla \psi}{\|\nabla \psi\|} = \frac{\psi'}{\psi} \frac{\partial}{\partial u} = \frac{\partial}{\partial u},\) as \(g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = 1\), also, \(\nabla \frac{\partial}{\partial u} \nabla \psi = \nabla \frac{\partial}{\partial u} \left(\psi' \frac{\partial}{\partial u}\right) = \psi'' \frac{\partial}{\partial u} + \psi' \nabla \frac{\partial}{\partial u} \frac{\partial}{\partial u}.

However, as \(0 = \nabla N N = \nabla \frac{\partial}{\partial u} \frac{\partial}{\partial u}\) we get
\[
\nabla \frac{\partial}{\partial u} \nabla \psi = \psi'' \frac{\partial}{\partial u} \tag{2.2.1}
\]

And,
\[
\nabla \frac{\partial}{\partial u} \nabla \psi = \psi' \nabla \frac{\partial}{\partial u} \frac{\partial}{\partial u} = \psi' \sum_{j=1}^{n} \Gamma^j_{in} \frac{\partial}{\partial u} \frac{\partial}{\partial u},\]
where \(u_n = u\) and \(\Gamma^j_{in}\) is given by:
\[
\Gamma^j_{in} = \frac{1}{2} \left[ \sum_{k=1}^{n} g^{jk} \left\{ \frac{\partial}{\partial u_i} (g_{nk}) + \frac{\partial}{\partial u} (g_{ik}) - \frac{\partial}{\partial u_k} (g_{in}) \right\} \frac{\partial}{\partial u_k} \right]
\]
\[
= \frac{1}{2} \left[ g^{ij} \left\{ \frac{\partial}{\partial u_i} (g_{ni}) + \frac{\partial}{\partial u} (g_{ii}) - \frac{\partial}{\partial u_i} (g_{in}) \right\} \frac{\partial}{\partial u_i} \right]
\]
\[
+ \frac{1}{2} \left[ g^{jn} \left\{ \frac{\partial}{\partial u_i} (g_{nn}) + \frac{\partial}{\partial u} (g_{in}) - \frac{\partial}{\partial u_i} (g_{in}) \right\} \frac{\partial}{\partial u_i} \right]
\]

However \(g_{in} = g_{ni} = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = 0\) and \(g_{nn} = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = 1\), give \(\frac{\partial}{\partial u_i} (g_{nn}) = 0\) and consequently that \(\Gamma^j_{in} = \frac{1}{2} g^{ij} \left(\frac{\partial}{\partial u}\right)^2 = g^{ij} \psi' \psi'' \frac{\partial}{\partial u_i}\), or \(\Gamma^j_{in} = g^{ij} \psi' \psi'' \frac{\partial}{\partial u_i}\) for \(j = i\) and 0 otherwise. Also as \(g^{ii} = \frac{1}{g_{ii}} = \frac{1}{(\psi')^2}\), we get, \(\Gamma^j_{in} = \frac{\psi''}{\psi} \frac{\partial}{\partial u_i}\) for \(j = i\) and 0 otherwise. Thus we arrive at
\[
\nabla \frac{\partial}{\partial u} \nabla \psi = \psi' \left\{ \psi'' \frac{\partial}{\partial u_i} \right\} = \psi'' \frac{\partial}{\partial u_i} \quad i = 1, 2, ..., n - 1 \tag{2.2.2}
\]
Thus using (2.2.1), (2.2.2) we get $H_\psi(\frac{\partial}{\partial u_i}) = \psi'' \frac{\partial}{\partial u_i}$, $i = 1, 2, \ldots, n$, or $H_\psi(X) = \psi'' X$ for all $X \in \chi(M)$, which is equivalent to
\[ \nabla^2 \psi = \psi'' g \] (2.2.3)

(i) $\implies$ (ii): Let $c = \psi(p)$ and let $M_c = \{q : \psi(q) = c\}$. Then $M_c$ is a regular level hypersurface of $\psi$. Choose a coordinate system $(u, u_1, \ldots, u_{n-1})$ on $M_c$ and extend this to geodesic parallel coordinates $(u, u_1, \ldots, u_{n-1})$ in a neighborhood of $p \in M$. These have the following properties:

— The $u$-lines are geodesics with $u$ as arc length, indeed, if $\alpha(u) = (u, c_1, c_2, \ldots, c_{n-1})$, where $c_1, c_2, \ldots, c_{n-1}$ are constants, then $\dot{\alpha}(u) = (1, 0, 0, \ldots, 0)$ and $\ddot{\alpha}(u) = (0, 0, \ldots, 0) = 0$

— $\frac{\partial}{\partial u}$ is orthogonal to every set $\{(u, u_1, \ldots, u_{n-1}) : u = \text{constant}\}$, (this set is called $u$-level), also different $u$-levels are parallel to each other and that the distance between them is just the difference of the $u$ values. By construction we see that the $u$-level containing $p$ coincides with $\psi$ level containing $p$. On the other hand we know that $\frac{\partial}{\partial u}$ is the normal of the $u$-levels, and $\nabla \psi$ is the normal of $\psi$-levels (this is by 2.2.1(iii)), but $\psi$ and $\frac{\partial}{\partial u}$ are proportional; thus $\psi$-levels are parallel to $u$-levels, with that the $u$-level containing $p$ is coincides with $\psi$ level containing $p$, hence $\psi$-levels coincide with $\psi$ levels and $\psi$ may expressed as a function of $u$ : $\psi = \psi(u)$ and $\nabla \psi = \psi' \frac{\partial}{\partial u}$.

Also, since $(u, u_1, \ldots, u_{n-1})$ are geodesic parallel coordinates we have
\[ g(\frac{\partial}{\partial u}, \frac{\partial}{\partial u_i}) = 1, \quad g(\frac{\partial}{\partial u}, \frac{\partial}{\partial u_i}) = 0, \quad \text{for } i = 1, 2, \ldots, n - 1. \] So we have to prove that \((\psi'(u))^2 g_{ij}(u, u_1, u_2, \ldots, u_{n-1})\) is independent of \(u\), and hence we can write this expression as \(g_{*ij}(u_1, u_2, \ldots, u_{n-1})\).

Note that
\[
\frac{\partial}{\partial u} g\left( \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) = g\left( \nabla \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) + g\left( \frac{\partial}{\partial u_i}, \nabla \frac{\partial}{\partial u_j} \right) = g\left( \nabla \frac{\partial}{\partial u_i}, \nabla \frac{\partial}{\partial u_j} \right) + g\left( \frac{\partial}{\partial u_i}, \nabla \frac{\partial}{\partial u_j} \right)
\]

\[
= \frac{1}{\psi} g\left( \nabla \frac{\partial}{\partial u_i}, \nabla \psi, \frac{\partial}{\partial u_j} \right) + g\left( \frac{\partial}{\partial u_i}, \nabla \frac{\partial}{\partial u_j} \nabla \psi \right) \]

\[
= \frac{1}{\psi} \left( 2 \nabla^2 \psi \left( \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) \right) = \frac{2}{\psi} \Delta \psi g_{ij} = \frac{2\psi''}{\psi'} g_{ij}
\]

where the last equalities follow from the equation (2.2.3). Thus, for fixed \(u_1, u_2, \ldots, u_{n-1}\), \(g_{ij} = g_{ij}(u)\) satisfies:

\[ g'_{ij} = \frac{2\psi''}{\psi'} g_{ij}, \quad \text{or} \]

\[ \left( \frac{g_{ij}}{\psi'} \right)[''] = \frac{\left(\psi'\right)^2 g'_{ij} - 2\psi' \psi'' g_{ij}}{\left(\psi'\right)^4} = \frac{g''_{ij}}{\left(\psi'\right)^2} - \frac{2\psi'' g_{ij}}{\left(\psi'\right)^3} = 0 \]

Now, we use the geometry of warped product to deduce the following information on the domain of solution of the differential equation (*).
Lemma 2.2.3  For the warped product \( ds^2 = du^2 + (\psi'(u))^2 ds^2_* \) the following equalities hold, where \( X, Y, Z \) always denote vectors orthogonal to \( \frac{\partial}{\partial u} \):

\[
(i) \ R(X, Y)Z = R_*(X, Y)Z - \left( \frac{\psi''}{\psi'} \right)^2 \{ g(Y, Z)X - g(X, Z)Y \}
\]

\[
R(X, Y) \frac{\partial}{\partial u} = 0
\]

\[
R(X, \frac{\partial}{\partial u}) \frac{\partial}{\partial u} = - \frac{\psi''}{\psi} X
\]

\[
(ii) \ \text{Ric}(Y, Z) = \text{Ric}_*(Y, Z) - \frac{1}{(\psi')^2} \left\{ (n - 2) \left( \frac{\psi''}{\psi'} \right)^2 + \psi' \psi'' \right\} g(Y, Z)
\]

\[
\text{Ric}(Y, \frac{\partial}{\partial u}) = 0
\]

\[
\text{Ric} \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right) = -(n - 1) \frac{\psi''}{\psi'}
\]

\[
(iii) \ \left( \frac{\psi'}{\psi} \right)^2 \rho = \frac{(n-2)}{n} \rho_* - \frac{(n-2)}{n} \left( \frac{\psi''}{\psi'} \right)^2 - \frac{2}{n} \psi' \psi''
\]

\[
(iv) \ g \text{ is Einstein (of constant sectional curvature)} \Leftrightarrow \text{g}_* \text{is Einstein ( of constant sectional curvature)} \text{ and } \rho = - \frac{\psi''}{\psi'}.
\]

If \( M \) is 3-dimensional then one has to read \( g_* \text{ is of constant Gauss curvature} \) instead of \( g_* \text{ is Einstein} \) in particular in this case \( \psi \) satisfies the equation \( \psi'' = -\rho \psi + B \), for some constant \( B \in R \).

Proof. (i): We use the Gauss equation for the level hypersurface \( \{ (u, u_1, u_2, \ldots, u_{n-1}) : u = \text{constant} \} \):

\[
\nabla_X Y = \nabla_X^* Y + h(X, Y)
\]
where \( h(X,Y) = g(h(X,Y), N)N = g(A(X), Y) \frac{\partial}{\partial N} \). Using 2.2.1(ii), we calculate 

\[
A(X) = -\nabla_X N = -\nabla_X \frac{\partial}{\partial N} = -\frac{\lambda}{\|\nabla \psi\|} X,
\]

where \( \lambda = \psi'' \) and 

\[
\|\nabla \psi\| = \sqrt{g(\nabla \psi, \nabla \psi)} = \sqrt{g(\psi' \frac{\partial}{\partial N}, \psi' \frac{\partial}{\partial N})} = \psi',
\]

and consequently we get 

\( A(X) = -\frac{\psi''}{\psi} X \). Thus we have 

\[
\nabla_X Y = \nabla_X^* Y - \frac{\psi''}{(\psi')^2} g(X, Y) \nabla \psi
\]

where we used \( \frac{\partial}{\partial N} = \nabla \psi \). Hence we obtain 

\[
\nabla_X(\nabla_Y Z) = \nabla_X(\nabla_Y^* Z - \frac{\psi''}{(\psi')^2} g(X, Y) \nabla \psi)
\]

\[
= \nabla_X^* (\nabla_Y^* Z) - \frac{\psi''}{(\psi')^2} g(X, \nabla_Y^* Z) \nabla \psi
\]

\[
- \frac{\psi''}{(\psi')^2} g(Y, Z) \nabla X (\nabla \psi) - X(\frac{\psi''}{(\psi')^2} g(Y, Z) \nabla \psi)
\]

But \( X(\frac{\psi''}{(\psi')^2} g(Y,Z)) = \frac{\psi''}{(\psi')^2} Xg(Y,Z) + X(\frac{\psi''}{(\psi')^2})g(Y,Z) = \frac{\psi''}{(\psi')^2} Xg(Y,Z) \), as \( X(\frac{\psi''}{(\psi')^2}) = 0 \). Thus we obtain 

\[
\nabla_X(\nabla_Y Z) = \nabla_X^* (\nabla_Y^* Z) - \frac{\psi''}{(\psi')^2} g(Y, Z) \nabla X (\nabla \psi)
\]

\[(2.2.4)\]

\[
- \frac{\psi''}{(\psi')^2} Xg(Y,Z) \nabla \psi - \frac{\psi''}{(\psi')^2} g(X, \nabla_Y Z) \nabla \psi
\]
Also we have

\[ \nabla_X (\nabla \psi) = \nabla_X^*(\nabla \psi) - \frac{\psi''}{(\psi')^2} g(X, \nabla \psi) \nabla \psi \]

\[ = \nabla_X^*(\nabla \psi) - \frac{\psi''}{(\psi')^2} \psi' g(X, \nabla \psi) \nabla \psi \]

\[ = \nabla_X^*(\nabla \psi) - \frac{\psi''}{\psi'} g(X, \frac{\partial}{\partial u}) \nabla \psi \]

and as \( g(X, \frac{\partial}{\partial u}) = 0 \), we get

\[ \nabla_X (\nabla \psi) = \nabla_X^*(\nabla \psi) \quad (2.2.5) \]

Since, \( \psi'' X = \lambda X = \nabla_X (\nabla \psi) = \nabla_X^*(\nabla \psi) \), this together with (2.2.5) we get

\[ \nabla_X (\nabla Y Z) = \nabla_X^*(\nabla Y Z) - \frac{(\psi'')^2}{(\psi')^2} g(Y, Z) X \]

\[ - \frac{\psi''}{(\psi')^2} \{ X g(Y, Z) + g(X, \nabla Y Z) \} \nabla \psi \]

Similarly we have,

\[ \nabla_Y (\nabla X Z) = \nabla_Y^*(\nabla X Z) - \frac{(\psi'')^2}{(\psi')^2} g(X, Z) Y \]

\[ - \frac{\psi''}{(\psi')^2} \{ Y g(X, Z) + g(Y, \nabla X Z) \} \nabla \psi \]
Finally,

\[ \nabla_{[X,Y]}Z = \nabla_{[X,Y]}^*Z - \frac{\psi''}{(\psi')}^2 g([X,Y], Z) \nabla \psi \]

\[ = \nabla_{[X,Y]}^*Z - \frac{\psi''}{(\psi')}^2 g(\nabla X Y, Z) \nabla \psi + \frac{\psi''}{(\psi')}^2 g(\nabla Y X, Z) \nabla \psi \]

Hence,

\[ R(X, Y)Z = R^*(X, Y)Z - \frac{(\psi'')^2}{(\psi')^2} \{ g(X, Z)Y - g(Y, Z)X \} \]

This proves the first equation in (i).

For the second equation, as \( \nabla X N = \nabla X \frac{\partial}{\partial u} = -A(X) = -\frac{\psi''}{\psi} X \), we get

\[ \nabla X (\nabla Y \frac{\partial}{\partial u}) = -\frac{\psi''}{\psi} \nabla X Y - X \frac{\psi''}{\psi} Y - \frac{\psi''}{\psi'} \nabla X Y \]

\[ = -\frac{\psi''}{\psi'} \nabla X Y - \frac{(\psi'')^2}{(\psi')^2} g(X, Y) \nabla \psi \]

Similarly we have, \( \nabla Y (\nabla X \frac{\partial}{\partial u}) = -\frac{\psi''}{\psi'} \nabla Y X - \frac{(\psi'')^2}{(\psi')^2} g(Y, X) \nabla \psi \), and

\[ \nabla_{[X,Y]} \frac{\partial}{\partial u} = \frac{\psi''}{\psi} [X, Y] = -\frac{\psi''}{\psi} \{ \nabla x Y - \nabla y X \} \]

\[ = -\frac{\psi''}{\psi} (\nabla X Y - \frac{\psi''}{(\psi')}^2 g(X, Y) \nabla \psi - \nabla Y X + \frac{\psi''}{(\psi')}^2 g(Y, X) \nabla \psi) \]

\[ = -\frac{\psi''}{\psi} (\nabla X Y - \nabla Y X) \]
Hence we conclude $R(X, Y) \frac{\partial}{\partial u} = 0$.

Also, from lemma 2.2.1 we have $0 = \nabla_N N = \nabla \frac{\partial}{\partial u}$, so $\nabla X (\nabla \frac{\partial}{\partial u}) = 0$ and $\nabla X \frac{\partial}{\partial u} = \frac{\psi''}{\psi} X$. Thus we get

$$\nabla \frac{\partial}{\partial u} (\nabla X \frac{\partial}{\partial u}) = \nabla \frac{\partial}{\partial u} \left( \frac{\psi''}{\psi} X \right) = \frac{\partial}{\partial u} \left( \frac{\psi''}{\psi} \right) X + \frac{\psi''}{\psi} \nabla \frac{\partial}{\partial u} X$$

$$= \left\{ \frac{\psi' \psi'' - (\psi')^2}{(\psi')^2} \right\} X + \frac{\psi''}{\psi} \nabla \frac{\partial}{\partial u} X$$

Finally,

$$\nabla [X, \frac{\partial}{\partial u}] = \frac{\psi''}{\psi'} [X, \frac{\partial}{\partial u}] = \frac{\psi''}{\psi'} \nabla X \frac{\partial}{\partial u} - \frac{\psi''}{\psi'} \nabla \frac{\partial}{\partial u} X$$

$$= \left( \frac{\psi''}{\psi'} \right)^2 X - \frac{\psi''}{\psi'} \nabla \frac{\partial}{\partial u} X$$

Thus we conclude $R(X, \frac{\partial}{\partial u}) \frac{\partial}{\partial u} = -\frac{\psi' \psi''}{(\psi')^2} X = -\frac{\psi''}{\psi} X$, and this proves (i).

(ii): If \{ $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_{n-1}}$ \} are local coordinates on $(M, g_*)$ then \{ $\frac{\partial}{\partial u_1}, \frac{1}{\psi'}, \frac{\partial}{\partial u_2}, \frac{1}{\psi'} \frac{\partial}{\partial u_3}, \ldots, \frac{1}{\psi'} \frac{\partial}{\partial u_{n-1}}$ \} are local coordinates on $(M, g)$, indeed we have, $g(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}) = g_*(\frac{\partial}{\partial u_1}, \frac{2}{\psi'} \frac{\partial}{\partial u_1}, g_*(\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3}) = $ $g_*(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}) = 1$, for $i = j$ and 0 otherwise; and $g(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}) = 1$. 

(iii): Using (i) and the properties of the sectional curvature we get

\[
\text{Ric}(Y, Z) = \sum_{i=1}^{n-1} g(R\left(\frac{1}{\psi'}, \frac{\partial}{\partial u_i}, Y, \frac{1}{\psi'} \frac{\partial}{\partial u_i}\right)) + g(R\left(\frac{\partial}{\partial u}, Y\right)\frac{\partial}{\partial u})
\]

\[
= \frac{1}{\psi'} \sum_{i=1}^{n-1} g(R\left(\frac{\partial}{\partial u_i}, Y, \frac{\partial}{\partial u_i}\right)) - \frac{1}{\psi'} \sum_{i=1}^{n-1} \{g(Y, Z)g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})
\]

\[
- g(\frac{\partial}{\partial u_i}, Z)g(Y, \frac{\partial}{\partial u_i})\} + g(R\left(\frac{\partial}{\partial u}, Y\right)\frac{\partial}{\partial u})
\]

\[
= \frac{1}{\psi'} \left\{(\psi')^2 \sum_{i=1}^{n-1} g(R\left(\frac{\partial}{\partial u_i}, Y, \frac{\partial}{\partial u_i}\right)) - \left(\psi''\right)^2 \{g(Y, Z)(n - 1)
\]

\[
- g(Y, Z)\} + g(R(Z, \frac{\partial}{\partial u})\frac{\partial}{\partial u}, Y)
\]

\[
= \text{Ric}_s(Y, Z) - \frac{1}{\psi'}^2 (n - 2)g(Y, Z) - \frac{1}{\psi'} g(Y, Z)
\]

\[
= \text{Ric}_s(Y, Z) - \frac{1}{\psi'}^2 (n - 2) \left(\psi''\right)^2 + \psi' \psi'' \} g(Y, Z)
\]

Also,

\[
\text{Ric}(Y, \frac{\partial}{\partial u}) = \sum_{i=1}^{n-1} g(R\left(\frac{1}{\psi'}, \frac{\partial}{\partial u_i}, Y, \frac{\partial}{\partial u}, \frac{\partial}{\partial u_i}\right)) + g(R\left(\frac{\partial}{\partial u}, Y\right)\frac{\partial}{\partial u}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{\psi'} \sum_{i=1}^{n-1} g(R\left(\frac{\partial}{\partial u_i}, Y, \frac{\partial}{\partial u}, \frac{\partial}{\partial u_i}\right)) + g(R\left(\frac{\partial}{\partial u}, Y\right)\frac{\partial}{\partial u}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{\psi'} \sum_{i=1}^{n-1} \{0\} + 0 = 0
\]
Finally,

\[
Ric \left( \frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u} \right) = \sum_{i=1}^{n-1} g(R(\frac{1}{\psi} \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u}, \frac{\partial}{\partial u_i})) + g(R(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u}))
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} g(R(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\} + g(R(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u}))
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \psi'' \sum_{i=1}^{n-1} (\frac{\partial}{\partial u_i}) \right\}
\]

\[
= -\frac{\psi''}{(\psi')}^2 \left\{ (\psi')^2 \sum_{i=1}^{n-1} g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\} = -(n-1)\frac{\psi''}{\psi}
\]

(iii):

\[
S = \sum_{i=1}^{n-1} Ric(\frac{1}{\psi} \frac{\partial}{\partial u_i}, \frac{1}{\psi} \frac{\partial}{\partial u_i}) + Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}) \right\} + Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} \left( Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}) - \frac{1}{(\psi')}^2 \left( (n-2)(\psi')^2 + \psi' \psi'' \right) g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right) \right\}
\]

\[
+ Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} \left( (n-2)(\psi')^2 + \psi' \psi'' \right) \left( \psi' \right)^2 g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\}
\]

\[
+ Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} \left( (n-1)(n-2)(\psi')^2 + (n-1)\psi' \psi'' \right) \left( \psi' \right)^2 g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\}
\]

\[
+ Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} \left( (n-1)(n-2)(\psi')^2 + (n-1)\psi' \psi'' \right) \left( \psi' \right)^2 g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\}
\]

\[
+ Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} \left( (n-1)(n-2)(\psi')^2 + (n-1)\psi' \psi'' \right) \left( \psi' \right)^2 g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\}
\]

\[
+ Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} \left( (n-1)(n-2)(\psi')^2 + (n-1)\psi' \psi'' \right) \left( \psi' \right)^2 g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\}
\]

\[
+ Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} \left( (n-1)(n-2)(\psi')^2 + (n-1)\psi' \psi'' \right) \left( \psi' \right)^2 g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\}
\]

\[
+ Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]

\[
= \frac{1}{(\psi')}^2 \left\{ \sum_{i=1}^{n-1} \left( (n-1)(n-2)(\psi')^2 + (n-1)\psi' \psi'' \right) \left( \psi' \right)^2 g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i})) \right\}
\]

\[
+ Ric(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u})
\]
Thus

\[
\frac{S}{n(n-1)} = \frac{1}{(\psi')^2} \left[ \frac{S_*}{n(n-1)} - \frac{(n-2)}{n} \left( \psi'' \right)^2 - \frac{\psi' \psi'''}{n} \right]
\]

or,

\[
\rho = \frac{1}{(\psi')^2} \left[ \frac{(n-2)}{n} \rho_* - \frac{(n-2)}{n} \left( \psi'' \right)^2 - \frac{2\psi' \psi'''}{n} \right]
\]

and consequently we have

\[
\left( \psi' \right)^2 \rho = \left[ \frac{(n-2)}{n} \rho_* - \frac{(n-2)}{n} \left( \psi'' \right)^2 - \frac{2\psi' \psi'''}{n} \right]
\]

(iv): For constant curvature case, by (i) we have

\[
R(X,Y)Z = R_*(X,Y)Z - \left( \frac{\psi''}{\psi'} \right)^2 \{g(Y,Z)X - g(X,Z)Y\}
\]

Thus for orthonormal \(X,Y\) we have

\[
g(R(X,Y)Y,X) = g(R_*(X,Y)Y,X) - \left( \frac{\psi''}{\psi'} \right)^2 \{g(Y,Y)g(X,X) - g(X,Y)g(Y,X)\}
\]

Or \(K = \left( \psi' \right)^2 K_* - \left( \frac{\psi''}{\psi'} \right)^2\). But, as \(K_*\) does not depend on \(X,Y\) by schurts theorem \(K\) is a constant.

For the Einstein case, it follow directly from (ii). Also by lemma 2.1.1, we have \(\nabla^2 \psi = (-\rho \psi + B)g\), which by lemma 2.2.2, reads as \(\psi'' g = (-\rho \psi + B)g\) or \(\psi'' = -\rho \psi + B\), which on differentiating gives \(\psi''' = -\rho \psi'\) that is \(\rho = -\frac{\psi'''}{\psi'}\).
Using the information obtained above, we have the following corollary:

**Corollary 2.2.4** Let \((M, g)\) be an Einstein space of scalar curvature \(\rho\), and let \(p \in M\) be a regular point of a function \(\psi\) satisfying \(\nabla^2 \psi = \frac{\Delta \psi}{n} g\), then in a neighborhood of \(p\) there are coordinates \((u, u_1, \ldots, u_{n-1})\) such that \(\psi = \psi(u)\) and \(\psi'' = -\rho \psi + B\), \(B\) being a constant.

**Proof.** From lemma 2.1.1, it follows that: there exist a constant \(B\) such that: \(\nabla^2 \psi = (-\rho \psi + B)g\). Also from lemma 2.2.2, it follows that: there exist local coordinates \((u, u_1, \ldots, u_{n-1})\) such that: \(\nabla^2 \psi = \psi''\), combining these two results we get the desired conclusion.

**Corollary 2.2.5** A 4-dimensional Einstein space admitting a non-constant solution \(\psi\) of \((*)\) is of constant sectional curvature.

**Proof.** Assume that for all \(p \in M\), \(p\) is critical point of \(\psi\), that is, \(\nabla \psi \mid_p = 0\), \(\forall p \in M\). But this means that, there is no non-constant solution \(\psi\) of the equation \(\nabla^2 \psi = \frac{\Delta \psi}{n} g\), which is a contradiction. So, there is at least one \(p \in M\) such that: \(\nabla \psi \mid_p \neq 0\). Now by Lemmas 2.2.2 and 2.2.3(iv), we get \(g\) on the ideal level is a 3-dimensional Einstein space, hence of constant sectional curvature on \(M\). Now by lemma 2.2.3(iv), \(g\) is of constant sectional curvature on \(M\).

**Corollary 2.2.6** A 4-dimensional Einstein space admitting a non-homothetic conformal (or concircular) mapping onto another Einstein space is of constant
sectional curvature.

**Proof.** If \((M,g)\) is a 4-dimensional Einstein space and, \(g \to \overline{g}\) is a non-homothetic conformal (or concircular) where \(\overline{g} = \psi^{-2}g\), and \((M,g)\) is also Einstein, then we have to prove that \((M,g)\) is of constant sectional curvature. Now by lemma 2.1.1 as \((M,\overline{g})\) is Einstein space, \(\nabla^2 \psi = \frac{\Delta \psi}{n}g\) holds and as the transformation \(g \to \overline{g}\) is a non-homothetic conformal (concircular) then by using corollary 10 in Künnel [6], \((M,\overline{g})\) is a space of constant sectional curvature and by Corollary 2.2.5, \((M,g)\) is of constant sectional curvature.

Now, we study the existence of the solution of \(\nabla^2 \psi = \frac{\Delta \psi}{n}g\) near a regular points, but the situation is quite different near a critical point. We state the following Lemma without proof.

**Lemma 2.2.7** Let \(p \in M\) be a critical point of a non-constant function \(\psi\) satisfying \(\nabla^2 \psi = \frac{\Delta \psi}{n}g\), then there exists a neighborhood \(U\) of \(p\) such that:

\[(i)\] \(p\) is the only critical point of \(\psi\) in \(U\)

\[(ii)\] the level hypersurfaces of \(\psi\) coincide in \(U\) with the geodesic distance spheres around \(p\), in particular the critical points of \(\psi\) are isolated.

Now we have the following relation, between the local solution of the differential equation (*) and the warped product of an interval with a unit sphere.
Lemma 2.2.8 The following conditions are equivalent:

(i) There exists a non-constant solution $\psi$ of $\nabla^2 \psi = \frac{\Delta \psi}{n} g$ in a neighborhood of $p \in M$ with $\nabla \psi |_{p} = 0$

(ii) There exist polar coordinates $(r, u_1, ..., u_{n-1})$ in a neighborhood of $p$ and an even function $\psi = \psi(r)$ with $\psi'(0) = 0$ and $\psi''(0) \neq 0$ such that

$$ds^2 = dr^2 + \frac{(\psi'(r))^2}{(\psi''(0))^2} \cdot ds_1^2$$

where $ds_1^2$ denotes the line element of the standard unit sphere $(S^{n-1}, g_1)$.

Proof. (ii) $\Rightarrow$ (i): Follows from lemma 2.2.2, for all points excepts at $r = 0$. The evenness of the function $\psi(r)$ (i.e. vanishing of the odd derivatives $\psi'(0), \psi''(0), ...$) implies that the right hand side of the equation $ds^2 = dr^2 + \frac{(\psi'(r))^2}{(\psi''(0))^2} \cdot ds_1^2$ has no proper singularity at $r = 0$. Then by continuity, the equation $(*)$ holds also at $r = 0$.

(i) $\Rightarrow$ (ii): It follows from lemma 2.2.2 and 2.2.7 that we can introduce local coordinates $(r, u_1, ..., u_{n-1})$ such that for $r \neq 0$, $ds^2 = dr^2 + (\psi'(r))^2 \cdot ds_1^2$, where $ds_1^2$ is the line element of the ideal level $(M_*, g_*)$. It remains to show:

(a) $\psi$ is an even function and $\psi''(0) \neq 0$

(b) $g_*$ is of constant sectional curvature $\left(\psi''(0)\right)^2$

Let $X, Y$ be two orthonormal vectors in $M$ which are tangent to a level
hypersurface \( M_* = \{ q \in M : \psi(q) = r_0 > 0 \} \), for sufficiently small \( r_0 \). For the sectional curvatures \( K_\sigma \) and \( K^*_\sigma \) of the \((X,Y)\)-plane in \((M,g)\) and \((M_,g_*)\) we have from lemma 2.2.3 (i)

\[
K_\sigma = g(R(X,Y)Y,X) = g(R_*(X,Y)Y,X) - \frac{\left(\psi''(r_0)\right)^2}{(\psi'(r_0))^2} = \frac{1}{(\psi'(r_0))^2}\{K^*_\sigma - \left(\psi''(r_0)\right)^2\}
\]

On the other hand, we know that \( K^*_\sigma \) is independent of \( r \) when \( r \) tends to zero [as \( K^*_\sigma \) is the sectional curvature of \((M_*,g_*)\) and \( M_* = \{ q : \psi(q) = r_0 > 0 \} \) (for sufficiently small \( r_0 \)].Because \( \psi'(0) = 0 \) (\( p \) is critical point of \( \psi \)) it follows from the above equation that, \((\psi'(r_0))^2K_\sigma = K^*_\sigma - \left(\psi''(r_0)\right)^2 \) and

\[
\lim_{r_0 \to 0}(\psi'(r_0))^2K_\sigma = \lim_{r_0 \to 0}(K^*_\sigma - (\psi''(r_0))^2)
\]

Or \( 0 = K^*_\sigma - (\psi''(0))^2 \). Hence \((M_*,g_*)\) is a space of constant curvature \((\psi''(0))^2\). This must be positive because by lemma 2.2.7, we may assume that \( M_* \) is geodesic distance sphere consequently \((\psi''(0))^2 > 0 \) and \( g_* = \frac{1}{(\psi'(0))^2}g_1 \). Also, by assumption the metric \( g \) has no singularity at \( r = 0 \), this implies that \( \psi(r) \) must be an even function and that the equation \( ds^2 = dr^2 + \frac{(\psi'(r))^2}{(\psi'(0))^2} ds_1^2 \), is valid for all \( r \geq 0 \).

The following corollary, shows that in a neighborhood of a critical point of the solution of the differential equation \((*)\) both manifolds are of constant curvature.
Corollary 2.2.9 Let \((M, g)\) and \((M, \bar{g})\) be two Einstein spaces, \(\bar{g} = \psi^{-2}g\) and let \(p \in M\) be a critical point of \(\psi\). Then in a neighborhood of \(p\) both spaces are of constant sectional curvature.

**Proof.** By applying lemma 2.1.1 we get that, there exist a function \(\psi : M \to \mathbb{R}\) such that \(\nabla^2 \psi = \frac{\Delta \psi}{n} g\) holds and by lemma 2.2.8 \((M_*, g_*)\) is of constant sectional curvature in a neighborhood of \(p\) (it has been already proved in the proof of lemma 2.2.8). Now by Lemma 2.2.3 \((iv)\), \((M, g)\) is of constant sectional curvature in a neighborhood of \(p\) and by proposition 2.1.2 we get that \((M, \bar{g})\) is of constant sectional curvature in a neighborhood of \(p\). Hence, we get a neighborhood of \(p\) where both spaces are of constant sectional curvature.

Now, we prove the following equivalences.

**Proposition 2.2.10** Let \((M, \bar{g}), (M, g)\); \(\bar{g} = \psi^{-2}g\) be given with a non-constant function \(\psi\). Then the following conditions are equivalent:

(i) \(\bar{R}(X, Y)Z = R(X, Y)Z\), for all \(X, Y, Z\)

(ii) \(\bar{Ric} = Ric\)

(iii) \(\nabla^2 \psi = \frac{\|\nabla \psi\|^2}{2\psi} g\)

(iv) \(g\) is a warped metric, \(ds^2 = du^2 + (\psi(u))^2 ds^2_*\) and there are constants \(a, b, c \in \mathbb{R}\) with \(b^2 - 4ac = 0\) satisfying \(\psi(u) = au^2 + bu + c\).
Moreover, each of the conditions implies that $\psi$ has no critical point. If one of the spaces is Einstein then both are Ricci-flat.

**Proof.** $(i) \Rightarrow (ii)$: is trivial.

$(ii) \Rightarrow (iii)$: By lemma 1.3.5$(iii)$, we have:

$$0 = \overline{\text{Ric}} - \text{Ric} = \left( \frac{\Delta \psi}{\psi} - (n - 1) \frac{\|\nabla \psi\|^2}{\psi^2} \right) g + (n - 2) \frac{\nabla^2 \psi}{\psi} \tag{2.2.6}$$

Taking the trace we get:

$$0 = n \left( \frac{\Delta \psi}{\psi} - (n - 1) \frac{\|\nabla \psi\|^2}{\psi^2} \right) + (n - 2) \frac{\Delta \psi}{\psi}$$

or $\frac{\Delta \psi}{n \psi} = \frac{\|\nabla \psi\|^2}{2 \psi^2}$. Using this in (2.2.6) we get

$$0 = \left( \frac{\Delta \psi}{\psi} - \frac{2}{n} (n - 1) \frac{\Delta \psi}{\psi} \right) g + (n - 2) \frac{\nabla^2 \psi}{\psi}$$

$$\Rightarrow 0 = - \frac{(n - 2)}{n} \left( \frac{\Delta \psi}{\psi} \cdot g \right) + (n - 2) \frac{\nabla^2 \psi}{\psi} \Rightarrow \frac{\nabla^2 \psi}{\psi} = \frac{\Delta \psi}{n \psi} g = \frac{\|\nabla \psi\|^2}{2 \psi^2} g,$$ or

$\nabla^2 \psi = \frac{\|\nabla \psi\|^2}{2 \psi} g$.

$(iii) \Rightarrow (i)$: If we write the equation in lemma 1.3.5 $(ii)$ in terms of $\psi$ we
get

$$\psi(\overline{R}(X, Y)Z - R(X, Y)Z) = -g(X, Z)H_{\psi}Y + g(Y, Z)H_{\psi}X$$

$$+ \nabla^2 \psi(Y, Z)X - \nabla^2 \psi(X, Z)Y$$

$$+ \frac{||\nabla \psi||^2}{\psi}(g(X, Z)Y - g(Y, Z)X)$$

$$= -\frac{||\nabla \psi||^2}{2\psi}g(X, Z)Y + \frac{||\nabla \psi||^2}{2\psi}g(Y, Z)X$$

$$+ \frac{||\nabla \psi||^2}{2\psi}g(Y, Z)X - \frac{||\nabla \psi||^2}{2\psi}g(X, Z)Y$$

$$+ \frac{||\nabla \psi||^2}{\psi}\{g(X, Z)Y - g(Y, Z)X\}$$

$$= 0$$

$$\Rightarrow \psi(\overline{R}(X, Y)Z - R(X, Y)Z) = 0.$$ But by assumption $\psi \neq 0$, so

$$\overline{R}(X, Y)Z - R(X, Y)Z = 0$$

which is $(i)$. 

Now assume that $p \in M$ is a point with $\nabla \psi|_p = 0$ and that $(iii)$ holds. Then $\nabla^2 \psi|_p = 0$ which contradicts Lemma 2.2.8 [because we have a non-constant function $\psi$ with $\nabla^2 \psi = \frac{\Delta \psi}{n}g$ in a neighborhood of $p \in M$ with $\nabla \psi|_p = 0$ and $\nabla^2 \psi|_p \neq 0$, which gives us that $\psi''(0) \neq 0$, and hence we will not get such expression $ds^2 = dr^2 + \frac{(\psi'(r))^2}{(\psi''(0))^2}ds_1^2$ thus, all the points are regular.

$(iii) \Leftrightarrow (iv)$: Follows from lemma 2.2.2, in the coordinates $(u, u_1, \ldots, u_{n-1})$ the equation: $\nabla^2 \psi = \frac{||\nabla \psi||^2}{2\psi}g$ reads as $\psi'' = \frac{(\psi')^2}{2\psi}$, or

$$2\psi\psi'' = (\psi')^2 \quad (2.2.7)$$
Differentiating again we get $2\psi\psi'' + 2\psi'\psi''' = 2\psi\psi'' \Rightarrow \psi\psi''' = 0$. But $\psi \neq 0$ and thus $\psi''' = 0$, that is, $\psi(u) = au^2 + bu + c$, for some constants $b,c$. Using this and (2.2.7) we get $b^2 - 4ac = 0$. Now if one of them is Einstein say $(M, g)$, then by corollary 2.2.4, $\psi'' = -\rho\psi + B$. But as $\psi'' = \text{constant} \Rightarrow \rho = 0$. That is

$$Ric = \overline{Ric} = 0$$

### 2.3 GLOBAL SOLUTIONS OF THE DIFFERENTIAL EQUATION

In this section we are interested in the following questions:

**I.** What are the conditions under which a complete Einstein space can be mapped conformally onto another (possibly non-complete) Einstein space?

**II.** What are the conditions under which complete Einstein spaces can be mapped conformally onto another complete Einstein space?

**III.** What are the conditions under which complete Riemannian manifolds admit a globally defined concircular mapping onto another Riemannian manifold?

**IV.** What are the conditions under which a complete Riemannian manifolds admit a globally defined solution of the equation $\nabla^2 \psi = \frac{\Delta \psi}{n} g$?
It has been observed that all above questions are dealing with the equation $\nabla^2 \psi = \frac{\Delta \psi}{n} g$. Therefore we will be interested in the answer for the last question, so we start with the following

**Theorem 2.3.1** Let $(M^n, g)$ be a complete connected Riemannian manifold admitting a non-constant solution $\psi$ of $\nabla^2 \psi = \frac{\Delta \psi}{n} g$. Then the number of critical points of $\psi$ is $N \leq 2$, and $M$ is conformally diffeomorphic to:

(i) the sphere $(S^n, g_1)$ if $N = 2$.

(ii) the euclidean space $(E^n, g_0)$ or hyperbolic space $(H^n, g_{-1})$ if $N = 1$.

(iii) the Riemannian product $I \times M_s$ if $N = 0$, where $(M_s, g_s)$ is a complete $(n - 1)$-manifold and $I \subseteq R$ is an open interval.

To prove this theorem, we need the following lemma:

**Lemma 2.3.2** Let $(M^n, g)$ and $\psi$ be as in theorem 2.3.1. If $P$ denotes the set of critical points of $\psi$ then $N = |P| \leq 2$, and $(M \setminus P, g)$ is isometric to the warped product $I \times M_s$ with the warped product metric

$$ds^2 = du^2 + (\psi'(u))^2 ds_s^2$$

(2.3.1)

where $(M_s, g_s)$ is a complete $(n - 1)$-manifold and $I \subseteq R$ is an open interval.

**Proof.** As $P$ denotes the set of critical points of $\psi$, by lemma 2.2.7 it is a set of isolated points. Thus by Lemma 2.2.2, we have, for each fixed point
$q \in M \setminus P$ there is a neighborhood $U$ such that (2.3.1) holds in $U$, where $ds^2_*$ is the line element of the level hypersurface

$$M_* = \{ x \in M : \psi(x) = \psi(q) \}$$

Now if $(x_n)$ is a Cauchy sequence in $(M_*, g_*)$, it is also a Cauchy sequence in $M$. But $M$ is complete so this sequence converges to a point $x_0 \in M$. But as $(x_n) \in M_*$, we have $\psi(x_n) = \psi(q)$ and consequently $\psi(\lim_{n \to \infty} x_n) = \psi(q)$. Hence $x_0 \in M_*$ and that $(M_*, g_*)$ is complete.

Thus we may assume that $u = (\alpha, \beta) \times M_*$, and let $\alpha_0, \beta_0$ to be the infimum and supremum of $\alpha, \beta$ respectively such that (2.3.1) holds for $(\alpha, \beta) \times M_*$. The integral curve through $q$ is the unique geodesic with tangent $\frac{\partial}{\partial u}$. Now if $\alpha_0$ is finite then there is a point $q_0$ on this geodesic with $\psi(q_0) = \psi(\alpha_0)$, and so if $q_0$ is a regular point, then by lemma 2.2.2 there is a neighborhood of $q_0$ such that (2.3.1) holds but $q_0$ is the infimum of $\alpha$ in $(\alpha, \beta) \times M_*$ such that (2.3.1) holds and so $q_0$ is a critical point and $q_0$ is a minimum of $\psi$ because by lemma 2.2.3, $\nabla^2 \psi$ is definite at critical point.

Similarly if $\beta_0$ is finite, we have the following possibilities:

**I.**

$\alpha_0, \beta_0 \in \mathbb{R}$, then as $M_*$ coincide in $U$ with the geodesic distance spheres around $q$, and $[\alpha_0, \beta_0]$ is a closed interval in $\mathbb{R}$, and hence is compact. Thus $(\alpha_0, \beta_0) \times M_*$ with the warped product metric (2.3.1) is a compact manifold if we add the two critical points of level $\psi(\alpha_0)$ and $\psi(\beta_0)$ and as $M$ is connected
Riemannian manifold then no other critical points can occur. Hence \( n = 2 \).

\[ II. \]

a) \( \alpha_0 \in R, \beta_0 = +\infty \), then as \([\alpha_0, \infty)\) is complete, and \( M_* \) also complete then \((\alpha_0, \infty) \times M_* \) is complete if we add the minimum of level \( \psi(\alpha_0) \). Hence \( N = 1 \).

b) \( \alpha_0 = -\infty, \beta_0 \in R \), then the same holds for \((-\infty, \beta_0) \times M_* \).

\[ III. \]

\( \alpha_0 = -\infty, \beta_0 = \infty \), then \( R \times M_* \) is complete with the warped product metric \( ds^2 = du^2 + (\psi'(u))^2 ds^*_2 \). Hence, \( M \cong R \times M_* \) and \( N = 0 \). And the proof of lemma 2.3.2 is complete.

**Proof of the theorem.** First case: if \( N = 0 \) then by last lemma; \( M \cong R \times M_* \) and the metric \( g \) satisfies

\[
    ds^2 = du^2 + (\psi'(u))^2 ds^*_2
\]

and hence,

\[
    d\tilde{s}^2 = \frac{1}{(\psi'(u))^2} du^2 + ds^*_2
\]

which is the Riemannian product \( I \times M_* \).
Second case: if $N = 1$, then the metric $g$ satisfies: $ds^2 = du^2 + (\psi'(u))^2 ds^2_a$, in $M \setminus \{p\}$, where $p$ is the only critical point and by lemma 2.2.8 it is satisfies:

$$ds^2 = dr^2 + \left(\frac{\psi'(r)}{\psi''(0)}\right)^2 ds^2_1$$

where $ds^2_1$ denotes the line element of the standard unit sphere $(S^{n-1}, g_1)$.

If we regard this as an expression in polar coordinates around $p$ we see the $M$ is diffeomorphic to $R^n$. By conformal change the metric transforms into the euclidean or hyperbolic metric depending on the growth of $\psi'(u)$ where $u$ tends to infinity.

Third Case: if $N = 2$, then $M$ is compact and by lemma 2.3.2 the metric $g$ satisfies

$$ds^2 = dr^2 + \left(\frac{\psi'(r)}{\psi''(0)}\right)^2 ds^2_1$$

in $M \setminus \{p, q\}$ where $p, q$ are the critical points, and so by Reeb's theorem $M$ is homeomorphic to the sphere and so compact. Hence, the metric can be conformally transformed into a metric of constant curvature. Hence $(M, g)$ is conformally flat manifold, so by theorem of N.H. Kuiper[7], $(M, g)$ is conformally equivalent to $(S^n, g_1)$.

**Corollary 2.3.3** Let $(M, g)$ be compact Riemannian manifold admitting a non-constant solution $\psi$ of $\nabla^2 \psi = \frac{\Delta \psi}{n} g$. Then $(M, g)$ is conformally diffeomorphic to $(S^n, g_1)$. 
Proof. This follows from theorem 2.3.1, as \((M, g)\) is compact then \(\psi\) has at least two critical points. Things are not going to be the same if we put new additional limitations. For compact manifolds we have the following:

\textbf{Theorem 2.3.4} Let \((M, g)\) be a compact Riemannian manifold of constant scalar curvature. Assume that it admits a non-constant solution \(\psi\) of the differential equation

\[ \nabla^2 \psi = \frac{\Delta \psi}{n} g \]

then \((M, g)\) is isometric with the standard sphere of certain radius.

Proof. By Corollary 2.3.3 we see that, \((M, g)\) is conformally diffeomorphic to the sphere and by lemma 2.3.2 we get that the metric on \(M \setminus \{p, q\}\) satisfies:

\[ ds^2 = dr^2 + \frac{(\psi'(r))^2}{(\psi''(0))^2} ds_1^2 \]  

(2.3.2)

Where \(ds_1^2\) is the line element of the standard unit sphere. Also as we know from the proof of Lemma 2.2.8 that \(\rho_* = (\psi''(0))^2\) and it is given to us that \(\rho = \text{constant}\), so by lemma 2.2.3\((iii)\)

\[ \left( \psi' \right)^2 \rho = \frac{(n-2)}{n} \rho_* - \frac{(n-2)}{n} \left( \psi'' \right)^2 - \frac{2}{n} \psi' \psi''' \]

And now we try to solve this differential equation for unknown function \(\psi' = \psi'(u)\) with the initial conditions: \(\psi'(0) = 0, \psi''(0) = \sqrt{\rho_*}\)

Let \(y = \psi'\). Then the equation reads as:
\[ y y'' + \frac{(n-2)}{n} (y')^2 + \frac{2}{3} \rho y^2 - \frac{(n-2)}{2} \rho_\ast = 0 \]

Multiplying by \(2y^{n-3}y'\), we get \(\left(y^{n-2} (y')^2\right)' + (\rho y^n - \rho_\ast y^{n-2})' = 0\), or \(y^{n-2} (y')^2 + \rho y^n - \rho_\ast y^{n-2} = c_1\), for some constant \(c_1\) in \(R\). Now as \(y(0) = \psi'(0)\), substitute this in last equation we get \(0 = c_1\), which gives \(y^{n-2} (y')^2 + \rho y^n - \rho_\ast y^{n-2} = 0\), or equivalently \(y^{n-2} (y')^2 + \rho y^n = \rho_\ast y^{n-2}\). Since \(y^{n-2} \neq 0\), we get \((y')^2 + \rho y^2 - \rho_\ast = 0\). Differentiating the last equation get \(2y'y'' + 2\rho y y' = 0\), that is \(2y''(y'' + \rho y) = 0\). However, \(y' \neq 0\), so we must have \(y'' + \rho y = 0\). Thus if we denote the \(n^{th}\)-derivative of \(y\) by \(D^n y = \frac{d^n y}{du^n}\) then the last equation reads as \((D^2 + \rho)y = 0\), and so the auxiliary equation of this ordinary differential equation is \(m^2 + \rho = 0\), \(\Rightarrow m = \pm \sqrt{-\rho}\); to solve this equation we have to discuss three cases:

**I.** If \(\rho > 0\), then \(y(u) = c \sin \sqrt{\rho}u + d \cos \sqrt{\rho}u\), where \(c, d\) are constants. Using the initial condition \(y(0) = 0\), we get \(d = y(0) = 0\). Thus \(\psi'(u) = y(u) = c \sin(\sqrt{\rho}u)\), for \(\rho > 0\).

**II.** If \(\rho = 0\), then \(y(u) = cu + d\). But as \(d = y(0) = 0\), we have \(y(u) = cu\).

**III.** If \(\rho < 0\), then \(y(u) = c \sinh \sqrt{-\rho}u + d \cosh \sqrt{-\rho}u\). Substitute that \(y(0) = 0\), to get: \(0 = y(0) = d\). Hence \(y(u) = c \sinh \sqrt{-\rho}u\). Thus

\[ \psi'(u) = \begin{cases} 
\frac{c \sin \sqrt{\rho}u}{c \sinh \sqrt{-\rho}u} & \rho > 0 \\
\frac{c \cdot u}{c \sinh \sqrt{-\rho}u} & \rho = 0 \\
\frac{c \cdot u}{c \sinh \sqrt{-\rho}u} & \rho < 0 
\end{cases} \quad (2.3.3) \]

where \(c\) is a constant.
Now by lemma 2.2.3(iv), as $g_*$ has constant sectional curvature $(\psi''(0))^2$, then $g$ is of constant sectional curvature $= -\frac{\psi''}{\psi'} = \rho$, and as we already know that $M$ diffeomorphic with the sphere, and hence $\rho > 0$.

Now by the formula for $\rho$ and (2.3.3)

$$\sqrt{p_*} = \begin{cases} \frac{\sqrt{pc}}{c} & \rho > 0 \\ \sqrt{-pc} & \rho = 0 \\ \sqrt{pc} & \rho < 0 \end{cases}$$

Thus, $c = \frac{\sqrt{p_*}}{\sqrt{\rho}}$. Hence

$$\psi'(u) = \frac{\sqrt{p_*}}{\sqrt{\rho}} \cdot \sin \sqrt{\rho} u$$

which gives us that

$$ds^2 = du^2 + \frac{\sin^2 \sqrt{\rho} u}{\rho} ds_1^2$$

which proves that $(M, g)$ is isometric with the sphere of radius $\frac{1}{\sqrt{\rho}}$.

Now we have the following theorem:

**Theorem 2.3.5** Let $C, B$ be constants and $(M, g)$ is a complete Riemannian manifold admitting a non-constant solution of $\nabla^2 \psi = (-C^2 \psi + B)g$. Then $(M, g)$ is isometric with the standard sphere of radius $1/C$.

**Proof.** If $P$ denotes the set of critical point of $\psi$, then by lemma 2.3.2 we have that $ds^2 = du^2 + (\psi'(u))^2 ds_*^2$ in $M \setminus P$, where $ds_*^2$ is the line element of
the level hypersurface \((M_\ast, g_\ast)\). Now, by lemma 2.2.2 we have: \(\nabla^2 \psi = \psi'' g\), and so \(\nabla^2 \psi = (-C^2 \psi + B)g\) becomes,

\[
\psi'' = -C^2 \psi + B
\]

(2.3.4)

to solve this equation, we will first solve the homogenous differential equation \(\psi'' + C^2 \psi = 0\). The auxiliary equation is \(m^2 + C^2 = 0\), or \(m = \pm \sqrt{-C^2}\). Thus the general solution of (2.3.4) is \(\psi(u) = a \cos(Cu) + b \sin(Cu) + d\), where \(a, b\) and \(C\) are constants. Substitute this in (2.3.4)

\[
\psi''(u) = -C^2 a \cos(Cu) - C^2 b \sin(Cu)
\]

\[
\Rightarrow -C^2 \{a \cos(Cu) + b \sin(Cu)\} + C^2 \{a \cos(Cu) + b \sin(Cu) + d\} = B,
\]

or \(C^2 d = B \Rightarrow d = \frac{B}{C^2}\). That is

\[
\psi(u) = a \cos(Cu) + b \sin(Cu) + \frac{B}{C^2}
\]

(2.3.5)

where \(a, b\) and \(C\) are constants.

Now as \(\psi(u)\) is bounded, then \(\psi\) has at least two critical points (minimum and maximum). Let \(u = 0\) be one of these critical points and hence we have by 2.2.7 and 2.2.8 that the levels are standard spheres and \(g\) satisfies:

\[
\ ds^2 = dr^2 + \left(\frac{\psi'(r)}{\psi(0)}\right)^2 ds^2_1,
\]

where \(ds^2_1\) denotes the line element of the standard unit sphere \((S^{n-1}, g_1)\). As \(u = 0\) is a critical then from \(\psi'(0) = 0 \Rightarrow b = 0\). That is \(\psi(u) = a \cos(Cu) + \frac{B}{C^2}\). Thus, \(ds^2 = du^2 + \frac{\sin^2(Cu)}{C^2} ds^2_1\), which gives us that \(M\) is isometric to the standard sphere of radius \(\frac{1}{C}\).

**Proposition 2.3.6** Assume that a complete manifold admits a non-constant
solution of $\nabla^2 \psi = Bg$. $0 \neq B \in R$. Then it is isometric with the euclidean space.

**Proof.** By lemma 2.3.2 we have that $M \setminus P$ satisfies globally $ds^2 = du^2 + (\psi'(u))^2 ds^2_*$, where $P$ denotes the set of critical points of $\psi$, and $ds^2_*$ is the line element of the level hypersurface of $(M_*, g_*)$, and by lemma 2.2.2 $Bg = \nabla^2 \psi = \psi''g$, or $\psi'' = B$. Hence we get $\psi(u) = au^2 + bu + c$, where $a, b, c$ are constants. As $B = \psi'' = 2a \Rightarrow a = \frac{B}{2}$, or $\psi(u) = \frac{B}{2} u^2 + bu + c$. Also as $B \neq 0$, then $\psi$ must have a critical point. Let $u = 0$ be the critical point, then $0 = \psi'(0) = b$ and hence $\psi(u) = \frac{B}{2} u^2 + c$. Also, we know from the proof of lemma 2.2.8 that $M_*$ is of constant sectional curvature $(\psi''(0))^2$, and that $ds^2_* = \frac{1}{(\psi''(0)^2)ds^2}$, $\frac{1}{B^2}ds^2_1$. Therefore

$$ds^2 = du^2 + u^2 ds^2_1$$

which is the metric of euclidean space in polar coordinates.

In the following theorem we try to get a solution of problem I and II.

**Main Theorem 2.3.7** (i) Let $(M, g)$ be a complete Einstein space. Then there exists a non-homothetic conformal di↵eomorphism $(M, g) \rightarrow (\overline{M}, \overline{g})$ onto another (possibly non-complete) Einstein space if and only if $(M, g)$ is isometric with a sphere, euclidean or hyperbolic space or it is the product $M = R \times M_*$ with a complete Ricci-flat space $(M_*, g_*)$, endowed with the warped product metric $ds^2 = du^2 + e^{2u}ds^2_*$, (up to scaling) which has negative scaler curvature.
(ii) If in addition $(\overline{M}, \overline{g})$ is complete, then each of the spaces is isometric with a standard sphere of certain radius.

**Proof.** If $\frac{1}{\psi^2}g = \tilde{g} = f^*\overline{g}$, then we know that $\psi : M \to R$ satisfies:

1. $\nabla^2 \psi = (-\rho \psi + B)g$ (by lemma 2.1.1).

2. $\psi$ have at most two critical points (by lemma 2.3.2).

So by lemma 2.2.8 we get that, $g$ satisfies $ds^2 = du^2 + (\psi'(u))^2ds_*^2$, on $M$ except at the critical points, where $ds_*^2$ is a complete metric on $(n-1)$-manifold $M_*$ and $(M_*, g_*)$ is isometric to a standard sphere. According to this we have two cases:

**I.** if $\psi$ has a critical point: If $p$ is a critical point of $\psi$ then by corollary 2.2.9 $(M, g)$ is a space of constant sectional curvature near $p$. And by the global warped product it follows that $(M, g)$ is globally a space of constant sectional curvature, and by theorem 2.3.1 we get that $M$ is conformally diffeomorphic to one of $S^n, E^n, H^n$.

**II.** if $\psi$ has no critical points: Then in this case, we know that by lemma 2.2.2 that $\psi = \psi(u)$, where $u$ is the arc length parameter on the geodesic trajectory. That is $\psi : R \to R$ also satisfies: $\psi'' = -\rho \psi + B$ by lemma 2.2.4.

So first we will solve the homogenous differential equation $\psi'' + \rho \psi = 0$. The auxiliary equation is $m^2 + \rho = 0$, or $m = \pm \sqrt{-\rho}$, then this is possible if and only if $\rho$ less than zero and $\psi(u) = e^{\pm \sqrt{-\rho} u} + c$, with $c \geq 0$ (this because of
the restriction that $\psi(u) \neq 0$ and $\psi'(u) \neq 0, \forall u \in R$. Hence,

$$ds^2 = du^2 - \rho e^{2\sqrt{-\rho}u} ds_*^2$$

Now as $(M, g)$ is an Einstein space then by lemma 2.2.3 $(M, g_*)$ is Einstein space with scalar curvature:

$$\rho_* = \left(\psi''\right)^2 + \frac{2}{(n-2)} \psi' \psi''' + \frac{n}{(n-2)} \rho \cdot \left(\psi'\right)^2$$

$$= \rho^2 e^{2\sqrt{-\rho}u} \left(1 + \frac{2}{(n-2)} - \frac{n}{(n-2)}\right) = 0$$

In the special case: $(M_*, g_*) = (E^n, g_0)$. We get $ds^2 = du^2 - \rho e^{2\sqrt{-\rho}u} ds_0^2$, which is hyperbolic space of curvature $\rho$. This complete the proof of (i)

Now to prove (ii), We know that $(M, g)$ is isometric with $S^n, H^n, E^n$ or the product $R \times M_*$ with a warped product metric $ds^2 = du^2 + e^{2u} ds_*^2$. Now we have to test which metrics $\tilde{g} = \frac{1}{\psi^2} \cdot g$ are complete. By scaling we may reduce the problem to the cases where $\rho = 1, 0, -1$. Then as last, the solution of $\psi'' = -\rho \psi + B$ are:

$$\psi(u) = \begin{cases} \frac{a \sin u + b \cos u + c}{au^2 + bu + c} & \rho > 0 \\ \frac{a \sinh u + b \cosh u + c}{a \sinh u + b \cosh u + c} & \rho = 0 \\ \frac{1}{(a \sinh u + b \cosh u + c)^2} & \rho < 0 \end{cases}$$

where $a, b, c$ are constants. Now as $(M, g)$ is complete, then every geodesic tangent to $\frac{\partial}{\partial u}$ has infinite length in $g$. We compute

$$\tilde{g} \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = \frac{1}{\psi(u)} \left(\frac{1}{(au^2 + bu + c)^2}, \rho > 0 \right) = \frac{1}{(a \sinh u + b \cosh u + c)^2}, \rho < 0$$

which gives us that in any cases either $\int_{-\infty}^{0} \tilde{g}(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})^2 du$, or $\int_{0}^{\infty} \tilde{g}(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})^2 du$ is finite, indeed: to find $\int_{-\infty}^{\infty} \frac{1}{au^2 + bu + c} du$ we may factor the denominator of this
integrand as $au^2 + bu + c = (u - L_1)(u - L_2)$, where $L_1, L_2$ are the zeros of $au^2 + bu + c$. Thus, to the factor $(u - L_1)$ which corresponds partial fraction of the form $\frac{A}{(u - L_1)}$. Similarly, to the factor $(u - L_2)$ there corresponds $\frac{B}{(u - L_2)}$, therefore, the partial fraction decomposition has the form:

$$\frac{1}{au^2 + bu + c} = \frac{A}{(u - L_1)} + \frac{B}{(u - L_2)}$$  \hspace{1cm} (2.3.6)

We find $A = \frac{1}{L_1 - L_2}$ and $B = \frac{1}{L_2 - L_1}$. The partial fraction decomposition is, therefore

$$\frac{1}{au^2 + bu + c} = \frac{1}{(L_1 - L_2)(u - L_1)} - \frac{1}{(L_1 - L_2)(u - L_2)}$$

Integrating, we have

$$\int_0^\infty \frac{1}{au^2 + bu + c} \, du = \frac{1}{L_2 - L_1} \left\{ \lim_{t \to \infty} \ln \frac{|t - L_1|}{|t - L_2|} \right\} - \ln \frac{|L_1|}{|L_2|}$$

Hence, this integral will be finite.

Now for the other integral

$$\int \frac{1}{(a \sinh u + b \cosh u + c)^2} \, du = \frac{1}{(a + b)^2} \left( \frac{1}{\sqrt{B}} \tan^{-1} \frac{t}{\sqrt{B}} \right) + C$$

where $t = c^u + A$, $A, B$ and $C$ being constants. Thus this integral is also finite, which contradicts the completeness of $(M, \tilde{g})$.

Now it remains for the case $\rho > 0$ that $(M, g)$ is isometric with a standard sphere. This follows by theorem 2.3.5 and hence $(M, \bar{g})$ is isometric with a standard sphere. This completes the proof of this Theorem.
2.4 EXAMPLES OF GLOBAL SOLUTIONS

Example 2.4.1 spaces of constant sectional curvature:

(a) **Sphere** $S^n(1)$:

Let $I = (0, \pi)$ and $f_1 : I \to \mathbb{R}$ be $f_1(r) = \sin r$, consider the warped product $M = I \times_{f_1} S^{n-1}$, and define the map $F_1 : M \to (S^n, \text{can})$ defined by $F_1(r, z) = (\cos r, (\sin r)z)$ where $z \in S^{n-1}$. Then we will show that $F_1$ is a local isometry, indeed $F_1 : M \to S^n$ is a diffeomorphism. Also, for $(x_1, x_2, ..., x_n) \in S^n$ and $(z_1, z_2, ..., z_{n-1}) \in S^{n-1}$ we have:

$x_1 = \cos r$, $x_2 = (\sin r)z_1$, ..., $x_{n-1} = (\sin r)z_{n-1}$ and hence $dx_1 = -(\sin r)dr$, $dx_2 = (\sin r)dz_1 + z_1(\cos r)dr$, ..., $dx_n = (\sin r)dz_{n-1} + z_{n-1}(\cos r)dr$.

And so the metric on $S^n$ is given by $ds^2 = dx_1^2 + dx_2^2 + ... + dx_n^2$

$$ds^2 = (\sin^2 r)dr^2 + (\sin^2 r)dz_1^2 + z_1^2(\cos^2 r)dr^2 + 2z_1(\sin r)(\cos r)drdz_1 + ... + (\sin^2 r)dz_{n-1}^2 + z_{n-1}^2(\cos^2 r)dr^2 + 2z_{n-1}(\sin r)(\cos r)drdz_{n-1} \quad (2.4.1)$$

Since $(z_1, z_2, ..., z_{n-1}) \in S^{n-1} \Rightarrow z_1^2 + z_2^2 + ... + z_{n-1}^2 = 1$ so $2z_1dz_1 + 2z_2dz_2 + ... + 2z_{n-1}dz_{n-1} = 0$. Substituting last two equations in (2.4.1) we get

$$ds^2 = dr^2 + \sin^2 r (dz_1^2 + dz_2^2 + ... + dz_{n-1}^2) = dr^2 + (\sin^2 r)ds_1^2$$

where $ds_1^2$ is the line element of $(S^{n-1}, g_1)$. 
By doing the same arguments for $f_2 : (\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow R$, $f_2 (r) = \cos r$ and $M = (\frac{\pi}{2}, \frac{\pi}{2}) \times f_2 S^{n-1}$ where $F_2 : M \rightarrow (S^n, \text{can})$ is given by $F_1 (r, z) = (\sin r, (\cos r)z)$, $z \in S^{n-1}$. We will get: $ds^2 = dr^2 + (\cos^2 r)ds_1^2$, where $ds_1^2$ is the line element of $(S^{n-1}, g_1)$. Then the corresponding solutions of $\nabla^2 \psi = \lambda g$ are:

$$\psi (u) = - \cos u + c, \text{ for } \psi' (u) = \sin u, \text{ and } \psi (u) = \sin u + c, \text{ for } \psi' (u) = \cos u,$$

these solutions are globally defined, but for global conformal factors $g \rightarrow g = \psi^{-2}g$, we need that $\psi^{-2}$, so we must have $\psi \neq 0$. Now for $\psi (u) = - \cos u + c$,

we have $\psi (u) = 0 \iff \cos u = c$, but $|\cos u| \leq 1, \forall u$. Thus if $|c| > 1$ then $\psi (u) \neq 0$.

Similarly for $\psi (u) = \sin u + c$. So, we have a global conformal transformation into (not necessarily onto) the spaces $S^n(1) \rightarrow \overline{M^n (K)}$, where the $K$ is the sectional curvature of $\overline{M^n}$ given by proposition 2.1.2

$$K = \psi^2 K + \frac{2}{n} \psi \triangle \psi - ||\nabla \psi||^2$$

Now for, $\psi (u) = - \cos u + c$, with $K = 1$, $\triangledown \psi = \psi' \frac{\partial}{\partial u} = \sin u \frac{\partial}{\partial u}$, and so, $||\nabla \psi||^2 = \sin^2 u$,

$$\Delta \psi = n \psi'' = n (\cos u) = n (\cos u)$$

which implies

$$K = (- \cos u + c)^2 + \frac{2}{n} (- \cos u + c)(n (\cos u)) - \sin^2 u$$

$$= c^2 - (\sin^2 u + \cos^2 u) = c^2 - 1$$
where \(|c| > 1 \Rightarrow c^2 - 1 > 0\), and hence \(\overline{M}^n = S^n\). Similarly we have for 
\(\psi(u) = \sin u + c\).

Hence, we have the following global conformal transformation:

\[
S^n(1) \rightarrow S^n(c^2 - 1)
\]

(b) **Euclidean space** \(E^n\):

(i) We know that: \(E^n \cong E \times E^{n-1}\), by the isometry \(F(r, z) = (r, z)\) (thus by doing the same argument as in step (a)) we will get that

\[
ds^2 = du^2 + ds_0^2
\]

where \(ds_0^2\) is the line element of euclidean space \(E^{n-1}\). And so, the corresponding solution of \(\nabla^2 \psi = \lambda g\) is \(\psi(u) = u + c\), where \(c\) is an arbitrary constant. But

\[
\psi(u) = 0 \iff u = -c
\]

Hence, this solution can not be conformal factors \(g \rightarrow \overline{g} = \psi^{-2}g\) because \(c\) is an arbitrary constant.

(ii) let \(M = (0, \infty) \times_r S^{n-1}\), define a map \(F : M \rightarrow R^n \setminus \{0\}\) by

\[
F(u, z) = uz, \text{ then } F \text{ is an isometry and that}
\]

\[
ds^2 = du^2 + u^2 ds_1^2
\]
And the corresponding solution of \( \nabla^2 \psi = \lambda g \) is given by: \( \psi(u) = \frac{u^2}{2} + c \), as \( \psi'(u) = u \). But

\[
\psi(u) = 0 \iff u^2 = -2c
\]

So, for \( c > 0 \), this solution will be the conformal factor \( g \to \bar{g} = \psi^{-2}g \), \( E^n \to \overline{M^n(K)} \). This implies

\[
\overline{K} = \psi^2 K + \frac{2}{n} \psi \Delta \psi - \| \nabla \psi \|^2, \quad K = 0
\]

\[
= 0 + 2\left( \frac{u^2}{2} + c \right) - u^2 = 2c
\]

But \( c > 0 \), and thus \( \overline{K} > 0 \), and so \( \overline{M^n} = S^n \). Therefore, we will have the global conformal transformation \( E^n \to S^n(2c) \).

(c) **The Hyperbolic space** \( H^n(-1) \):

On the Hyperbolic space \( H^n(-1) \), we can express the metric in one of the following three ways and consequently we have three models for it:

(I) \( ds^2 = du^2 + (\sinh u)^2 ds_1^2 \).

(II) \( ds^2 = du^2 + e^{2u} ds_5^2 \).

(III) \( ds^2 = du^2 + (\cosh u)^2 ds_{-1}^2 \).

Note that on warped product \( M = B \times_f F \), with \( g = g_B + f^2 g_F \), we have:

(a) \( R(V, X)W = \frac{g(V, W)}{f} \nabla_X (\nabla f), \quad X \in \chi(B) \) and \( V, W \in \chi(F) \).

(b) \( R(V, W)U = R^F(V, W)U - \frac{\| \nabla f \|^2}{f^2} \{ g(U, W)V - g(U, V)W \} \)
Model (I): We have $M = (-\varepsilon, \varepsilon) \times_f S^{n-1}$ with $f^2 = (\sinh u)^2$, and $g = du^2 + f^2 ds^2_1$. We have $\nabla f = \cosh u \frac{\partial}{\partial u}$ and $\nabla \frac{\partial}{\partial u} \nabla f = \sinh u \frac{\partial}{\partial u}$. Consequently we get $R(V, \frac{\partial}{\partial u})V = (\frac{g(V,V)}{\sinh u}) \nabla \frac{\partial}{\partial u} \nabla f = g(V,V) \frac{\partial}{\partial u}$. Thus for $V, W \in \chi(F)$ with $g^F(V,V) = \frac{1}{(\sinh u)^2}$, $g^F(W,W) = \frac{1}{(\sinh u)^2}$ and $g^F(V,W) = 0$ we get

$$K(V, \frac{\partial}{\partial u}) = -g(V,V)g(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}) = -1$$

Next,

$$R(V,W)W = R^F(V,W)W - \frac{\cosh^2 u}{\sinh^2 u} \{g(W,W)V - g(W,V)W\}$$

which gives

$$K(V,W) = g(R^F(V,W)W, V) - \frac{\cosh^2 u}{\sinh^2 u} g(W,W)g(V,V)$$

$$= (\sinh u)^2 g^F(R^F(V,W)W, V) - \frac{\cosh^2 u}{\sinh^2 u}$$

$$= (\sinh u)^2 R^F(V,W,V) - \frac{\cosh^2 u}{\sinh^2 u}$$

$$= \frac{1}{(\sinh u)^2} - \frac{\cosh^2 u}{\sinh^2 u} = -\left(\frac{\cosh^2 u - 1}{\sinh^2 u}\right) = -1$$

Hence, $M = H^n(-1)$.

Model (II): We have $M = (-\varepsilon, \varepsilon) \times_f R^{n-1}$ with $f^2 = e^{2u}$ & $g = du^2 + f^2 ds^2_0$. Thus $\nabla f = e^u \frac{\partial}{\partial u}$ and $\nabla \frac{\partial}{\partial u} \nabla f = e^u \frac{\partial}{\partial u}$. Consequently for $V, W \in \chi(R^{n-1})$ with $euc(V,V) = euc(W,W) = \frac{1}{e^{2u}}$ and $g(V,W) = 0$ we get

$$R(V, \frac{\partial}{\partial u})V = (\frac{g(V,V)}{e^u}) \nabla \frac{\partial}{\partial u} \nabla f = g(V,V) \frac{\partial}{\partial u}$$

which implies

$$K(V, \frac{\partial}{\partial u}) = -1$$
Next
\[ R(V, W)W = 0 - \frac{e^{2u}}{e^{2u}} \{ g(W, W)V - g(W, V)W \} \]

which implies
\[ K(V, W) = -1 \]

Hence, \( M = H^n(-1) \).

**Model (III):** In this model we have \( M = (- \in, \in) \times f H^{n-1} \), with \( f^2 = (\cosh u)^2 \) and \( g = du^2 + f^2 ds_{-1}^2 \). In this case we have \( \nabla f = \sinh u \frac{\partial}{\partial u} \),
\[ \nabla f \partial f = \cosh u \frac{\partial}{\partial u} \text{ and } R(V, \frac{\partial}{\partial u})V = g(V, V) . \]

for \( V, W \in \chi(F) \) with \( g^F(V, V) = \frac{1}{(\cosh u)^2} \) and \( g^F(W, W) = \frac{1}{(\cosh u)^2} \),
\[ g^F(V, W) = 0 \] we get
\[ K(V, \frac{\partial}{\partial u}) = -1 \]

However,
\[ R(V, W)W = R^F(V, W)W - \frac{\sinh^2 u}{\cosh^2 u} \{ g(W, W)V - g(W, V)W \} \]

which implies
\[ K(V, W) = (\cosh u)^2 R^F(V, W, W, V) - \frac{\sinh^2 u}{\cosh^2 u} \]
\[ = \frac{1}{(\cosh u)^2} - \frac{\sinh^2 u}{\cosh^2 u} = -1 \]
\[ = - \frac{\cosh^2 u}{\cosh^2 u} = -1 \]

Hence \( M = H^n(-1) \). Thus we have three models for \( H^n(-1) \) as warped product.
Example 2.4.2 Einstein spaces which are not of constant sectional curvature:

(a) Let \((M, g_* )\) be a complete \(( n - 1 )\)-dimensional Einstein space \(( n \geq 5 )\) which is not of constant sectional curvature and whose scalar curvature is \( \rho_* = -1 \), then \(( R \times M, g )\) with

\[
ds^2 = du^2 + (\cosh u)^2 ds_*^2
\]

is a complete Einstein space and by lemma 2.2.3 (iv), it has a scalar curvature \( \rho = -\frac{\psi''}{\psi'} \). Now as \( \psi' = \cosh u \). Integrating we get \( \psi(u) = \sinh u + c \), for a constant \( c \in R \). And so, \( \psi''(u) = \sinh u \) and \( \psi'''(u) = \cosh u \), that is \( \rho = -\frac{\cosh u}{\cosh u} = -1 \). Also, as \( \psi(u) = 0 \iff \sinh u = -c \). But \( c \) is an arbitrary constant in \( R \), thus it is not globally defined conformal factor.

(b) Let \((M, g_* )\) be as above but with scalar curvature \( \rho_* = 0 \) (i.e.\((M, g_* )\) is Ricci-flat), then \(( R \times M, g )\) with

\[
ds^2 = du^2 + e^{2u} ds_*^2
\]

is a complete Einstein space of scalar curvature \( \rho = -\frac{\psi'''}{\psi'} \). Now \( \psi'(u) = e^u \), that is \( \psi(u) = e^u + c \), for some constant \( c \in R \). Thus \( \psi''(u) = e^u = \psi'''(u) \) and consequently \( \rho = -1 \). Also as \( \psi(u) = 0 \iff e^u = -c \), this means that if we choose \( c \geq 0 \), and as \( e^u > 0 \), then \( \psi(u) > 0 \). Hence, \( \psi(u) = e^u + c \), \( c \geq 0 \) is globally defined and \( \psi(u) \neq 0 \), thus, it induces a globally defined conformal transformation

\[
g \to \overline{g} = \psi^{-2} g
\]
where \((R \times M_s, \bar{g})\) is a non-complete Einstein space with
\[
ds^2 = (e^u + c)^2 du^2 + \left(\frac{e^u}{e^u + c}\right)^2 ds^2_\ast
\]
Also, by lemma 1.3.5 (iv) the scaler curvature \(\bar{p}\) of \(\bar{g}\) will be
\[
\bar{p} = (e^u + c)(-1) + \frac{2}{n}(e^u + c)(ne^u) - e^{2u} = -c^2
\]
In particular for \(c = 0\) we get
\[
ds^2 = e^{-2u}du^2 + ds^2_\ast
\]

**Example 2.4.3**

(a) let \((M, g)\) be \(M = R \times S^2\), and the metric \(g\) be
\[
ds^2 = du^2 + e^{2u}ds^2_1
\]
Thus \(\psi(u) = e^u + c, c \geq 0\). In a special case \(c = 0\) we get, \(\psi(u) = e^u\). Now we will compute the radial curvature (which is the sectional curvature of every plane containing \(\nabla \psi\)) given by the relation (see the proof of lemma 2.2.1)
\[
K = \frac{-g(\nabla \psi, \nabla \lambda)}{g(\nabla \psi, \nabla \psi)}
\]
where \(\psi(u) = e^u \Rightarrow \nabla \psi = \psi' \frac{\partial}{\partial u} = e^u \frac{\partial}{\partial u}\), and \(\frac{\Delta \psi}{n} = \psi'' = e^u, \nabla \lambda = \nabla \left(\frac{\Delta \psi}{n}\right) = (\psi'')' \frac{\partial}{\partial u} = e^u \frac{\partial}{\partial u}\).
So
\[
K = -\frac{e^{2u}}{e^{2u}} = -1 \tag{2.4.2}
\]
But the sectional curvature of the plane $\sigma$ spanned by the vectors $\{X, Y\}$ where $\{X, Y\}$ are orthogonal to $\frac{\partial}{\partial u}$ is given by

$$K_\sigma = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

$$= \frac{R_*(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} - \left(\frac{\psi''}{\psi}\right)^2 \left[ \frac{g(X, X)g(Y, Y) - g(X, Y)^2}{g(X, X)g(Y, Y) - g(X, Y)^2} \right]$$

$$= \frac{R_*(X, Y, Y, X)}{\left(\frac{\psi'}{\psi}\right)^4 \{g_*(X, X)g_*(Y, Y) - g_*(X, Y)^2\}} - \left(\frac{\psi''}{\psi} \right)^2 = \frac{1}{e^{4u}K_*} - 1$$

(as $\psi(u) = e^u \Rightarrow \psi''(u) = \psi'(u)$). But, $K_*$ is the sectional curvature of $S^2$, which is 1.

$$K_\sigma = e^{-4u} - 1 \quad (2.4.3)$$

which is non-constant.

Using equations (2.4.2) and (2.4.3) we get that the sectional curvature is not constant.

(b) let $(M_*, g_*)$ be an arbitrary complete $(n - 1)$-manifold, and $\psi (u)$ be a smooth function $\psi : R \rightarrow R$ satisfying:

(i) $\psi (u) > 0$ and $\psi' (u) > 0$, for all $u \in R$.

(ii) $\int_0^\infty \frac{du}{\psi(u)} = +\infty$.

(iii) $\lim_{u \rightarrow -\infty} \psi (u) = 0$.

Then, the metric

$$ds^2 = du^2 + (\psi'(u))^2 ds_*^2$$
is a complete metric on $M = R \times M_\ast$, and by lemma 2.2.2 $\psi$ satisfies $\nabla^2 \psi = \frac{\Delta \psi}{n}g$, thus it is induces a globally defined conformal transformation $(M, g) \to (M, \overline{g})$, where $\overline{g} = \psi^{-2}g$. Also, note that

$$\int_0^\infty \frac{du}{\psi(u)} = +\infty,$$

as $\lim_{u \to -\infty} \psi(u) = 0$, then for each $\varepsilon > 0$, there exists $H = H (\varepsilon)$ such that $|\psi(u)| < \varepsilon$ for $u > H$. Thus,

$$\int_{-\infty}^0 \frac{du}{\psi(u)} \geq \int_{-\infty}^0 \frac{du}{\varepsilon} = \int_{-\infty}^0 du = \lim_{t \to -\infty} \int_{-t}^0 du = \lim_{t \to -\infty} (t) = \infty$$

and, the geodesic tangent to $\frac{\partial}{\partial u}$ has infinite length, and so, $\overline{g}$ is complete.

**Remark 2.4.4** We can choose any arbitrary $(n-1)$- manifold $M_\ast$ (possibly non-complete), except that $M_\ast$ should be complete if we want $(M, g)$ and $(M, \overline{g})$ to be complete.

**Example 2.4.5** The differential equation on surfaces:

(a) A complete 2-manifold admitting a non-constant solution $\psi$ of $\nabla^2 \psi = Kg$:

We define the metric on $(R^2, g)$ by

$$ds^2 = du^2 + 4(\tanh^2 u)dv^2$$

in polar coordinates $(u, v)$. Then $g$ is complete, and if $\psi' (u) = 2 \tanh u$, then

$$\psi(u) = \int \psi' (u) du = 2 \int (\tanh u) du = 2 \int \frac{\sinh u}{\cosh u} du = 2 \int \frac{dt}{t}$$
where \( t = \cosh u \), \( \Rightarrow \psi(u) = 2 \ln |\cosh u| + c \), where \( c \in R \). Choose \( c = 0 \) \( \Rightarrow \psi(u) = 2 \ln |\cosh u| \). But

\[
\ln |\cosh u| = 0 \iff |\cosh u| = 1 \iff u = 0
\]

Thus \( \psi(u) = 2 \ln |\cosh u| \) induce a conformal transformation (not globally defined) \((M, g) \rightarrow (M, \overline{g})\), where \( \overline{g} = \psi^{-2} g \) and it satisfies:

\[
\nabla^2 \psi = \psi'' g
\]

given that \( \psi''(u) = 2 \sech^2 u = \frac{2}{\cosh^2 u} \).

The Gaussian curvature \( K(= \rho) \) is given by

\[
K(u) = -\frac{\psi'''(u)}{\psi'(u)} = \frac{4(\sech^2 u)(\tanh u)}{2 \tanh u} = 2 \sech^2 u = \psi''(u)
\]

that is \( -\frac{\psi'''(u)}{\psi'(u)} = \psi''(u) \), or \( \psi'(u) \psi''(u) + \psi'''(u) = 0 \). Integrating this equation we get

\[
(\psi'(u))^2 + \psi''(u) = \text{constant}
\]

(b) A complete 2-manifold admitting a non constant solution of \( \nabla^2 \psi = K \psi g \):

In the metric \( ds^2 = du^2 + (\psi'(u))^2 dv^2 \), the equation \( \nabla^2 \psi = K \psi g \) becomes

\[
\psi'' = K \psi = -\frac{\psi'''(u)}{\psi'(u)} \psi, \text{ or } \psi'(u) \psi''(u) = -\psi'''(u) \psi(u) .
\]

Using this we have

\[
\frac{d}{du} \left( \psi(u) \psi''(u) \right) = \psi'(u) \psi''(u) + \psi(u) \psi'''(u)
\]

\[
= -\psi'''(u) \psi(u) + \psi'''(u) \psi(u) = 0
\]
Thus $\psi (u) \psi'' (u) = \text{constant}.$

Also for the equation $\nabla^2 \psi = -K \psi$ we have

$$\psi'' = -K \psi \quad (2.4.4)$$

That is $\psi' (u) \psi'' (u) = \psi''' (u) \psi (u),$ or $\psi' (u) \psi'' (u) - \psi''' (u) \psi (u) = 0.$ Using this we have

$$\frac{d}{du} \left( \frac{\psi (u)}{\psi'' (u)} \right) = \left( \frac{\psi' (u) \psi'' (u) - \psi''' (u) \psi (u)}{\left( \psi'' (u) \right)^2} \right) = 0$$

$$\Rightarrow \frac{\psi (u)}{\psi'' (u)} = \text{constant} \quad (2.4.5)$$

and by (2.4.4) and (2.4.5) we get that $K$ is constant.

### 2.5 Conformal Diffeomorphisms Preserving Ricci Tensor

In this final section, we are interested in classifying complete Riemannian manifolds $(M, g),$ which admit a non-constant function $\psi \in C^\infty(M),$ with $\overline{g} = \psi^{-2} g$ such that the difference $\operatorname{Ric}_{\overline{g}} - \operatorname{Ric}_g$ of Ricci tensors is a constant multiple of the metric $g$ or $\overline{g}.$ It turns out that such Riemannian manifolds are either standard spaces of constant curvature or is a warped product $R \times_{\exp} M_s$ of the line and a Ricci-flat manifold $M_s.$

We shall use the notation $[g]$ to denote the equivalence class of all symmetric tensors which are point wise scalar multiples of a given metric $g.$
First we prove that the problem that \( Ric_{\tilde{g}} - Ric_g \) is a constant multiple of \( g \) or \( \tilde{g} \) is equivalent to finding a non-constant solutions of (*)

**Lemma 2.5.1** Two conformally equivalent metrics \( g \) and \( \tilde{g} = \frac{1}{\phi^2}g \) satisfies the relation

\[
[Ric_{\tilde{g}} - Ric_g] = [g] = [\tilde{g}]
\]

if and only if the function \( \phi \) satisfies the equation

\[
\nabla^2 \phi = \frac{\Delta \phi}{n}g
\]

**Proof.** This follows from formula

\[
[Ric_{\tilde{g}} - Ric_g] = \frac{1}{\phi^2}[(n - 2)\phi \nabla^2 \phi + \{\phi \Delta \phi - (n - 1)\|\nabla \phi\|^2\}g]
\]

given in lemma 1.3.5.

The lemma 2.5.1 holds only under the assumption \( n \geq 3 \).

**Lemma 2.5.2** A function \( \phi : M \to R \) satisfies \( \nabla^2 \phi = \lambda g \) for some \( \lambda : M \to R \) in a neighborhood of a point with \( g(\nabla \phi, \nabla \phi) \neq 0 \), if and only if \( g \) is locally a warped product metric \( ds^2 = dt^2 + \phi^2(t)ds^2_\ast \), where \( \phi, \lambda \) are functions dependent on \( t \) only satisfying \( \phi'' = n\lambda \), and \( ds^2_\ast \) is independent of \( t \).

**Proof.** This lemma was discussed in last sections.
Lemma 2.5.3 Let \((M, g)\) be a complete indefinite Riemannian manifold admitting a globally defined non-constant solution \(\phi\) of

\[
\nabla^2 \phi = \frac{\Delta \phi}{n} g
\]

then \(\phi\) has a zero.

Proof Along a null geodesic \(\gamma(s)\) one calculates

\[
\frac{d^2}{ds^2}(\phi(\gamma(s))) = \frac{d}{ds}(\gamma(\phi)) = \frac{d}{ds} \langle \dot{\gamma}, \nabla \phi \rangle = \dot{\gamma}(\dot{\gamma}, \nabla \phi) = \nabla_{\dot{\gamma}}(\nabla \phi, \dot{\gamma}) = (\frac{\Delta \phi}{n} \dot{\gamma}, \dot{\gamma}) = 0
\]

Therefore \(\phi(\gamma(s))\) is linear in \(s\) and \(\frac{d}{ds}(\phi(\gamma(s))) = g(\nabla \phi, \dot{\gamma})\). If we choose \(\gamma\) such that \(g(\nabla \phi, \gamma) \neq 0\) at a point \(p\), then it follows that \(\phi\) has a zero along \(\gamma\).

Theorem 2.5.4 Let \((M, g)\) be complete and admitting a global conformal transformation \(\bar{g} = \frac{1}{\phi^2} g\) satisfying

\[
\text{Ric}_{\bar{g}} - \text{Ric}_g = c(n - 1)g
\]

for some constant \(c\). Then one of the following three cases occurs:

1. \(\phi\) is constant.

2. \((M, g)\) and \((M, \bar{g})\) are simply connected Riemannian spaces of constant sectional curvature.

3. \((M, g)\) is a warped product \(R \times e^t M_s\) where \(\phi(t) = e^t\) and \((M_s, g_s)\) is a complete Ricci-flat \((n - 1)\)-dimensional Riemannian manifold.
**Proof.** By assumption $\phi : M \to R$ is a function which is positive everywhere. By lemma 2.5.3 we may assume that $\phi$ is non-constant function on a manifold with positive definite metric $g$. Let $p \in M$ be a point with $\nabla \phi(p) \neq 0$, by lemma 2.5.1 the equation:

$$c(n-1)g = \text{Ric}_g - \text{Ric}_g = \frac{1}{\phi^2}[(n-2)\phi \nabla^2 \phi$$

$$+(\phi \triangle \phi - (n-1)\|\nabla \phi\|^2)g]$$

(where the last equality comes from 1.3.5), Implies

$$c(n-1)\phi^2 = (n-2)\phi \Delta \phi + n\phi \Delta \phi - n(n-1)\|\nabla \phi\|^2$$

$$\Rightarrow c(n-1)\phi^2 = \frac{2(n-1)}{n} \phi \Delta \phi - (n-1)\|\nabla \phi\|^2 \quad (2.51)$$

And hence,

$$\frac{2}{n} \phi \Delta \phi - \|\nabla \phi\|^2 - c\phi^2 = 0$$

By lemma 2.5.2 along the unit speed geodesic $\gamma(t)$ in the direction of $\nabla \phi = \phi' \frac{\partial}{\partial t}$ we have:

$$2\phi \phi'' - \left(\phi'\right)^2 - c\phi^2 = 0 \quad (2.5.2)$$

Differentiating once more we get:

$$2\phi \phi'' + 2\phi \phi''' - 2\phi' \phi'' - 2c\phi \phi' = 0$$

And so,

$$\phi \phi'' - c\phi \phi' = 0 \quad (2.5.3)$$

or, (since $\phi \neq 0$)

$$\phi''' = c\phi' \quad (2.5.4)$$
Therefore, there is a constant $a$ satisfying
\[ \phi'' = c\phi + a \] (2.5.5)

But $\phi'(p) \neq 0$, implies equivalently
\[ \left( \left( \phi' \right)^2 \right)' = c \cdot (\phi^2)' + 2a\phi' \] (2.5.6)

This means that there is a constant $b$ satisfying
\[ \left( \phi' \right)^2 = c\phi^2 + 2a\phi + b \] (2.5.7)

Now, by substituting (2.5.5), (2.5.7) in (2.5.2) we get $b = 0$, thus
\[ \left( \phi' \right)^2 = \phi(c\phi + 2a) \] (2.5.8)

**Case (i):** \( c = 0 \)

Using equation (2.5.5), we find that $\phi$ is a polynomial of degree at most 2. Then in this case the equation (2.5.8) reduces to \( \left( \phi' \right)^2 = 2a\phi \), which implies that $\phi$ is quadratic and that $\phi$ and $\phi'$ have a common zero along $\gamma$. By the completeness of $g$ the metric $\overline{g} = \frac{1}{\phi^2}g$ has a singularity there, which is a contradiction. Alternatively, if $\phi(t) = At^2 + Bt + C$, then (2.5.8) implies that the discriminant $4AC - B^2$ is zero. Therefore $\phi(t)$ is the square of a linear function.

**Case (ii):** \( c < 0 \)

In this case every solution $\phi$ of (2.5.5) is periodic and therefore attains its maximum and minimum. At each of these points the equation $\phi(c\phi + 2a) = 0$ is
is satisfied by (2.5.8). Hence, $\phi = 0$ and $c\phi + 2a = 0$ must be satisfied at the minimum and maximum respectively, which for the case $\phi = 0$ leads to a contradiction, and in the case $c\phi + 2a = 0$, we get that $\phi$ is a constant, which also gives a contradiction.

Alternatively, by equation (2.5.5) we have $\phi'' = c\phi + a$. To solve this equation, let $\phi = y$, and so

$$y'' - cy = a \quad (2.5.9)$$

The corresponding homogenous differential equation to this is $y'' - cy = 0$, which has the auxiliary equation $m^2 - c = 0$, and so, $m = \pm \sqrt{c}$. And, thus the complementary function $y_c$ is $y_c = \alpha\cos(\sqrt{-ct}) + \beta\sin(\sqrt{-ct})$, where $\alpha, \beta$ are constants. To get the particular solution $y_p$ of (2.5.5) we will use method of undetermined coefficients to arrive at $y_p = c_1$, where $c_1$ is a constant. Thus the general solution is

$$y = \alpha\cos\sqrt{-ct} + \beta\sin\sqrt{-ct} + c_1$$

But as $y_p$ is also a particular solution of (2.5.9), we get $-cc_1 = a$, or $c_1 = \frac{-a}{c}$. And hence, the general solution of (2.5.5) is

$$\phi(t) = \alpha\cos\sqrt{-ct} + \beta\sin\sqrt{-ct} - \frac{a}{c}$$

Consequently $\phi'(t) = -\sqrt{-c}\alpha\sin\sqrt{-ct} + \sqrt{-c}\beta\cos\sqrt{-ct}$. Now substitute the value of $\phi$ and $\phi'$ in (2.5.8) to get $(\alpha^2 + \beta^2)(\cos^2\sqrt{-ct} + \sin^2\sqrt{-ct}) = \frac{a^2}{c^2}$, or $\alpha^2 + \beta^2 = \frac{a^2}{c^2}$. 
Case \((iii) : c > 0,\)

In this case as previous case we will arrive at
\[
\phi(t) = \alpha \cosh(\sqrt{c}t) + \beta \sinh(\sqrt{c}t) - \frac{a}{c},
\]
where \(\alpha^2 - \beta^2 = \frac{a^2}{c^2}.\)

In particular \(\alpha^2 \geq \beta^2.\) If \(\alpha^2 > \beta^2,\) then \(\phi\) has a critical point along \(\gamma\) (as \(\phi' = \sqrt{\phi(c\phi + 2a)}\), so \(\phi = -\frac{2a}{c}\) will be critical point). This implies that \(\gamma\) has a point \(q\) with \(\nabla\phi \big|_q = 0\) (note that \(\nabla \phi = \phi' \frac{\partial}{\partial t}\)). Furthermore, \(\phi\) satisfies globally \(\nabla^2 \phi = \phi''g\), which by (2.5.5) we will give \(\nabla^2 \phi = (c\phi + a)g\), with \(c > 0.\) Then a result of Tashiro[11] implies that \((M, g)\) is isometric with the hyperbolic space of constant sectional curvature \(-c.\)

If \(\alpha^2 = \beta^2,\) then \(a = 0,\) and consequently, \(\phi(t) = \alpha e^{\pm \sqrt{c}t}\) is a solution without a critical point along \(\gamma.\) This implies that
\[
d s^2 = d t^2 + e^{2\sqrt{c}t} d s_*^2
\]
is a complete metric on \(M = R \times M_*.\) It follows that \(\bar{g} = e^{-2\sqrt{c}t}g\) is the product metric \(d t^2 + d s_*^2\) on \((0, \infty) \times M_*.\) Now, for \(U, V \in \chi(M_*)\) we have
\[
\text{Ric}(U, V) = f^2 \text{Ric}_{g^*}(U, V) - (g(U, V) f^2)
\]
where \(f^2 = \frac{\Delta f}{f^2} + (d - 1) \frac{\|\nabla f\|^2}{f^2},\) \(f = e^{\sqrt{c}t}\) and \(d = \text{dim} M_* = n - 1.\) Thus \(\nabla f = f' = \sqrt{c}e^{\sqrt{c}t} = \sqrt{c}f\), and \(\Delta f = f'' = cf.\) Therefore \(f^2 = c + (n - 2)c = (n - 1)c\) and
\[
\text{Ric}(U, V) = f^2 \text{Ric}_{g^*}(U, V) - (n - 1)cg(U, V)
\]
Now, as \(d s^2 = d t^2 + d s_*^2,\) we find
\[
\text{Ric}_g(U, V) = \text{Ric}_{g^*}(U, V)
\] (2.5.11)
Consequently
\[ c(n - 1)g(U, V) = [\text{Ric}_{\overline{g}} - \text{Ric}_g](U, V) \]
\[ = \text{Ric}_g(U, V) - f^2 \text{Ric}_g(U, V) + (n - 1)c g(U, V) \]
\[ = (1 - f^2) \text{Ric}_g(U, V) + c(n - 1)g(U, V) \]
which is impossible unless
\[ \text{Ric}_g = 0 \] (2.5.12)
and this completes the proof of theorem 2.5.4.

**Remark 2.5.5** In theorem 2.5.4, case (i) corresponds to \( c = 0 \), and cases (ii) and (iii) correspond to \( c > 0 \). In case (ii), \( \overline{g} \) must be flat and \( g \) must be hyperbolic, and in case (iii) \( \text{Ric}_\overline{g} = 0 \). A non-constant \( \phi \) occurs only for Einstein space.

Indeed, if \( \phi \) is constant, then by using the relation
\[ \text{Ric}_{\overline{g}} - \text{Ric}_g = \frac{1}{\phi^2} [(n - 2)\phi \nabla^2 \phi + (\phi \Delta \phi - (n - 1)\|\nabla \phi\|^2)g] = 0 \]
As \( \nabla \phi = \phi \cdot \frac{\partial}{\partial u} = 0 \), and \( \Delta \phi \) is the trace of \( \nabla^2 \phi = 0 \), we have \( 0 = \text{Ric}_g - \text{Ric}_\overline{g} = c(n - 1)g \), by definiteness of \( g \) we get \( c = 0 \).

Now, if \( \phi \) is non-constant, and case (ii) occurs, then \( \text{Ric}_g - \text{Ric}_\overline{g} = c(n - 1)g \), reads as \( \phi^{-2}\overline{k}(n - 1)g - k(n - 1)g = c(n - 1)g \), or \( \overline{k} = \phi^2(k + c) \). But as \( \phi \) is non-constant, and \( \overline{k}, k \) and \( c \) are constants \( \Rightarrow \phi^2 = 0 \), so \( \phi = 0 \). Thus \( \overline{k} = 0 \) and then \( \overline{g} \) is flat, and as \( c > 0 \) (this is because we already proved in
the theorem that $c$ cannot be less than zero) then $k = -c$, will be negative and hence $g$ is hyperbolic. In case $(iii)$, using (2.5.11) and (2.5.12), we get $Ric_{g^*} = Ric_g = 0$. Thus, a non-constant $\phi$ occurs only for Einstein space.

**Theorem 2.5.6** Let $(M, g)$ be complete and assume that both $g$ and $\bar{g} = \frac{1}{\phi^2}g$ are Einstein metrics, then the same conclusion as in theorem 2.5.4 holds, that is one of the case $(i)$, $(ii)$, $(iii)$ occurs.

**Proof.** The proof of this theorem follows on similar way as theorem 2.3.7.

**Theorem 2.5.7** Let $(M, g)$ be complete and admitting a global conformal transformation $\bar{g} = \frac{1}{\phi^2}g$ satisfying

$$Ric_{\bar{g}} - Ric_g = c(n - 1)\bar{g} = \frac{c(n - 1)}{\phi^2}g$$

for a constant $c$. Then either $\phi$ is a constant or $(M, g)$ is isometric to the euclidean space.

**Proof.** This follows the pattern of the proof of theorem 2.5.4, In particular $g$ must be positive definite if $\phi$ is non-constant. We start with the equation:

$$\frac{c \cdot (n - 1)}{\phi^2}g = Ric_{\bar{g}} - Ric_g = \frac{1}{\phi^2}[(n - 2)\phi \nabla^2 \phi + (\phi \Delta \phi - (n - 1) \|\nabla \phi\|^2)g]$$

which implies

$$\frac{2(n - 1)}{n} \phi \Delta \phi - (n - 1) \|\nabla \phi\|^2 - c(n - 1) = 0$$

(2.5.13)
If $\nabla \phi \neq 0$ at $p$, then along the geodesic $\gamma$ in direction $\nabla \phi$ we have

$$2(n-1)\phi'' - (n-1)\left(\phi'\right)^2 - c(n-1) = 0$$

or

$$2\phi'' - \left(\phi'\right)^2 - c = 0 \quad (2.5.14)$$

which implies by differentiating

$$2\phi'' = 0 \quad (2.5.15)$$

Since $\phi \neq 0$, we have

$$\phi(t) = At^2 + Bt + C \quad (2.5.16)$$

If we put this into (2.5.14) we get

$$4AC = B^2 = c \quad (2.5.17)$$

The case $c = 0$ leads to a zero of $\phi$ as in the proof of theorem 2.5.4; the case $c < 0$ leads to two zeros of $\phi$, a contradiction. If $c > 0$, then $\phi$ has no zero but it has a critical point along $\gamma$. This is a critical point for $\phi$ on $M$, and $\phi$ satisfies the equation $\nabla^2 \phi = 2Ag$. By theorem of Tashiro [11], this implies that $(M,g)$ is isometric with the euclidean space. In particular, if $\phi$ non-constant, then $c$ must be positive and $\tilde{g}$ is a space of constant sectional curvature $c$.

**Corollary 2.5.8** A globally defined transformation $g \to \tilde{g}$ s.t $\text{Ric}_g - \text{Ric}_\tilde{g} = 0$ of a complete Riemannian manifold is a homothety.

This result is just the case $c = 0$ in theorems 2.5.4 and 2.5.7.
Theorem 2.5.9 Let \((M, g)\) be complete and admitting a globally defined concircular transformation \(\overline{g} = \frac{1}{\phi^2} g\). Assume that \(S, \overline{S}\) are constants. Then one of three cases \((i), (ii), (iii)\) as in theorem 2.5.4 occurs.

Proof. The equation

\[
\text{Ric}_{\overline{g}} - \text{Ric}_g = \frac{1}{n} \left( \frac{\overline{S}}{\phi^2} - S \right) g
\]

implies

\[
\frac{1}{\phi^2} [ (n - 2) \phi \nabla^2 \phi + \left( \phi \Delta \phi - (n - 1) \| \nabla \phi \| ^2 \right) g] = \frac{1}{n} \left( \frac{\overline{S}}{\phi^2} - S \right) g
\]

that is,

\[
(n - 2) \phi \left( \frac{\Delta \phi}{n} \right) g + \left( \phi \Delta \phi - (n - 1) \| \nabla \phi \| ^2 \right) g = \frac{1}{n} \left( \frac{\overline{S}}{\phi^2} - S \right) g
\]

which gives

\[
2 \phi \left( \frac{\Delta \phi}{n} \right) - \| \nabla \phi \|^2 + \frac{S \phi^2}{n(n - 1)} - \frac{\overline{S}}{n(n - 1)} g = 0
\]

and finally,

\[
2 \phi \phi'' - \left( \phi' \right)^2 + \frac{S}{n(n - 1)} \phi^2 - \frac{\overline{S}}{n(n - 1)} = 0
\]

Along a unit speed geodesic in direction grad \(\phi\). Differentiating once more leads to

\[
2 \phi \phi''' + \frac{2S}{n(n - 1)} \phi \phi' = 0
\]
or

\[
\phi''' + \rho \phi' = 0
\]

(2.5.20)
where \( \rho = \frac{S}{n(n-1)} \) denotes the scalar curvature. As in the proof of theorem 2.5.4 we conclude that

\[
\left( \phi' \right)^2 = -\rho \phi^2 + 2a\phi - \bar{\rho}
\]

for a constant \( a \).

In any case the solution \( \phi \) of (2.5.20) and (2.5.21) either has a zero (which is impossible because \( \phi \) is conformal factor) or a critical point, except for solutions of the type

\[
\phi'(t) = \alpha e^{\sqrt{-\rho}t}
\]

(2.5.22)

Leading to the same warped product metric as in (2.5.10). If there is a critical point, then the levels around it, are round spheres and thus \( (M, g) \) is a standard space of constant sectional curvature. (see lemma 2.2.3 and lemma 2.2.8).
CHAPTER 3

SCHWARZIAN DERIVATIVE

For a smooth function $\phi : M \to \mathbb{R}$ on a Riemannian manifold $(M, g)$ we define a tensor

$$B_g(\varphi) = H_\varphi - d\varphi \otimes d\varphi - \frac{1}{n} \{\Delta \varphi - \|\text{grad} \varphi\|^2\} g$$

where $H_\varphi$ is the Hessian of $\varphi$, $n = \dim M$ and $\Delta$ is the Laplacian operator, in this chapter we will study the properties of this tensor called the Schwarzian tensor and will study the solution of the differential equation $B_g(\varphi) = p$, where $p : \chi(M) \times \chi(M) \to \mathcal{C}^\infty(M)$ is a field of symmetric $(0, 2)$ tensors of trace 0. The substition $u = e^{-\varphi}$ change the equation $B_g(\varphi) = 0$ into the linear equation $H_u = \frac{1}{n}(\Delta u) g$. We let $U(M)$ denote the space of solutions of this equation, then it turns out that we can use the space $U(M)$ to realize the metric $g$ as a warped product.

3.1 ELEMENTARY PROPERTIES OF SCHWARZIAN TENSOR

In this section we discuss some elementary properties of the Schwarzian tensor. We also introduce Möbius transformations between manifolds and the Möbius group of a manifold, and make a preliminary study of integrability of the equation $B(\phi) = 0$ in relation to the local geometry of the manifold.
We start with the following lemma:

**Lemma 3.1.1** Let $\phi, \sigma : M \to R$ be smooth functions on $(M, g)$. Then

$$B_{\tilde{g}}(\phi + \sigma) = B_{\tilde{g}}(\phi) + B_{\tilde{g}}(\sigma)$$

where $\tilde{g} = e^{2\phi}g$.

**Proof.** The relation between covariant derivatives with respect to Riemannian connections for $g$ and $\tilde{g}$ is given by

$$\hat{\nabla}_X Y = \nabla_X Y + X(\phi)Y + Y(\phi)X - g(X, Y)\nabla \phi$$

where $\nabla$ is the gradient with respect to $g$. We have

$$\text{Hess}_{\tilde{g}}(\sigma)(X, Y) = XY(\sigma) - \hat{\nabla}_X Y(\sigma)$$

$$= XY(\sigma) - \nabla_X Y(\sigma) - X(\phi)Y(\sigma)$$

$$- Y(\phi)X(\sigma) + g(X, Y)\nabla \phi(\sigma)$$

$$= \text{Hess}_g(\sigma)(X, Y) - X(\phi)Y(\sigma) - Y(\phi)X(\sigma) + g(X, Y)g(\nabla \phi, \nabla \sigma)$$

(3.1.1)

Let $\hat{\nabla}, \hat{\Delta}$ be the gradient and the Laplacian with respect to $\tilde{g}$, then for

a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ on $(M, \tilde{g})$, $\{e^\phi e_1, e^\phi e_2, \ldots, e^\phi e_n\}$

is the local orthonormal frame on $(M, g)$. Using (3.1.1) we get
\[ \hat{\Delta} (\sigma) = \sum_{i=1}^{n} \text{Hess}_g(\sigma)(e_i, e_i) \]
\[ = \sum_{i=1}^{n} \{ \text{Hess}_g(\sigma)(e_i, e_i) - e_i(\phi) e_i(\sigma) - e_i(\phi) e_i(\sigma) + g(e_i, e_i) g(\nabla, \nabla) \} \]
\[ = \sum_{i=1}^{n} \{ \text{Hess}_g(\sigma)(e_i, e_i) \} \]
\[ - 2\hat{g}(\hat{\nabla} \phi, \hat{\nabla} \sigma) + n e^{-2\phi} g(\nabla \phi, \nabla \sigma) \]

Note that: \( X(\phi) = \hat{g}(\hat{\nabla} \phi, X) = g(\nabla \phi, X) \), that is \( e^{2\phi} g(\nabla \phi, X) = g(\nabla \phi, X) \), and consequently \( e^{2\phi}(\hat{\nabla} \phi) = \nabla \phi \). Thus, \( \hat{\nabla} \phi = e^{-2\phi} \nabla \phi \). Similarly for \( \sigma \), we have \( \hat{\nabla} \sigma = e^{-2\phi} \nabla \sigma \). Therefore

\[ \hat{\Delta} \sigma = e^{-2\phi} \Delta \sigma - 2e^{-2\phi} g(\nabla \phi, \nabla \sigma) + n e^{-2\phi} g(\nabla \phi, \nabla \sigma) \]
\[ = e^{-2\phi} \{ \Delta \sigma + (n - 2)g(\nabla \phi, \nabla \sigma) \} \tag{3.1.2} \]

which implies

\[ B_{\hat{g}}(\sigma)(X, Y) = \text{Hess}_{\hat{g}}(\sigma)(X, Y) - X(\sigma) Y(\sigma) \]
\[ = \frac{1}{n} \{ \hat{\Delta} \sigma - \left\| \nabla \sigma \right\|_{\hat{g}}^2 \} \hat{g}(X, Y) \]
\[ = \text{Hess}_g(\sigma)(X, Y) - X(\phi) Y(\sigma) - Y(\phi) X(\sigma) \]
\[ + g(X, Y) g(\nabla \phi, \nabla \sigma) - X(\sigma) Y(\phi) \]
\[ - \frac{e^{-2\phi}}{n} \{ \Delta \sigma + (n - 2)g(\nabla \phi, \nabla \sigma) - \left\| \nabla \sigma \right\|^2 \} e^{2\phi} g(X, Y) \]

or,

\[ B_{\hat{g}}(\sigma)(X, Y) = B_{g}(\sigma)(X, Y) - X(\phi) Y(\sigma) - Y(\phi) X(\sigma) + \frac{2}{n} g(\nabla \phi, \nabla \sigma) g(X, Y) \]
that is

\[ B_g(\sigma) = B_g(\sigma) - d\phi \otimes d\sigma - d\sigma \otimes d\phi + \frac{2}{n} g(\nabla\phi, \nabla\sigma)g \quad (3.1.3) \]

Also we have

\[
Hess_g(\phi + \sigma) - d(\phi + \sigma) \otimes d(\phi + \sigma) = Hess_g(\phi) + Hess_g(\sigma) - d\phi \otimes d\phi \\
- d\phi \otimes d\sigma - d\sigma \otimes d\phi \\
- d\sigma \otimes d\sigma \quad (3.1.4)
\]

Using (3.1.3) & (3.1.4) we have

\[
B_g(\phi + \sigma) = Hess_g(\phi + \sigma) - d(\phi + \sigma) \otimes d(\phi + \sigma) \\
- \frac{1}{n}\{\Delta(\phi + \sigma) - \|\nabla(\phi + \sigma)\|^2\}g \\
= Hess_g(\phi) - d\phi \otimes d\phi - \frac{1}{n}\{\Delta\phi - \|\nabla\phi\|^2\}g \\
+ Hess_g(\sigma) - d\sigma \otimes d\sigma - \frac{1}{n}\{\Delta\sigma - \|\nabla\sigma\|^2\}g \\
- d\phi \otimes d\sigma - d\sigma \otimes d\phi + \frac{2}{n} g(\nabla\phi, \nabla\sigma)g \\
= B_g(\phi) + B_g(\sigma) - d\phi \otimes d\sigma - d\sigma \otimes d\phi \\
+ \frac{2}{n} g(\nabla\phi, \nabla\sigma)g \\
= B_g(\phi) + B_g(\sigma)
\]

**Definition 3.1.2** The Schwarzian \( \vartheta_g(f) \) of a conformal diffeomorphism \( f : (M, g) \rightarrow (M', g') \) is defined as \( \vartheta_g(f) = B_g(\log \|df\|) \), where \( B_g \) is the Schwarzian tensor on \( (M, g) \). A conformal diffeomorphism \( f \) is said to be Möbius transformation if \( \vartheta_g(f) = 0 \). A conformal metric \( \tilde{g} = e^{2\phi} g \) is said to
be a Möbius metric with respect to $g$ if it satisfies the Möbius equation

$$B_g (\phi) = 0$$

**Example 3.1.3** Let $f : (M, g) \to (M', \tilde{g})$ be a homothety, that is, $f$ is diffeomorphism and $f^* \tilde{g} = e^{2\phi} g$ with $\phi$ a constant, then by the definition of $B_g (\phi)$ we get $0 = B_g (\phi) = \vartheta_g (f)$. Thus, any homothety is a Möbius transformation.

**Remark 3.1.4** (i) The Möbius transformations of a manifold to itself from a group, we denote this group by $M\tilde{o}b(M)$. It contains the group of homotheties of $M$ and it is contained in the group of all conformal mappings of $M$.

(ii) Let $h : (M, g) \to (M', g')$ and $f : (M', g') \to (M'', g'')$ be conformal diffeomorphisms with $h^* g' = e^{2\phi} g$ and $f^* g'' = e^{2\sigma} g'$, where, $\phi : M \to R$ and $\sigma : M' \to R$. Then $(f \circ h)^* g'' = e^{2(\phi + \sigma \circ h)} g$, and by lemma 3.1.1, we have the following, for smooth functions $\phi, \sigma \circ h : M \to R$ on $(M, g)$ with $\tilde{g} = e^{2\phi} g$

$$B_g(\phi + \sigma \circ h) = B_g(\phi) + B_g(\sigma \circ h)$$

But as $\tilde{g} = h^* g'$, we have

$$\vartheta_g(f \circ h) = \vartheta_g(h) + h^* \vartheta_{g'}(f) \quad (3.1.5)$$
(iii) From equation (3.1.5), it follows that for any conformal transformation \( h \) the equation \( \vartheta(f \circ h) = \vartheta(h) \) holds if and only if \( f \) is Möbius.

(iv) Let \( id : (M, g) \to (M, g) \) be the identity map. Then as \( id \) is a homothety map we have \( \vartheta_g(id) = 0 \). Thus using (3.1.5) we get

\[
0 = \vartheta(id) = \vartheta(f \circ f^{-1}) = \vartheta(f^{-1}) + (f^{-1})^* \vartheta(f)
\]

Hence,

\[
\vartheta(f^{-1}) = - (f^{-1})^* \vartheta(f)
\]

(3.1.6)

for any conformal diffeomorphism \( f \).

(v) Let \( f : (M, g) \to (M, g) \) be a conformal diffeomorphism and let \( \tilde{g} = e^{2\phi} g \) (here \( \tilde{g} \) is not necessarily \( f^* g \)), and consider the composition of conformal maps

\[
(M, g) \xrightarrow{id} (M, \tilde{g}) \xrightarrow{f} (M, \tilde{g}) \xrightarrow{id^{-1}} (M, g)
\]

Then

\[
\vartheta_g(f) = \vartheta_g(id^{-1} \circ f \circ id) = \vartheta_g(id) + (id)^* (\vartheta_{\tilde{g}}(f) + f^* \vartheta_{\tilde{g}}(id^{-1}))
\]

however as \( \vartheta_g(id) = 0 \) and \( \vartheta_{\tilde{g}}(id^{-1}) = - (id^{-1})^* \vartheta_g(id) = 0 \), we get

\[
\vartheta_g(f) = \vartheta_{\tilde{g}}(f)
\]

We summarize these remarks as the following theorem:
Theorem 3.1.5 Composites and inverses of Möbius transformation are Möbius, and the Möbius transformations of Riemannian manifold \((M, g)\) is unchanged by a Möbius change of metric.

Consider the equation
\[ B(\phi) = p \]  
(3.1.7)

where \(p : \chi(M) \times \chi(M) \to C^\infty(M)\) is a field of symmetric \((0, 2)\) tensors of trace 0. The equation (3.1.7) is a non-linear equation but by lemma 3.1.1, if \(\phi_0\) is a particular solution of (3.1.7) and if \(\tilde{g} = e^{2\phi_0} g\), then we can solve (3.1.7) by solving the homogeneous equation \(B_{\tilde{g}}(\sigma) = 0\) on the Riemannian manifold \((M, \tilde{g})\).

Definition 3.1.6 We say that the equation (3.1.7) is fully integrable at \(q \in M\). If for every \(X \in T_q M\) there exist a locally defined solution \(\phi\) with \(\nabla\phi(q) = X\).

In the present section, we will concentrate on the homogeneous equation
\[ B(\phi) = 0 \]  
(3.1.8)

On making the substitution \(u = e^{-\phi}\) in (3.1.8) we get \(Y(u) = -e^{-\phi}Y(\phi)\),
and \(XY(\phi) = e^{-\phi}Y(\phi)X(\phi) - e^{-\phi}XY(\phi)\). Thus

\[
Hess(u)(X, Y) = XY(u) - \nabla_X Y(u)
= e^{-\phi}X(\phi)Y(\phi) - e^{-\phi}XY(\phi) + e^{-\phi}\nabla_X Y(\phi)
= -e^{-\phi}Hess(\phi)(X, Y) + e^{-\phi}X(\phi)Y(\phi)
\]

Now using this with (3.1.8) we arrive at

\[
Hess(u)(X, Y) = -e^{-\phi}[X(\phi)Y(\phi) + \frac{1}{n}\{\triangle\phi - \|\nabla\phi\|^2\}g(X, Y)] + e^{-\phi}X(\phi)Y(\phi)
= -\frac{e^{-\phi}}{n}\{\triangle\phi - \|\nabla\phi\|^2\}g(X, Y)
\]

Next we compute

\[
\triangle u = \sum_{i=1}^{n} Hess(u)(e_i, e_i) = e^{-\phi}\{\triangle\phi - \|\nabla\phi\|^2\}
\]

where \(\{e_1, e_2, \ldots, e_n\}\) is a local orthonormal frame, and consequently we have

\[
\triangle\phi = e^\phi \triangle u + \|\nabla\phi\|^2
\]

Substituting this in (3.1.9) we get

\[
Hess(u)(X, Y) = -\frac{e^{-\phi}}{n}\{e^\phi \triangle u + \|\nabla\phi\|^2 - \|\nabla\phi\|^2\}g(X, Y)
= \frac{\triangle u}{n}g(X, Y)
\]

Hence, the equation (3.1.8) takes the form

\[
Hess(u) = \lambda g
\]
where \( \lambda = \frac{\Delta u}{n} \), and consequently the non-homogeneous equation \( B(\phi) = p \) takes the form

\[
Hess(u) + up = \lambda g
\]  

(3.1.11)

Note that (3.1.10) may be written as

\[
\nabla_X \nabla u = \lambda X
\]  

(3.1.12)

for any vector field \( X \in \chi(M) \). We compute

\[
\nabla_X \nabla u = \nabla_X \left( e^{-\phi} \nabla \phi \right) = e^{-\phi} (\nabla_X \nabla \phi - X(\phi) \nabla \phi)
\]

and thus we have

\[
\nabla_X \nabla \phi - X(\phi) \nabla \phi = \nu X
\]  

(3.1.13)

where \( \nu = \frac{\Delta u}{n} = \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} \).

The equations (3.1.8) and (3.1.10) are equivalent at least for families of solutions. For we can always add a large constant to a solution of \( Hess(u) = \lambda g \) to guarantee that we can form \( \phi = -\log u \) in a given neighborhood. There may, however, be non-constant global solutions of (3.1.10) but none of (3.1.8), that is the substitution goes only one way. Recall that, we let \( U(M) \) to denote the set of all solutions to (3.1.10).

**Lemma 3.1.7** Let \( N \) be the unit vector field in the direction of \( \nabla \phi \) in a neighborhood of \( x \). For \( y \) near \( x \), the sectional curvature of every plane containing \( N_y \) has the same value \( K(y) \), and \( K \) is constant on each leaf of the foliation.
Proof. Let \( X \in \chi(M) \), and write \( B(\phi) = 0 \) in the form

\[
\nabla_X \nabla \phi - X(\phi) \nabla \phi = \nu X \tag{3.1.14}
\]

As \( N = \frac{\nabla \phi}{\|\nabla \phi\|} \), we have \( g(\nabla \phi, N) = \|\nabla \phi\| g\left(\frac{\nabla \phi}{\|\nabla \phi\|}, N\right) = \|\nabla \phi\| g(N, N) = \|\nabla \phi\| \). Thus

\[
X \|\nabla \phi\| = X \sqrt{g(\nabla \phi, \nabla \phi)} = \frac{2 \sqrt{g(\nabla_X \nabla \phi, \nabla \phi)}}{2 \sqrt{g(\nabla \phi, \nabla \phi)}} = g(\nabla_X \nabla \phi, N)
\]

\[
= g(X(\phi) \nabla \phi + \nu X, N) = g(\nabla \phi, N) + g(\nu X, N)
\]

\[
= \|\nabla \phi\|^2 g(X, N) + \nu g(X, N) = g(X, \nabla \phi) \|\nabla \phi\| + g(\nu X, N)
\]

\[
= \|\nabla \phi\|^2 g(X, N) + \nu g(X, N) = g(X, N) \{\nu + \|\nabla \phi\|^2\} \tag{3.1.15}
\]

Thus, if \( X \perp N \) then

\[
X \|\nabla \phi\| = 0 \tag{3.1.16}
\]

Thus we conclude that \( \|\nabla \phi\| \) is a constant on each leaf.

Using equations (3.1.14) and (3.1.16) we arrive at

\[
\nabla_X N = \nabla_X \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right) = \frac{1}{\|\nabla \phi\|} \nabla_X \nabla \phi + X \left( \frac{1}{\|\nabla \phi\|} \right) \nabla \phi
\]

\[
= \frac{1}{\|\nabla \phi\|} \{X(\phi) \nabla \phi + \nu X\} - \frac{X \|\nabla \phi\|}{\|\nabla \phi\|^2} \nabla \phi
\]

\[
= -\frac{\nu g(X, N)N}{\|\nabla \phi\|} + \frac{\nu X}{\|\nabla \phi\|}
\]

and we conclude that for \( X \perp N \)

\[
\nabla_X N = \frac{\nu X}{\|\nabla \phi\|} \tag{3.1.17}
\]
Also, from (3.1.5) for $X = N$, we have $N \| \nabla \phi \| = \nu + \| \nabla \phi \|^2$, or

$$
\nu = N \| \nabla \phi \| - \| \nabla \phi \|^2
$$

Next we have,

$$
\nabla_N N = \frac{1}{\| \nabla \phi \|} \nabla \nabla \phi (\nabla \phi) = \frac{1}{\| \nabla \phi \|} \left\{ \frac{1}{\| \nabla \phi \|} \nabla \nabla \phi \nabla \phi - \frac{\nabla \phi (\| \nabla \phi \|)}{\| \nabla \phi \|^2} \nabla \phi \right\}
$$

Now using equation (3.1.13), we compute

$$
\nabla \nabla \phi \nabla \phi = \nabla \phi (\phi) \nabla \phi + \nu \nabla \phi = \| \nabla \phi \|^2 \nabla \phi + \nu \nabla \phi = \{\| \nabla \phi \|^2 + \nu\} \nabla \phi
$$

and using (3.1.17), we have

$$
\nabla \phi (\| \nabla \phi \|) = g (\nabla \phi, N) \{\nu + \| \nabla \phi \|^2\}
$$

Thus

$$
\nabla_N N = \frac{1}{\| \nabla \phi \|} \left\{ (\nu + \| \nabla \phi \|^2) N - (\nu + \| \nabla \phi \|^2) N \right\} = 0
$$

Using above equation together with (3.1.17) for $X \perp N$, we conclude that

$$
\langle [X, N], N \rangle = -\langle \nabla_N X, N \rangle = -N \langle X, N \rangle + \langle X, \nabla_N N \rangle = -N \langle X, N \rangle = 0
$$

that is,

$$
[X, N] \perp N \quad (3.1.18)
$$

Therefore

$$
X\nu = X \left( N \| \nabla \phi \| - \| \nabla \phi \|^2 \right) = XN (\| \nabla \phi \|) - X(\| \nabla \phi \|^2)
$$

$$
= [X, N] (\| \nabla \phi \|) + N X (\| \nabla \phi \|) - 2 \| \nabla \phi \| X (\| \nabla \phi \|) \quad (3.1.19)
$$

$$
= [X, N] (\| \nabla \phi \|) + N (X \| \nabla \phi \|) - 2 \| \nabla \phi \| X (\| \nabla \phi \|) = 0
$$
Where the last equality comes from (3.1.16) as $X \perp N$ and $[X, N] \perp N$. Hence, we conclude that $\nu$ is also constant on each leaf.

Now let $X \in T_yM$ be a unit vector orthogonal to $N_y$ and extend $X$ to be orthogonal to $N$ everywhere. The plane spanned by $X_y$ and $N_y$ then has the sectional curvature

$$K = R(X_y, N_y, N_y, X_y)$$
$$= g(\nabla_X \nabla_N N - \nabla_N \nabla_X N - \nabla_{[X,N]}N, X)$$
$$= -g(\nabla_N \left(\frac{\nu X}{\|
abla \phi\|}\right) + \frac{\nu}{\|
abla \phi\|}[X, N], X)$$
$$= -\frac{\nu}{\|
abla \phi\|} g(\nabla_X N, X) - N \left(\frac{\nu}{\|
abla \phi\|}\right)$$
$$- \frac{\nu}{\|
abla \phi\|} g(\nabla_X N, X) + \frac{\nu}{\|
abla \phi\|} g(\nabla_N X, X)$$
$$= -N \left(\frac{\nu}{\|
abla \phi\|}\right) - \left(\frac{\nu}{\|
abla \phi\|}\right)^2$$

Thus, $K$ depends on $y$ alone and not on the vector $X_y$, also note that for $Y \perp N$ we have by (3.1.16), (3.1.18) and (3.1.19) that

$$Y(K) = -YN \left(\frac{\nu}{\|
abla \phi\|}\right) - Y \left(\frac{\nu}{\|
abla \phi\|}\right)^2$$
$$= [N, Y] \left(\frac{\nu}{\|
abla \phi\|}\right) - N \left(Y \left(\frac{\nu}{\|
abla \phi\|}\right)\right) - Y \left(\frac{\nu}{\|
abla \phi\|}\right)^2$$
$$= 0$$

This proves that $K$ is a constant on each leaf of the foliation, which completes the proof of the lemma.

Now, we will prove the following theorem:
Theorem 3.1.8 Let \((M, g)\) have dimension \(n \geq 2\). The equation \(B(\phi) = 0\) is fully integrable at \(x \in M\), if and only if \(M\) has constant curvature near \(x\).

Proof. First we will prove necessity. Assume that \(B(\phi) = 0\) is fully integrable at \(x \in M\), then we have to show that \(M\) has constant curvature near \(x\). If \(\dim M = 2\). Consider \(\phi_1, \phi_2\) two solutions of \(B(\phi) = 0\) such that \(\nabla \phi_1, \nabla \phi_2\) are linearly independent (this is guaranteed by full integrability of the equation \(B(\phi) = 0\)). Since the foliations corresponding to \(\phi_1\), and \(\phi_2\) are then transverse, since the Gaussian curvature is constant on leaves of either (by lemma 3.1.7), it is constant near \(x\).

Next, when \(\dim M \geq 3\), it is sufficient to show that when \(y\) is near \(x\) the sectional curvature of all planes through \(y\) coincide. This holds when \(y = x\), since the lemma 3.1.7, implies that any plane in \(T_xM\) with non trivial intersection will have the same sectional curvature. Thus, the problem now reduces to showing that \(B(\phi) = 0\) is fully integrable at all points \(y\) near \(x\). But, as the equation \(B(\phi) = 0\) is fully integrable at \(y\) if and only if the corresponding linear equation \(Hess(\phi) = \lambda g\) is fully integrable at \(y\), and for a linear equation full integrability follows from the given condition.

The proof of sufficiency in theorem 3.1.8, needs following two lemmas.

Lemma 3.1.9 \(B(\phi) = 0\) is fully integrable on \(R^n\) with respect to the euclidean metric, the set \(U(R^n)\) consists of all functions of the form

\[
u(x) = a |x|^2 + b \cdot x + c \quad (3.1.20)
\]
where we write $b. x$ for the usual euclidean inner product.

As above a positive function $u \in U(M)$ gives a solution to $B(\phi) = 0$, with $\phi = - \log u$.

**Proof.** Equation (3.1.10) reads as

$$
\begin{bmatrix}
\frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 x_n} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{\partial^2 u}{\partial x_n x_2} & \cdots & \frac{\partial^2 u}{\partial x_n x_n}
\end{bmatrix}
= \frac{\Delta u}{n}
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \frac{\Delta u}{n} & 0 \\
\vdots & \ddots & 0 \\
0 & 0 & \frac{\Delta u}{n}
\end{bmatrix}
$$

that is $\partial_i \partial_i u = \partial_j \partial_j u, \partial_i \partial_j u = 0 \quad i \neq j$. This implies $\partial_i \partial_i \partial_1 u = 0$, thus

$$
u = f(x_2, x_3, \ldots, x_n) x_1^2 + h(x_2, x_3, \ldots, x_n) x_1 + k(x_2, x_3, \ldots, x_n)
$$

Since $\partial_i \partial_j u = 0, j \neq 1, f$ and $h$ are constants, applying this argument to all indices, we obtain

$$
u = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i=1}^{n} b_i x_i + c
$$

Also as $\partial_i \partial_i u = \partial_j \partial_j u$, all the coefficients $a_i$ coincide proving that any function in $U(R^n)$ has the form in (3.1.20). The function $\phi = - \log u$ then satisfies $B(\phi) = 0$.

**Lemma 3.1.10** If $B(\phi) = 0$ in $R^n$, then the metric $e^{2\phi}(eu)$ has constant curvature $K_\phi$ and $\phi$ can be chosen so that $K_\phi$ is any given value.

**Proof.** Note that $R^n$ is flat that is all the sectional curvatures are zero. Now the sectional curvature of a metric $e^{2\phi}(eu)$ of the plane spanned by $\frac{\partial}{\partial x_i}$,
\( \frac{\partial}{\partial x_j} \) is given by remark 1.3.6 as

\[
\hat{R} \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_j} = 0 + \left[ g_{ij} \text{Hess}_\phi \left( \frac{\partial}{\partial x_j} \right) - g_{jj} \text{Hess}_\phi \left( \frac{\partial}{\partial x_j} \right) \right] \\
- \left[ \nabla^2 \phi \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j} \right) - \frac{\partial}{\partial x_j} (\phi) \frac{\partial}{\partial x_j} (\phi) + g_{jj} \| \nabla \phi \|^2 \right] \frac{\partial}{\partial x_i} \\
+ \nabla^2 \phi \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right) - \frac{\partial}{\partial x_i} (\phi) \frac{\partial}{\partial x_i} (\phi) + g_{ii} \| \nabla \phi \|^2 \frac{\partial}{\partial x_j} \\
+ \left[ \frac{\partial}{\partial x_i} (\phi) g_{jj} - \frac{\partial}{\partial x_j} (\phi) g_{ij} \right] \nabla \phi
\]

where \( g_{ij} = g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = 0 \) and \( g_{ii} = g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right) = 1 \). Thus

\[
g \left( \hat{R}_{ijj}, \frac{\partial}{\partial x_i} \right) = - \nabla^2 \phi \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right) - \nabla^2 \phi \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right) \\
+ \left( \frac{\partial}{\partial x_j} (\phi) \right)^2 + \left( \frac{\partial}{\partial x_i} (\phi) \right)^2 - \| \nabla \phi \|^2
\]

However, as \( \frac{\partial}{\partial x_j} (\phi) = \frac{\partial}{\partial x_j} \phi = \partial_j \phi \), we get

\[
g \left( \hat{R}_{ijj}, \frac{\partial}{\partial x_i} \right) = -\partial_i \partial_j \phi + (\partial_i \phi)^2 - \partial_j \partial_j \phi + (\partial_j \phi)^2 - \| \nabla \phi \|^2
\]

which implies

\[
\hat{K}_{ij} = e^{2\phi} g \left( \hat{R}_{ijj}, \frac{\partial}{\partial x_i} \right) \\
= e^{2\phi} \left( g_{ii} g_{jj} - g_{ji}^2 \right) \\
= e^{2\phi} \left\{ -\partial_i \partial_j \phi + (\partial_i \phi)^2 - \partial_j \partial_j \phi + (\partial_j \phi)^2 - \| \nabla \phi \|^2 \right\}
\]

Next, as \( u = e^{-\phi} \), we get \( \partial_t u = -u \partial_t \phi \) and \( \partial_t \partial_t u = -u \{ \partial_t \partial_t \phi - (\partial_t \phi)^2 \} \) and consequently (3.1.21) takes the form

\[
\hat{K}_{ij} = u \left( \partial_i \partial_t u + \partial_j \partial_t u \right) - \| \nabla u \|^2
\]
Now if $u$ is as in (3.1.20) then we get $\partial_i \partial_i u = 2a$ and \( \| \nabla u \|^2 = 4a^2 |x|^2 + 4abx + b^2 \) and this implies

\[
\hat{K}_{ij} = 4a^2 |x|^2 + 4abx + 4ac - 4a^2 |x|^2 - 4abx - b^2 \\
= 4ac - b^2 = \frac{2u(0) \triangle u}{n} - \| \nabla u(0) \|^2
\]

and thus by Schur’s theorem $\hat{K}$ is a constant, that is the metric $e^{2\phi}(euc) = u^{-2}(euc)$ is of constant curvature $\hat{K} = 4ac - b^2$. This completes the proof of the lemma.

Now, we will prove the sufficiency in theorem 3.1.8:

Assume that $(M, g)$ has a constant curvature $K$ near a point $x \in M$ and then we have to show that $B(\phi) = 0$ is fully integrable near $x$. For this, take $(R^n, euc)$ and choose $\phi(x) = -\log \left( a |x|^2 + b.x + c \right)$, $a, c \in R$ and $b \in R^n$. Solving $B(\phi) = 0$ on $R^n$ such that $g_1 = e^{2\phi}(euc)$ has curvature $K$, we get that $(R^n, g_1)$ and $(M, g)$ both are of constant curvature $K$ near $x$, thus there exist a local isometry $f : M \rightarrow R^n$, near $x$, thus to prove that $B(\phi) = 0$ is fully integrable near $x$ on $(M, g)$, we have to show that $B_{g_1}(\sigma) = 0$ is fully integrable near $f(x)$ on $(R^n, g_1)$, where $\phi = f^{-1}(\sigma)$.

Define

\[
\sigma = -\log \left( \frac{A|x|^2 + B.x + C}{a |x|^2 + b.x + c} \right), \quad A, C \in R, B \in R^n
\]
which is equivalent to

\[
\sigma = -\log(A|x|^2 + Bx + C) + \log(a|x|^2 + b.x + c)
\]

\[
= -\log(A|x|^2 + Bx + C) + \phi
\]

and consequently \(\sigma + \phi = \gamma\), where \(\gamma(x) = -\log(A|x|^2 + Bx + C)\) and \(B_{\text{euc}}(\gamma) = 0\). This implies by lemma 3.1.1, that

\[
B_{\text{euc}}(\gamma) = B_{\text{euc}}(\sigma + \phi) = B_{\text{euc}}(\phi) + B_{g_1}(\sigma)
\]

that is \(0 = 0 + B_{g_1}(\sigma)\). Hence \(B_{g_1}(\sigma) = 0\) is fully integrable at \(f(x)\), and consequently \(B_{g_2}(\phi) = 0\) is fully integrable at \(x\).

This complete the proof of theorem 3.1.8.

Examples 3.1.11:

(1) The spherical metric on \(R^n\), given by

\[
g_1 = \left(\frac{2}{1 + |x|^2}\right)^2 (\text{euc})
\]

is a Möbius metric on \(R^n\), as for

\[
\sigma_1(x) = -\log\left(\frac{1}{1 + |x|^2}\right)
\]

\(g_1 = e^{2\sigma_1}(\text{euc})\) and \(B_{g_1}(\sigma_1) = 0\). We have already proved this in the proof of sufficiency in theorem 3.1.8, for any \(\sigma\) of the form

\[
\sigma(x) = -\log\left(\frac{A|x|^2 + Bx + C}{a|x|^2 + b.x + c}\right), \quad A, C \in R, B \in R^n
\]
defined on \((M, g)\), that the equation \(B_{g_1}(\sigma) = 0\) holds.

(2) The Poincare’s metric for the Poincare’s ball and a Poincare’s half-space are given by
\[
g_2 = \left(\frac{2}{1 - |x|^2}\right)^2 (euc)
\]
and
\[
g_3 = \frac{1}{(x^n)^2} (euc)
\]
respectively. Here in this case
\[
\sigma_2(x) = -\log\left(\frac{1}{1 - |x|^2}\right)
\]
and
\[
\sigma_3(x) = -\log\left(\frac{1}{x_n}\right)
\]
satisfy \(g_2 = e^{2\sigma_2} (euc)\) and \(g_3 = e^{2\sigma_3} (euc)\) and that \(B_{g_2}(\sigma_2) = 0\) and \(B_{g_3}(\sigma_3) = 0\) hold.

That is, these are all examples of Möbius metrics on \(R^n\) or on domains in \(R^n\). In fact, one can say more. Suppose we insist that \(\phi\), as given in (3.1.20), is to be defined in all of \(R^n\). Then there are essentially two possibilities for the metric \(g_1 = e^{2\phi} (euc)\). If \(a = 0\), then \(b = 0\) and \(c > 0\) so that \(g_1\) is homothetic to the euclidean metric. If \(a \neq 0\), then \(a, c > 0\), and after a translation of \(R^n\) by \(b/2a\), \(\phi\) takes the form
\[
\phi(x) = -\log\left(a|x|^2 + \frac{K}{4a}\right)
\]
where \(K = 4ac - |b|^2\) is the curvature of \(g_1\), as above. So \(K > 0\).
3.2 SCHWARZIAN AND THE SECOND FUNDAMENTAL FORM

In this section we wish to illustrate the usefulness of isolating the Schwarzian tensor when computing the change in geometric quantities under a conformal change of metric. Also we do some calculation on the second fundamental form $h$ of a submanifold of $M$, with respect to conformal metrics $g$ and $\hat{g} = e^{2\phi} g$.

To fix notation, let $g = \langle , \rangle$ be a metric on $M$, let $P$ be a submanifold of $M$, and let $T^\perp P$ be its normal bundle. Tangent vector fields along $P$ will be denoted $X$, $Y$, etc.; normal vector fields will be denoted $V$, $U$, etc. The Riemannian connection for $P$ will be denoted by $\nabla'$ and the normal connection by $\nabla''$; $\nabla$ is the Riemannian connection for $M$, then

(a) $\nabla_X Y = \nabla' Y + h(X,Y)$
(b) $\nabla_X V = \nabla'' V - A_V(X)$ \hfill (3.2.1)

where $h$ is the second fundamental form of $P$ in $M$ and $A$ is the Weingarten map which satisfies

(c) $\langle A_V(X), Y \rangle = g(h(X,Y), V)$ \hfill (3.2.2)

Now if $\hat{g} = e^{2\phi} g$, then using Koszul’s formula we have find the following expression for the covariant derivative for the Riemannina connection with respect the metric $\hat{g}$

$$2\hat{g}\left(\nabla' X, Z\right) = X\hat{g}(Y,Z) + Y\hat{g}(X,Z) - Z\hat{g}(X,Y)$$
$$-\hat{g}([X,Y],Z) - \hat{g}([Y,Z],X) - \hat{g}([X,Z],Y)$$
which in light of \( X\tilde{g}(Y, Z) = 2e^{2\phi}X(\phi)g(Y, Z) + e^{2\phi}Xg(Y, Z) \), takes the form

\[
2\tilde{g}\left(\nabla'_X Y, Z\right) = e^{2\phi}\{2g\left(\nabla'_X Y, Z\right) + 2X(\phi)g(Y, Z) \\
+ 2Y(\phi)g(X, Z) - 2Z(\phi)g(X, Y)\}
\]

If \( (\nabla\phi)^T, (\nabla\phi)^N \) are the tangential and normal parts of \( \nabla\phi \), then as \( Z(\phi) = g\left((\nabla\phi)^T, Z\right) \), we get

\[
2\tilde{g}\left(\nabla'_X Y, Z\right) = e^{2\phi}\{2g\left(\nabla'_X Y, Z\right) + 2X(\phi)g(Y, Z) \\
+ 2Y(\phi)g(X, Z) - 2g(X, Y)g\left((\nabla\phi)^T, Z\right)\}
\]

- \( g\left((\nabla\phi)^T, Z\right) = 2\tilde{g}\left(\nabla'_X Y, Z\right) + 2X(\phi)\tilde{g}(Y, Z) \\
- 2Y(\phi)\tilde{g}(X, Z) - 2g(X, Y)\tilde{g}\left((\nabla\phi)^T, Z\right)\)

Hence, \( \nabla'_X Y = \nabla' Y + X(\phi)Y + Y(\phi)X - \langle X, Y \rangle \nabla^T \phi \). Using this relation we calculate

\[
\hat{h}(X, Y) = \nabla_X Y - \nabla'_X Y \\
= (\nabla_X Y + X(\phi)Y + Y(\phi)X - \langle X, Y \rangle \nabla \phi) \\
- (\nabla' Y + X(\phi)Y + Y(\phi)X - \langle X, Y \rangle \nabla^T \phi) \\
= (\nabla_X Y - \nabla' Y) - \langle X, Y \rangle (\nabla\phi - \nabla^T \phi)
\]

Or

\[
\hat{h}(X, Y) = h(X, Y) - \langle X, Y \rangle (\nabla\phi)^N
\]
Using this relation we have

\[
\langle \hat{A}_V (X), Y \rangle = \langle V, \hat{h} (X, Y) \rangle = \langle V, h (X, Y) - \langle X, Y \rangle (\nabla \phi)^N \rangle \\
= \langle V, h (X, Y) \rangle - \langle X, Y \rangle \langle V, (\nabla \phi)^N \rangle \\
= \langle A_V (X), Y \rangle - \langle X, Y \rangle V (\phi)
\]

consequently we arrive at

\[
\hat{A}_V (X) = A_V (X) - V (\phi) X
\]

Finally, by 1.3.6 (i) we have

\[
\hat{\nabla}''_{X} V = \hat{\nabla}_X V + \hat{A}_V (X) = \hat{\nabla}_X V + A_V (X) - V (\phi) X \\
= \nabla X V + X (\phi) V + V (\phi) X \\
- \langle X, V \rangle (\nabla \phi) + A_V (X) - V (\phi) X
\]

However as \( \langle X, V \rangle = 0 \), we conclude

\[
\nabla''_{X} V = (\nabla X V + A_V (X)) + X (\phi) V = \nabla''_{X} V + X (\phi) V
\]

We summarize these remarks as following formulas

\[
(a) \quad \hat{\nabla}^{' '}_{X} Y = \nabla^{' '} Y + X (\phi) Y + Y (\phi) X - \langle X, Y \rangle \nabla^T \phi \\
(b) \quad \hat{h} (X, Y) = h (X, Y) - \langle X, Y \rangle (\nabla \phi)^N \\
(c) \quad \hat{A}_V (X) = A_V (X) - V (\phi) X \\
(d) \quad \hat{\nabla}''_{X} V = \nabla''_{X} V + X (\phi) V
\] (3.2.2)

where \((\nabla \phi)^T\) and \((\nabla \phi)^N\) are the tangent and normal components of \(\nabla \phi\).
**Definition 3.2.1** The covariant derivative of the second fundamental form $h$ is defined by

$$
(\nabla_Z h)(X, Y) = \nabla_Z^2 h(X, Y) - h\left(\nabla'_Z X, Y\right) - h\left(X, \nabla'_Z Y\right)
$$

(3.2.3)

We want to compare this with $\left(\hat{\nabla}_Z \hat{h}\right)(X, Y)$, to do this, note that

$$
\nabla_Z^2 \left((\nabla \phi)^N\right) = \left\{\nabla_Z \left(\nabla \phi - (\nabla \phi)^T\right)\right\}^N
$$

(3.2.4)

and consequently we calculate

$$
\left(\hat{\nabla}_Z \hat{h}\right)(X, Y) = \nabla_Z \hat{h}(X, Y) - \hat{h}\left(\nabla'_Z X, Y\right) - \hat{h}\left(X, \nabla'_Z Y\right)
$$

(3.2.5)

Next we calculate

$$
\nabla_Z^2 \hat{h}(X, Y) = \nabla_Z \hat{h}(X, Y) + Z \left(\phi\right) \hat{h}(X, Y)
$$

$$
= \nabla_Z \left(h(X, Y) - \langle X, Y \rangle (\nabla \phi)^N\right)
+ Z \left(\phi\right) \left(h(X, Y) - \langle X, Y \rangle (\nabla \phi)^N\right)
$$

$$
= \nabla_Z \hat{h}(X, Y) - \langle X, Y \rangle \nabla_Z (\nabla \phi)^N
$$

$$
- Z \langle X, Y \rangle (\nabla \phi)^N + Z \left(\phi\right) h(X, Y)
$$

$$
- Z \left(\phi\right) \langle X, Y \rangle (\nabla \phi)^N
$$

$$
= \nabla_Z \hat{h}(X, Y) - \langle X, Y \rangle \nabla_Z (\nabla \phi)^N
$$

$$
- \left\langle \nabla'_Z X, Y \right\rangle (\nabla \phi)^N - \left\langle X, \nabla'_Z Y \right\rangle (\nabla \phi)^N
$$

$$
+ Z \left(\phi\right) h(X, Y) - Z \left(\phi\right) \langle X, Y \rangle (\nabla \phi)^N
$$
and thus, by (3.2.4)

\[
\nabla_Z \hat{h} (X, Y) = \nabla_Z'' h (X, Y) - \langle X, Y \rangle (\nabla_Z \nabla \phi)^N \\
+ \langle X, Y \rangle h \left( Z, (\nabla \phi)^T \right) - \left( \nabla_Z' X, Y \right) (\nabla \phi)^N \\
- \left( X, \nabla_Z Y \right) (\nabla \phi)^N + Z (\phi) h (X, Y) \\
- Z (\phi) \langle X, Y \rangle (\nabla \phi)^N
\]

(3.2.6)

Also we have

\[
\hat{h} \left( \nabla'_Z X, Y \right) = h \left( \nabla'_Z X, Y \right) - \left( \nabla'_Z X, Y \right) (\nabla \phi)^N \\
= h \left( \nabla'_Z X + Z (\phi) X + X (\phi) Z - (Z, X) (\nabla \phi)^T, Y \right) \\
- \left( \nabla'_Z X + Z (\phi) X + X (\phi) Z - (Z, X) (\nabla \phi)^T, Y \right) (\nabla \phi)^N \\
= h \left( \nabla'_Z X, Y \right) + Z (\phi) h (X, Y) \\
+ X (\phi) h (Z, Y) - (Z, X) h \left( (\nabla \phi)^T, Y \right) \\
- \left( \nabla'_Z X, Y \right) (\nabla \phi)^N - Z (\phi) \langle X, Y \rangle (\nabla \phi)^N \\
- X (\phi) \langle Z, Y \rangle (\nabla \phi)^N + (Z, X) \left( (\nabla \phi)^T, Y \right) (\nabla \phi)^N
\]

(3.2.7)

Similarly

\[
\hat{h} \left( X, \nabla'_Z Y \right) = h \left( X, \nabla'_Z Y \right) + Z (\phi) h (Y, X) + Y (\phi) h (X, Z) \\
- \langle Z, Y \rangle h ((\nabla \phi)^T, X) - \left( X, \nabla'_Z Y \right) (\nabla \phi)^N \\
- Z (\phi) \langle Y, X \rangle (\nabla \phi)^N - Y (\phi) \langle Z, X \rangle (\nabla \phi)^N \\
+ \langle Z, Y \rangle \left( (\nabla \phi)^T, X \right) (\nabla \phi)^N
\]

(3.2.8)
Substituting (3.2.6), (3.2.7) and (3.2.8) in (3.2.5) we get

\[
\left(\nabla_Z \hat{h}\right)(X, Y) = \nabla_Z h(X, Y) - \langle X, Y \rangle (\nabla Z \nabla \phi)^N \\
+ \langle X, Y \rangle h\left(Z, (\nabla \phi)^T\right) - \left\langle \nabla_Z X, Y \right\rangle (\nabla \phi)^N \\
- \left\langle X, \nabla_Z Y \right\rangle (\nabla \phi)^N + Z(\phi) h(X, Y) \\
- Z(\phi) \langle X, Y \rangle (\nabla \phi)^N - h\left(\nabla_Z X, Y \right) \\
- Z(\phi) h(X, Y) - X(\phi) h(Z, Y) \\
+ \langle Z, X \rangle h\left((\nabla \phi)^T, Y \right) + \left\langle \nabla_Z X, Y \right\rangle (\nabla \phi)^N \\
+ Z(\phi) \langle X, Y \rangle (\nabla \phi)^N + X(\phi) \langle Z, Y \rangle (\nabla \phi)^N \\
- \langle Z, X \rangle \left\langle (\nabla \phi)^T, Y \right\rangle (\nabla \phi)^N - h\left(X, \nabla_Z Y \right) \\
- Z(\phi) h(Y, X) - Y(\phi) h(Z, X) + \langle Z, Y \rangle h\left((\nabla \phi)^T, X \right) \\
+ \left\langle X, \nabla_Z Y \right\rangle (\nabla \phi)^N + Z(\phi) \langle Y, X \rangle (\nabla \phi)^N \\
+ Y(\phi) \langle Z, X \rangle (\nabla \phi)^N - \langle Z, Y \rangle \left\langle (\nabla \phi)^T, X \right\rangle (\nabla \phi)^N \\
= (\nabla_Z h)(X, Y) + \alpha(X, Y, Z) \\
- \langle X, Y \rangle (\nabla Z \nabla \phi)^N + Z(\phi) \langle X, Y \rangle (\nabla \phi)^N \\
- Z(\langle X, Y \rangle) (\nabla \phi)^N + \left\langle \nabla_Z X, Y \right\rangle (\nabla \phi)^N \\
+ \left\langle X, \nabla_Z Y \right\rangle (\nabla \phi)^N + X(\phi) \langle Z, Y \rangle (\nabla \phi)^N \\
- \langle Z, Y \rangle \left\langle (\nabla \phi)^T, X \right\rangle (\nabla \phi)^N + Y(\phi) \langle Z, X \rangle (\nabla \phi)^N \\
- \langle Z, X \rangle \left\langle (\nabla \phi)^T, Y \right\rangle (\nabla \phi)^N \\
(3.2.9)
\]

Where

\[
\alpha(X, Y, Z) = h\left((\nabla \phi)^T, \langle X, Y \rangle Z + \langle Y, Z \rangle X + \langle Z, X \rangle Y \right) \\
- (Z(\phi) h(X, Y) + X(\phi) h(Y, Z) + Y(\phi) h(Z, X)) \\
(3.2.10)
\]
and the last seven terms in (3.2.9) will be cancelled.

Now we define

$$M(\phi)Z = \nabla_Z \nabla \phi - Z(\phi) \nabla \phi - \frac{1}{n}\{\triangle \phi - \|\nabla \phi\|^2\}Z$$ \hspace{1cm} (3.2.11)$$

then at any point \(x\), \(M(\phi)\) is a linear transformation of \(T_xM\) which satisfies

$$\langle M(\phi)Z, W \rangle = B(\phi)(Z, W), \quad Z, W \in T_xM$$ \hspace{1cm} (3.2.12)$$

Now as \(\{M(\phi)Z\}^N = (\nabla_Z \nabla \phi)^N - Z(\phi)(\nabla \phi)^N\), the equation (3.2.9) takes the form

$$\left(\nabla_Z \hat{h}\right)(X, Y) = (\nabla_Z h)(X, Y) + \alpha(X, Y, Z) - \langle X, Y \rangle \{M(\phi)Z\}^N$$ \hspace{1cm} (3.2.13)$$

where, \(\alpha(X, Y, Z)\) and \(\{M(\phi)Z\}\) are as given in (3.2.10) and (3.2.11) respectively.

Equation (3.2.13) is the main formula in this section. Now suppose that \(M\) is a surface and let \(\Gamma\) be a curve on \(M\). Choose unit tangent and unit normal vector fields (with respect to \(g\)) as \(T\) and \(N\), respectively along \(\Gamma\). Then the geodesic curvature \(k\) is given by

$$h(T, T) = kN$$

If we let \(\hat{T} = e^{-\phi}T, \hat{N} = e^{-\phi}N\) then (3.2.2 b) with \(X = Y = T\) gives

$$\hat{h}(T, T) = h(T, T) - (\nabla \phi)^N$$
from which it follows that $e^{2\phi} k\tilde{N} = kN - (\nabla\phi)^N$, and as $(\nabla\phi)^N = \left\langle (\nabla\phi)^N, N \right\rangle N = N (\phi) N$, we get $e^{2\phi} k\tilde{N} = kN - N (\phi) N$, or

$$e^{\phi} kN = kN - N (\phi) N$$

Taking inner product with $N$ we arrive at

$$e^{\phi} k = k - N (\phi)$$

which is a well-known relation. Since, $g (T, T) = 1$ and $g (N, N) = 1$, it follows that $g \left( \nabla'_T T, T \right) = 0$ and $g \left( \nabla''_T N, N \right) = 0$. Thus we conclude that $\nabla'_T T$ and $\nabla''_T N$ vanish, hence (3.2.3) implies that

$$(\nabla h) (T, T) = \nabla'' h (T, T) = \nabla'' (kN) = T (k) N + k \nabla'' N = T (k) N$$

Using a similar argument we can prove that

$$\left( \hat{\nabla}_T \hat{h} \right) \left( \hat{T}, \hat{T} \right) = \hat{T} \left( \hat{k} \right) \hat{N}$$

Now

$$\alpha (T, T, T) = h \left( (\nabla\phi)^T, 3T \right) - 3T (\phi) h (T, T) = 3h \left( (\nabla\phi)^T, T \right) - 3T (\phi) h (T, T)$$

but as $(\nabla\phi)^T = \left\langle (\nabla\phi)^T, T \right\rangle T = T (\phi) T$, we get

$$\alpha (T, T, T) = 3T (\phi) h (T, T) - 3T (\phi) h (T, T) = 0$$

The above results together with (3.2.12) and (3.2.13) we get

$$e^{3\phi} \hat{T}(\hat{k})\hat{N} = T(k)N - \{M(\phi)T\}^N$$
$$e^{3\phi} e^{-\phi} \hat{T}(\hat{k})N = T(k)N - \{M(\phi)T\}^N$$
$$e^{2\phi} \hat{T}(\hat{k})N = T(k)N - \{M(\phi)T\}^N$$
and thus conclude

\[ e^{2\phi} \mathcal{T}(\hat{k}) = T(k)N - B(\phi)(T, N) \]  \hfill (3.2.14)

Thus, for Möbius metrics the derivatives of \( k \) and \( \hat{k} \) are therefore proportional, and (3.2.14) gives a relationship between the derivative of the euclidean curvature of a curve and that of its image under conformal mapping, and it follows that a conformal diffeomorphism between surfaces maps all curves of constant geodesic curvature to curves of constant geodesic curvature if and only if it is a Möbius transformation.

**Definition 3.2.2** For a submanifold \( P \) of \( M \), a point \( x \in P \) is called umbilic (with respect to the metric \( g \)) if there exists \( A \in T^+_xP \) such that

\[ h(X, Y) = \langle X, Y \rangle A \quad \forall X, Y \in T_xP \]  \hfill (3.2.15)

Thus at an umbilic point \( x \) of \( P \), we have

\[ h((\nabla \phi)^T, X) = X(\phi)A \]  \hfill (3.2.16)

Now by (3.2.2 b) and (3.2.15), for any smooth function \( \phi \) on \( M \), we have

\[ \hat{h}(X, Y) = \langle X, Y \rangle \{ A - (\nabla \phi)^N \} \]  \hfill (3.2.17)

for the metric \( \tilde{g} = e^{2\phi}g \), and thus \( x \) is also an umbilic point for \( P \) with respect to the metric \( \tilde{g} \). Also, by substituting (3.2.16) in (3.2.10) we get \( \alpha(X, Y, Z) = 0 \), for \( X, Y, Z \in T_xP \). Hence at \( x \), (3.2.13) becomes

\[ (\tilde{\nabla}_Z \hat{h})(X, Y) = (\nabla_Z h)(X, Y) - \langle X, Y \rangle \{ M(\phi)Z \}^N \]  \hfill (3.2.18)
**Definition 3.2.3** The mean curvature vector $H$ along $P$ is defined as $H = \frac{1}{p} \text{tr}(h)$, where $p = \dim P$.

Now using (3.2.2 b) we will get

$$e^{2\phi} \hat{H} = H - (\nabla \phi)^N$$

(3.2.19)

For $x \in P$, if $x$ is an umbilic point and $\{e_1, e_2, ..., e_p\}$ is a local orthonormal frame (with respect to $g$) on $P$ then $\{e^{-\phi}e_1, ..., e^{-\phi}e_p\}$ is a local frame on $(M, \tilde{g})$, so by (3.2.2 a)

$$\nabla'_Z e_i = \nabla'_Z e_i + Z(\phi)e_i + e_i(\phi)Z - \langle Z, e_i \rangle (\nabla \phi)^T$$

But $\nabla'_Z e_i = 0$ as $g(e_i, e_i) = 1$, we compute

$$\sum_{i=1}^p \left( \nabla'_Z e_i, e_i \right) = pZ(\phi) + Z(\phi) - Z(\phi) = pZ(\phi)$$

And

$$h(\nabla'_Z e_i, e_i) = Z(\phi)h(e_i, e_i) + e_i(\phi)h(Z, e_i) - \langle Z, e_i \rangle h((\nabla \phi)^T, e_i)$$

Therefore

$$\sum_{i=1}^p h(\nabla'_Z e_i, e_i) = pZ(\phi)H + h(Z, (\nabla \phi)^T) - h(Z, (\nabla \phi)^T)$$

$$= pZ(\phi)H$$

Thus, using (3.2.2 b) with $X = Y = e_i$, we have

$$\sum_{i=1}^p h(\nabla'_Z e_i, e_i) = \sum_{i=1}^p \left( \nabla'_Z e_i, e_i \right) - \sum_{i=1}^p \langle \nabla'_Z e_i, e_i \rangle (\nabla \phi)^N$$

$$= pZ(\phi)H - pZ(\phi)(\nabla \phi)^N$$
Putting this value in (3.2.3) we get
\[ p(\nabla_Z^2 \hat{h})(e_i, e_i) = \sum_{i=1}^{p} \nabla_Z^2 \hat{h}(e_i, e_i) - 2pZ(\phi)H - 2pZ(\phi)(\nabla \phi)^N \]
which implies
\[ p(\nabla_Z^2 \hat{h})(e_i, e_i) = p\nabla_Z^2 (e^{2\phi \hat{H}}) - 2pZ(\phi)H + 2pZ(\phi)(\nabla \phi)^N \]
\[ = p\nabla_Z^2 H - p\{M(\phi)Z\}^N \]
where the last equality follows from (3.2.18). Thus we have
\[ p\nabla_Z^2 (e^{2\phi \hat{H}}) - 2pZ(\phi)H + 2pZ(\phi)(\nabla \phi)^N = p\nabla_Z^2 H - p\{M(\phi)Z\}^N \]
or
\[ 2e^{2\phi}Z(\phi)\hat{H} + e^{2\phi}\nabla_Z^2 \hat{H} - 2Z(\phi)H + 2Z(\phi)(\nabla \phi)^N \]
\[ = \nabla_Z^2 H - \{M(\phi)Z\}^N \]
But
\[ 2e^{2\phi}Z(\phi)\hat{H} - 2Z(\phi)H + 2Z(\phi)(\nabla \phi)^N \]
\[ = 2Z(\phi)\{e^{2\phi} \hat{H} - (H - (\nabla \phi)^N)\} \]
\[ = 0 \]
where we used (3.2.19). Thus at the umbilic point we have
\[ e^{2\phi}\nabla_Z^2 \hat{H} = \nabla_Z^2 H - \{M(\phi)Z\}^N \] (3.2.20)
If the submanifold \( P \) is a hypersurface, it is more convenient to use the scalar second fundamental from \( L \), defined by
\[ h(X, Y) = l(X, Y)N \]
where $N$ is a unit normal field along $P$, then if $\hat{N} = e^{-\phi}N$, we have

$$\hat{h}(X, Y) = \hat{l}(X, Y)\hat{N}$$

Substitute these in (3.2.3) and (3.2.13) and take the inner product with respect to $g$ we get

$$e^{-\phi}(\nabla Z\hat{l})(X, Y) = (\nabla Z\hat{l})(X, Y) + \hat{\alpha}(X, Y, Z) - \langle X, Y \rangle B(\phi)(Z, N) \quad (3.2.21)$$

where

$$\hat{\alpha}(X, Y, Z) = \langle \alpha(X, Y, Z), N \rangle$$

that is $\hat{\alpha}$ is the same as $\alpha$ with $h$ replaced by $l$.

**Definition 3.2.4** The mean curvature scaler $L$, given either by $H = \beta N$, or $L = \frac{1}{p} tr(l)$, where $p = \dim P$.

According to this definition and by taking inner product of (3.2.19) with $N$, and using $(\nabla \phi)^N = \langle \nabla \phi^N, N \rangle N = N(\phi)N$, we get

$$e^\phi \hat{L} = L - N(\phi)$$

at any point. While at umbilic point, (3.2.20) leads to the relation

$$e^{2\phi} \nabla''_Z(\hat{L}\hat{N}) = \nabla''_Z(LN) - \{M(\phi)Z\}^N$$

Taking the inner product in above equation with $N$ we get

$$e^{3\phi} g(\nabla''_Z(\hat{L}\hat{N}), \hat{N}) = g(\nabla''_Z(LN), N) - B(\phi)(Z, N)$$
which gives

\[ e^\phi \hat{g}(L \left( \nabla''_Z \hat{N} \right) + Z \left( \hat{L} \right) \hat{N}, \hat{N}) = g(L \left( \nabla''_Z N \right) + (ZL)N, N) - B(\phi)(Z, N) \]

But \( \nabla''_Z N = 0 \), as \( g(N, N) = 0 \) and

\[
\nabla''_Z \hat{N} = \nabla''_Z (e^{-\phi} N) \\
= e^{-\phi} \nabla''_Z N - e^{-\phi} Z(\phi) N \\
= e^{-\phi} \nabla''_Z N + e^{-\phi} Z(\phi) N - e^{-\phi} Z(\phi) N \\
= e^{-\phi} \nabla''_Z N = 0
\]

Thus, for umbilic point we get

\[ e^\phi Z(\hat{L}) = Z(L) - B(\phi)(Z, N) \quad (3.2.22) \]

We have the following theorem:

**Theorem 3.2.5** If \( \dim M \geq 3 \) and \((M, g)\) is of constant curvature, then the Möbius and conformal groups coincide.

**Proof.** Clearly from the definition of the Möbius transformation it follows that a Möbius group is contained in the conformal group. For if \( f : (M, \hat{g}) \rightarrow (M, g) \) is a conformal transformation with \( f^* g = \hat{g} = e^{2\phi} g \), then we can show that \( f \) is a Möbius transformation, that is, \( \varphi_g(f) = B_g(\phi) = 0 \). For this let \( x \in M \) and let \( Z, N \in T_x M \) with \( \langle Z, N \rangle = 0 \), \( \|N\| = 1 \). As \( M \) is of constant curvature, we can find umbilic hypersurface \( P \) (i.e. \( P \) of \( \dim \geq 2 \))
with \( x \in P \), with extensions \( Z, N \) tangent and normal to \( P \) respectively (see O’Neill [9]), and \( P \) has constant mean curvature \( L \). So that, with \( ZL = 0 \).

Now by (3.2.17), umbilic points are preserved by conformal mappings so as \( P \) is umbilic hypersurface, and \( f \) is a conformal transformation, it follows that \( f(P) \) is also totally umbilic and so also has constant mean curvature.

Since \( f : (M, \tilde{g}) \rightarrow (M, g) \) is an isometry, we conclude that \( \tilde{L} \) is constant along \( P \); hence \( Z\tilde{L} = 0 \) and \( B(\phi)(Z, N) = 0 \) (by (3.2.22)), but as \( Z \) and \( N \) are arbitrary orthogonal vector fields in \( T_xM \), this proves \( B(\phi)(x) = 0 \) and consequently that \( B_g(\phi) = 0 \).

### 3.3 SCHWARZIAN AND THE RIEMANN CURVATURE TENSOR

There are several applications of Schwarzian tensor. One of the advantages of the Schwarzian tensor is that it helps in predicting how one of the natural components of the Riemannian curvature tensor changes under a conformal change of metric. In this section we analyze how the curvature tensor can be decomposed in different components and how these components change with the conformal change of the metric.

**Definition 3.3.1** Let \( E \) be a real vector space of dimension \( n \geq 2 \) with inner product \( g = \langle \cdot, \cdot \rangle \). Then a curvature tensor on \( E \) is a bilinear mapping \( R : E \times E \rightarrow \text{End}(E) \) such that:

\[
(i) \quad R(X, Y) = -R(Y, X)
\]

\[
(ii) \quad \langle R(X, Y)Z, W \rangle = -\langle Z, R(X, Y)W \rangle
\]
(iii) \( R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \)

where \( \text{End}(E) = \{ T : E \to E, T \text{ is a linear transformation} \} \)

**Example 3.3.2:**

1) Consider \( kI_g : E \times E \to \text{End}(E) \) defined by

\[
 kI_g(X,Y)Z = k\{ (Y,Z)X - (X,Z)Y \}
\]

Then the sectional curvature of the curvature tensor \( kI_g \) is

\[
 \frac{k\{ (Y,Y) \langle X,X \rangle - \langle X,Y \rangle^2 \}}{\{ (Y,Y) \langle X,X \rangle - \langle X,Y \rangle^2 \}} = k
\]

2) Take any symmetric \( \beta : E \times E \to \mathbb{R} \), bilinear trace-free map and define \( R_\beta : E \times E \to \text{End}(E) \) by

\[
 R_\beta(X,Y)Z = \langle Y,Z \rangle \sum_{j=1}^{n} \beta(X,e_j) e_j - \langle X,Z \rangle \sum_{j=1}^{n} \beta(Y,e_j) e_j \\
\quad + \beta(Y,Z)X - \beta(X,Z)Y
\]

where \( \{e_1,...,e_n\} \) is an orthonormal basis for \( E \), \( n \geq 3 \). In dimension two, \( R_\beta = 0 \) for all \( \beta \).

3) Note that \( kI_g + R_\beta \) is also curvature tensor.

Next we define what is known as is the Ricci contraction:

**Definition 3.3.3** For any curvature tensor \( R \) we define the *Ricci contraction*:
tion by

\[ \text{Ric}(R)(Y, Z) = \sum_{j=1}^{n} \langle R(e_j, Y)Z, e_j \rangle \]

which maps the space \( \mathcal{R} \) of curvature tensors into the space of bilinear, symmetric functions on \( E \times E \).

**Example 3.3.4** We shall find the *Ricci* contraction of the curvature tensor \((kI_g + k\beta)\), we get

\[
\text{Ric}(kI_g + k\beta)(X, Y) = \sum_{i=1}^{n} \langle kI_g(e_i, X)Y, e_i \rangle + \sum_{i=1}^{n} \langle R\beta(e_i, X)Y, e_i \rangle \\
= \sum_{i=1}^{n} k\{ \langle X, Y \rangle \langle e_i, e_i \rangle - \langle Y, e_i \rangle \langle X, e_i \rangle \} \\
+ \sum_{i=1}^{n} \{ \langle X, Y \rangle \sum_{j} \beta(e_i, e_j) \langle e_j, e_i \rangle \\
- \langle e_i, Y \rangle \sum_{j} \beta(X, e_j) \langle e_j, e_i \rangle + \beta(X, Y) \langle e_i, e_i \rangle \\
- \beta(e_i, Y) \langle X, e_i \rangle \} \\
= k(n-1) \langle X, Y \rangle + \langle X, Y \rangle \sum_{i} \beta(e_i, e_i) \\
- \sum_{i} \langle e_i, Y \rangle \beta(X, e_i) + n\beta(X, Y) - \beta(X, Y)
\]

But \( \sum_{i} \beta(e_i, e_i) = 0 \), and thus

\[
\text{Ric}(kI_g + R\beta)(X, Y) = k(n-1) \langle X, Y \rangle - \beta(X, Y) \\
+ n\beta(X, Y) - \beta(X, Y) \\
= k(n-1) \langle X, Y \rangle + (n-2)\beta(X, Y)
\]

That is

\[
\text{Ric}(kI_g + R\beta) = (n-1)kg + (n-2)\beta
\]
Remark 3.3.5 If \( \alpha \) is a symmetric bilinear function on \( E \times E \), then for \( R \) defined by
\[
R = \frac{\text{tr} \, (\alpha)}{n(n-1)} I_g + R_\beta
\]
with \( \beta \) defined by
\[
\beta = \frac{1}{(n-2)} \left[ \alpha - \frac{\text{tr} \, (\alpha)}{n} \right], \text{ trace} \beta = 0
\]
satisfies
\[
\text{Ric}(R) = \text{Ric}(\frac{\text{tr} \, (\alpha)}{n(n-1)} I_g + R_\beta)
\]
( which is the \( \text{Ric}(kI_g + R_\beta) \) with
\[
k = \frac{\text{tr} \, (\alpha)}{n(n-1)} = \frac{1}{n(n-1)} \left( \frac{\text{tr} \, (\alpha)}{n(n-1)} \right) g + (n-2) \beta
\]
\[
= \left( \frac{\text{tr} \, (\alpha)}{n} \right) g + \frac{(n-2)}{(n-2)} \left\{ \alpha - \left( \frac{\text{tr} \, (\alpha)}{n} \right) g \right\}
\]
\[
= \left( \frac{\text{tr} \, (\alpha)}{n} \right) g + \alpha - \left( \frac{\text{tr} \, (\alpha)}{n} \right) g = \alpha
\]
Thus, for any symmetric, bilinear function \( \alpha \) on \( E \times E \), there exist a \( R \in \mathbb{R} \) such that
\[
\text{Ric}(R) = \alpha
\]
which gives us that the Ricci contraction map is surjective for \( n \geq 3 \). Thus, if we denote the space of bilinear, symmetric functions on \( E \times E \) by \( \text{SB}(E \times E) \) then we have proved that
\[
\text{Ric} : \mathbb{R} \rightarrow \text{SB}(E \times E)
\]
is onto for \( n \geq 3 \), and this implies
\[
\mathbb{R} = \text{Im}(\text{Ric}) \oplus \ker(\text{Ric})
\]
which gives

$$\mathcal{R} = \{kI_g : k \in R\} \oplus \{R_\beta : \text{trac}(\beta) = 0\} \oplus \{C \in \mathcal{R} : \text{Ric}(C) = 0\} \quad (3.3.1)$$

where we call the space \{C \in \mathcal{R} : \text{Ric}(C) = 0\} = \{C\} as the space of Weyl conformal tensors, which vanishes when \( n = 3 \).

**Theorem 3.3.6** Let \((M, g)\) be a Riemannian manifold of dimension \( n \geq 2 \) and let \( \tilde{g} = e^{2\phi} g \). If \( R = kI_g + R_\beta + C \) and \( \tilde{R} = \hat{k}I_g + \hat{R}_\beta + \hat{C} \) are the Riemannian curvature tensors with respect to \( g \) and \( \tilde{g} \), respectively; then:

\[(a) \quad \hat{k} = e^{-2\phi}(k - \frac{2}{n}\Delta\phi - \frac{(n-2)}{n} \|\nabla\phi\|^2),\]
\[(b) \quad \hat{\beta} = \beta - B(\phi) \quad (n \geq 3),\]
\[(c) \quad \hat{C} = C \quad (n \geq 3),\]

where \( \Delta\phi, \|\nabla\phi\|, \) and \( B(\phi) \) are computed with respect to \( g \).

**Proof.** We know that

\[
\text{Ric}(kI_g + R_\beta + C) = \text{Ric}(kI_g + R_\beta) + \text{Ric}(C)
\]
\[
= \text{Ric}(kI_g + R_\beta)
\]
\[
= (n - 1)kg + (n - 2)\beta
\]

Taking the trace of the Ricci contraction of the curvature tensor \( R = kI_g + R_\beta + C \), we get \( S = (n - 1)nk \), as \( \sum_{j=1}^{n} \beta(e_j, e_j) = 0 \). Thus

\[
\rho = \frac{S}{n(n - 1)} = k
\]
Similarly: for $\hat{R} = \hat{k}I + \hat{R}_\beta + \hat{C}$,

$$\hat{\rho} = \hat{k}$$

Substitute these values in the formula

$$\hat{\rho} = e^{-2\phi}(\rho - \frac{2}{n}\Delta\phi - \left(\frac{n-2}{n}\right)\|\nabla\phi\|^2)$$

and thus we have

$$\hat{k} = e^{-2\phi}(k - \frac{2}{n}\Delta\phi - \left(\frac{n-2}{n}\right)\|\nabla\phi\|^2)$$

which proves (a). Also, by remark 1.3.6, we have

$$\hat{\text{Ric}}(\hat{R})(Y, Z) = \text{Ric}(R)(Y, Z) - (n-2)\nabla^2\phi(Y, Z) + (n-2)Z(\phi)Y(\phi)$$

$$-\{(n-2)\|\nabla\phi\|^2 + \Delta\phi\}g(Y, Z)$$

which gives

$$(n-1)\hat{k}g(Y, Z) + (n-2)\hat{\beta}(Y, Z) = (n-1)kg(Y, Z) + (n-2)\beta(Y, Z)$$

$$-(n-2)\nabla^2\phi(Y, Z) + (n-2)Z(\phi)Y(\phi)$$

$$-\{(n-2)\|\nabla\phi\|^2 + \Delta\phi\}g(Y, Z)$$

Now, using (a) we have

$$(n-1)e^{-2\phi}(k - \frac{2}{n}\Delta\phi - \left(\frac{n-2}{n}\right)\|\nabla\phi\|^2)e^{2\phi}g(Y, Z)$$

$$- (n-1)kg(Y, Z) + (n-2)\beta(Y, Z)$$

$$= (n-2)\beta(Y, Z) - (n-2)\nabla^2\phi(Y, Z) + (n-2)Z(\phi)Y(\phi)$$

$$-\{(n-2)\|\nabla\phi\|^2 + \Delta\phi\}g(Y, Z)$$
By canceling the similar terms and dividing the equation by \((n - 2)\) one get

\[
\hat{\beta}(Y, Z) = \beta(Y, Z) - \{\nabla^2 \phi(Y, Z) - Z(\phi)Y(\phi) \\
- \frac{1}{n}(\Delta \phi - \|\nabla \phi\|^2)g(Y, Z)\}
\]

or \(\hat{\beta} = \beta - B(\phi)\), which is (b).

Finally, using the relation between \(R\) and \(\hat{R}\) given by remark 1.3.6 we have

\[
(kI_{\hat{g}} + \hat{R}_{\hat{\beta}} + \hat{C})(X, Y)Z = (kI_g + R_{\beta} + C)(X, Y)Z + \{g(X, Z)Hess_{\phi}(Y) \\
- g(Y, Z)Hess_{\phi}(X)\} - \{\nabla^2 \phi(Y, Z) \\
+ g(Y, Z)\|\nabla \phi\|^2 - Y(\phi)Z(\phi)\}X \\
+ \{\nabla^2 \phi(X, Z) + g(X, Z)\|\nabla \phi\|^2 - X(\phi)Z(\phi)\}Y \\
+ \{X(\phi)g(Y, Z) - Y(\phi)g(X, Z)\}\nabla \phi
\]

Substitute the value of \(kI_{\hat{g}}, \hat{R}_{\hat{\beta}}, \hat{k}\) and \(\hat{\beta}\) get

\[
(k - 2\frac{\Delta \phi}{n} - \frac{n-2}{n}\|\nabla \phi\|^2)\{g(Y, Z)X - g(X, Z)Y\} + \langle Y, Z \rangle \sum_j (\beta(X, e_j) \\
- B(\phi)(X, e_j))e_j - \langle X, Z \rangle \sum_j (\beta(Y, e_j) - B(\phi)(Y, e_j))e_j + (\beta(Y, Z) \\
- B(\phi)(Y, Z))X - (\beta(X, Z) - B(\phi)(X, Z))Y + C(X, Y)Z
\]

\[= kI_g(X, Y)Z + R_{\beta}(X, Y)Z + C(X, Y)Z + \{g(X, Z)Hess_{\phi}(Y) - g(Y, Z)Hess_{\phi}(X)\} \\
- \{\nabla^2 \phi(Y, Z) + g(Y, Z)\|\nabla \phi\|^2 - Y(\phi)Z(\phi)\}X + \{\nabla^2 \phi(X, Z) \\
+ g(X, Z)\|\nabla \phi\|^2 - X(\phi)Z(\phi)\}Y + \{X(\phi)g(Y, Z) - Y(\phi)g(X, Z)\}\nabla \phi
\]

which gives us
\[(k I_g + R_{\beta})(X, Y)Z + \dot{C}(X, Y)Z \]
\[= (k I_g + R_{\beta})(X, Y)Z + C(X, Y)Z + \frac{2}{n} (\Delta \phi) I_g (X, Y) Z + \left( \frac{n-2}{n} \right) \| \nabla \phi \|^2 I_g (X, Y) Z \]
\[+ (Y, Z) \sum_j B(\phi)(X, e_j) e_j - (X, Z) \sum_j B(\phi)(Y, e_j) e_j \]
\[+ B(\phi)(Y, Z)X - B(\phi)(X, Z)Y + \{ g(X, Z) Hess_{\phi}(Y) - g(Y, Z) Hess_{\phi}(X) \} \]
\[- \{ \nabla^2 \phi(Y, Z) + g(Y, Z) \| \nabla \phi \|^2 - Y(\phi)Z(\phi) \} X \]
\[+ \{ \nabla^2 \phi(X, Z) + g(X, Z) \| \nabla \phi \|^2 - X(\phi)Z(\phi) \} Y \]
\[+ \{ X(\phi)g(Y, Z) - Y(\phi)g(X, Z) \} \nabla \phi \]

(3.3.2)

However,

\[B(\phi)(X, e_j) = Hess_{\phi}(X, e_j) - X(\phi) e_j(\phi) - \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} g(X, e_j)\]

implies

\[\sum_j B(\phi)(X, e_j) e_j = \sum_j g(Hess_{\phi}(X), e_j) e_j - X(\phi) \sum_j g(\nabla \phi, e_j) e_j \]
\[- \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} \sum_j g(X, e_j) e_j \]
\[= Hess_{\phi}(X) - X(\phi) \nabla \phi - \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} X\]

Similarly, we have

\[B(\phi)(Y, e_j) = Hess_{\phi}(Y) - Y(\phi) \nabla \phi - \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} Y\]

Also,

\[\frac{2}{n} (\Delta \phi) I_g (X, Y) Z + \left( \frac{n-2}{n} \right) \| \nabla \phi \|^2 I_g (X, Y) Z - \{ \nabla^2 \phi(Y, Z) + g(Y, Z) \| \nabla \phi \|^2 \nabla \phi \|^2 - Y(\phi)Z(\phi) \} X \]
\[+ \{ \nabla^2 \phi(X, Z) + g(X, Z) \| \nabla \phi \|^2 - X(\phi)Z(\phi) \} Y \]
\[= \left[ \nabla^2 \phi(X, Z) - X(\phi)Z(\phi) - \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} g(X, Z) \right] Y \]
\[- \left[ \nabla^2 \phi(Y, Z) - Y(\phi)Z(\phi) - \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} g(Y, Z) \right] X \]
\[+ \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} g(Y, Z) X - \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} g(X, Z) Y \]
\[= B(\phi)(X, Z)Y - B(\phi)(Y, Z)X + \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} g(Y, Z) X \]
\[- \frac{1}{n} \{ \Delta \phi - \| \nabla \phi \|^2 \} g(X, Z) Y \]
Thus, (3.3.2) becomes

\[ \hat{C}(X, Y)Z = C(X, Y)Z + \left[ \text{Hess}_\phi(X) - X(\phi)\nabla\phi - \frac{1}{n}\{\Delta\phi - \|\nabla\phi\|^2\}X \right. \\
\left. + \frac{1}{n}\{\Delta\phi - \|\nabla\phi\|^2\}X + X(\phi)\nabla\phi - \text{Hess}_\phi(X)\right]g(Y, Z) \]

\[ - \left[ \text{Hess}_\phi(Y) - Y(\phi)\nabla\phi - \frac{1}{n}\{\Delta\phi - \|\nabla\phi\|^2\}Y \right. \\
\left. + \frac{1}{n}\{\Delta\phi - \|\nabla\phi\|^2\}Y + Y(\phi)\nabla\phi - \text{Hess}_\phi(Y)\right]g(X, Z) \]

Hence,

\[ \hat{C}(X, Y)Z = C(X, Y)Z \]

This complete the proof of the theorem 3.3.6.

Remark 3.3.7 As an Einstein manifold is a Riemannian manifold for which the Ricci tensor is a constant multiple of the metric. Thus, a space of dimension three or more is Einstein if \( \beta \) vanishes and \( k \) is a constant.

Theorem 3.3.6 implies the following corollary.

Corollary 3.3.8 Let \((M, g)\) be a connected Einstein manifold of dimension \( n \geq 3 \) and let \( \tilde{g} = e^{2\phi}g \). Then \((M, \tilde{g})\) is Einstein if and only if \( B(\phi) = 0 \). In particular, the conformal and Möbius groups of \( M \) coincide.

The results of this section can be useful for the study of the differential equation \( B(\phi) = p \) on a Riemannian manifold. This differential equation, on substitution \( u = e^{-\phi} \) takes the form of the linear equation

\[ \text{Hess}(u) + up = \lambda g, \quad \lambda = \frac{\triangle u}{n} \quad (3.3.3) \]
If we take \( W = \nabla u \), then \( \langle X, W \rangle = \langle X, \nabla u \rangle = X(u) \) and thus,

\[
0 = X(u) - \langle X, W \rangle \tag{3.3.4}
\]

If \( \{e_1, \ldots, e_n\} \) is a geodesic frame on \( M \), then from (3.3.3) we have

\[
g(\nabla_X W, Y) + u p(X, Y) = \lambda g(X, Y)
\]

which implies

\[
g(\nabla_X W, Y) + u \sum_i g(Y, e_i) p(X, e_i) = \lambda g(X, Y)
\]

that is

\[
\nabla_X W + u \sum_i p(X, e_i) e_i = \lambda X \tag{3.3.5}
\]

Now let us calculate \( Ric(X, W) = \sum_{i=1}^n g(R(e_i, X) W, e_i) \), where

\[
R(e_i, X) W = \nabla_{e_i} \nabla_X W - \nabla_X \nabla_{e_i} W - \nabla_{[e_i, X]} W
\]

Using (3.3.5) we have

\[
\nabla_{e_i} (\nabla_X W) = \nabla_{e_i} \left( -u \sum_i p(X, e_i) e_i + \lambda X \right)
\]

\[
= -u \sum_i \nabla_{e_i} p(X, e_i) e_i - e_i (u) \sum_i p(X, e_i) e_i
\]

\[
+ \lambda \nabla_{e_i} X + e_i (\lambda) X \tag{3.3.6}
\]

and \( \nabla_{e_i} W = -u \sum_i p(e_i, e_i) e_i + \lambda e_i \), but as the trace of \( p \) is the trace of \( B(\phi) \), which vanishes, and thus \( \nabla_{e_i} W = \lambda e_i \), which gives

\[
\nabla_X (\nabla_{e_i} W) = X(\lambda) e_i + \lambda \nabla_X e_i \tag{3.3.7}
\]
Finally,

\[
\nabla_{[e_i, X]} W = -u \sum_i p([e_i, X], e_i) e_i + \lambda [e_i, X] \\
= -u \sum_i p(\nabla_{e_i} X, e_i) e_i + u \sum_i p(\nabla X e_i, e_i) e_i \\
+ \lambda \nabla_{e_i} X - \lambda \nabla_X e_i
\]

(3.3.8)

and thus using (3.3.6), (3.3.7) and (3.3.8) we get

\[
\text{Ric}(X, W) = -u \sum_i \nabla_{e_i} p(X, e_i) - e_i (u) \sum_i p(X, e_i) + \sum_i \lambda \langle \nabla_{e_i} X, e_i \rangle \\
+ \sum_i e_i (\lambda) \langle X, e_i \rangle - X (\lambda) \sum_i \langle e_i, e_i \rangle - \lambda \sum_i \langle \nabla X e_i, e_i \rangle \\
+ u \sum_i p(\nabla_{e_i} X, e_i) - u \sum_i p(\nabla X e_i, e_i) \\
- \lambda \sum_i \langle \nabla_{e_i} X, e_i \rangle + \lambda \sum_i \langle \nabla X e_i, e_i \rangle
\]

(3.3.9)

Now

\[
\sum_i e_i (\lambda) \langle X, e_i \rangle = \sum_i g(e_i, \nabla \lambda) g(X, e_i) = g\left(\sum_i g(X, e_i) e_i, \nabla \lambda\right) \\
= g(X, \nabla \lambda) \\
= X (\lambda)
\]

and

\[
\sum_i e_i (u) p(X, e_i) = \sum_i g(e_i, \nabla u) p(X, e_i) = p\left(X, \sum_i g(\nabla u, e_i) e_i\right) \\
= p(X, \nabla u) = p(X, W)
\]
also,
\[
\sum_i u \left( \nabla_X e_i, e_i \right) = u \left( \sum_i \left( \sum_j g(\nabla_X e_i, e_j) e_j, e_i \right) \right) = u \sum_i \left( \nabla_X e_i, e_j \right) p(e_i, e_j) = -u \sum_i g(e_i, \nabla_X e_j) p(e_j, e_i)
\]

Now, in the last sum the first factor is skew symmetric and the second factor is symmetric in \(i\) and \(j\), thus
\[
\sum_{ij} g(\nabla_X e_i, e_j) p(e_i, e_j) = 0
\]

which gives that
\[
u \sum_i p(\nabla_X e_i, e_i) = 0
\]

Furthermore, as \(\{e_1, \ldots, e_n\}\) is a geodesic frame on \(M\), then at a point we have \(\nabla_{e_i} e_i = 0\), and consequently we compute
\[
-u \sum_i \nabla_{e_i} p(X, e_i) + u \sum_i p(\nabla_{e_i} X, e_i) = -u \sum_i (\nabla_{e_i} p)(X, e_i)
\]
\[
= -u (\text{div} p)(X)
\]

Substitute these values in equation (3.3.9) we get
\[
Ric(X, W) = -p(X, W) - (n - 1) X(\lambda) - u (\text{div} p(X))
\]

but,
\[
Ric(X, W) = (n - 1) kg(X, W) + (n - 2) \beta(X, W)
\]
and thus
\[(n - 1) kg (X, W) + (n - 2) \beta (X, W) = -p(X, W) - (n - 1) X(\lambda) - u(div p(X))\]
or
that
\[
0 = X(\lambda) + kg(X, W) + \frac{(n - 2)}{(n - 1)} \beta(X, W)
+ \frac{1}{(n - 1)} [p(X, W) + u(div p(X))] \tag{3.3.10}
\]
Thus, we obtain the first-order system of equations
\[
0 = X(u) - \langle X, W \rangle
\]
\[
\nabla X W + u \sum_{i} p(X, e_i) e_i = \lambda X
\]
\[
0 = X(\lambda) + kg(X, W) + \frac{(n - 2)}{(n - 1)} \beta(X, W) + \frac{1}{(n - 1)} [p(X, W) + u(div p(X))]
\]
the term involving \(\beta\) being interpreted as zero when \(n = 2\). This system is
clearly equivalent to equation (3.3.3).
3.4 THE MOBIUS EQUATION AND WARPED PRODUCTS

Consider a Riemannian manifold \((M, g)\). Recall that \(U(M)\) denotes the space of solutions for the equation

\[
Hess u = \lambda_u g, \quad \lambda_u = \frac{\Delta u}{\dim M}
\]  

(3.4.1)

this equation is derived from the Möbius equation by substitution \(u = e^{-\phi}\).

In this section, we show how one can use the full space \(U(M)\) to express the metric \(g\) as a special kind of warped product near a regular points of \(u \in U(M)\).

Definition 3.4.1 For a connected Riemannian manifold \(M\) and a real number \(K\), we denote by \(U_K(M)\) the space of functions \(u \in U(M)\) such that \(\lambda_u + Ku\) is a constant.

If \(u, v \in U_K(M)\), then the expression

\[
\langle\langle u, v \rangle \rangle_K = u\lambda_v + v\lambda_u - \langle\nabla u, \nabla v \rangle + Kwv
\]

is also a constant. Indeed for \(X \in \chi(M)\), we have

\[
X \langle\langle u, v \rangle \rangle_K = X(u)\lambda_v + uX(\lambda_v) + X(v)\lambda_u + vX(\lambda_u) - \langle\nabla_X \nabla u, \nabla v \rangle - \langle\nabla u, \nabla_X \nabla v \rangle + KX(u)v + KuX(v)
\]
and as equation (3.4.1) implies $\nabla_X \nabla u = \lambda_u X$, we get

$$X \langle \langle u, v \rangle \rangle_K = X (u) \lambda_v + uX (\lambda_v) + X (v) \lambda_u + vX (\lambda_u) - \lambda_u X (v) - \lambda_v X (u) + KX (u) v + KuX (v)$$

$$= uX (\lambda_v) + vX (\lambda_u) + KX (u) v + KuX (v)$$

$$= u (X (\lambda_u + K v)) + v (X (\lambda_u + K u))$$

However as $u, v \in U_K M$, $\lambda_u + K v$ is a constant. And $\lambda_u + K u$ is a constant, which implies $X (\lambda_u + K v) = 0$ and $X (\lambda_u + K u) = 0$. Hence we have

$X \langle \langle u, v \rangle \rangle_K = 0$, for any $X$, that is $\langle \langle u, v \rangle \rangle_K = 0$ is a constant. Moreover

$$\langle \langle u, 1 \rangle \rangle_K = 1 \lambda_u + Ku1 = \lambda_u + Ku$$

**Theorem 3.4.2** (a) If $I$ is an interval, then $U_K I$ is three-dimensional, and $\langle \langle \cdot, \cdot \rangle \rangle_K$ is a non-degenerate inner product with signature $(+, -, -)$.

(b) Let $(M, g)$ be a connected space of constant curvature $K$ and dimension $n \geq 2$. Then $U_K M = U M$.

Whenever $u \in U M$ is positive, $\langle \langle u, u \rangle \rangle_K$ is the constant curvature of the metric $u^{-2}g$. If $M$ is also simply connected, then $U M$ has dimension $n + 2$, and $\langle \langle \cdot, \cdot \rangle \rangle_K$ is a non-degenerate inner product with signature $(+, -, \ldots, -)$.

**Proof.** (a) In an interval, $U_K$ consists of the solution of $\triangle u + Ku = 0$, that is solutions of the differential equation

$$u'' + Ku = \text{constant}$$

which takes the form
and so is three-dimensional. Now, the auxiliary equation of (3.4.3) is $m^3 + Km = 0$, or $m(m^2 + K) = 0$ and that gives $m = 0$, or $m = \pm \sqrt{-K}$, and consequently we have two cases:

(i) $K > 0$, then the solutions of (3.4.3) are \{1, \cos \sqrt{K} t, \sin \sqrt{K} t\}, so

$$\langle \langle u_0, u_0 \rangle \rangle_K = \langle \langle 1, 1 \rangle \rangle_K = K > 0$$

Also, for $u_1(t) = \cos \sqrt{K} t$ as $\nabla u_1 = -\sqrt{K} \sin \sqrt{K} t \frac{\partial}{\partial t}$, and $\nabla \frac{\partial}{\partial t} \nabla u_1 = -K \cos \sqrt{K} t \frac{\partial}{\partial t} = \lambda u_1 \frac{\partial}{\partial t}$. Hence,

$$\langle \langle u_1, u_1 \rangle \rangle_K = \langle \langle \cos \sqrt{K} t, \cos \sqrt{K} t \rangle \rangle_K = -2K \cos^2 \sqrt{K} t$$

$$-K \sin^2 \sqrt{K} t \frac{\partial}{\partial t} + K \cos^2 \sqrt{K} t$$

$$= -K \cos^2 \sqrt{K} t - K \sin^2 \sqrt{K} t \frac{\partial}{\partial t} = -K < 0$$

Similarly

$$\langle \langle u_2, u_2 \rangle \rangle_K = \langle \langle \sin \sqrt{K} t, \sin \sqrt{K} t \rangle \rangle_K = -K < 0.$$ 

Thus, the signature of the inner product is $(+, -, -)$.

(ii) $K < 0$, following the same arguments as in (i) we will see that the signature of the inner product is $(+, -, -)$.
Using theorem 3.3.6(a) \( u^{-2}g \) has constant curvature \( \tilde{K} = e^{-2\phi}(K - \frac{2}{n} \Delta \phi - \frac{(n-2)}{n} \|\nabla \phi\|^2) \), where \( u = e^{-\phi} \)

which gives us \( \nabla \phi = -e^\phi \nabla u \) and \( \Delta \phi = -e^{2\phi} \|\nabla u\|^2 - e^\phi \Delta u \)

Hence, \( \tilde{K} = e^{-2\phi}K + 2e^{-\phi} \frac{\Delta u}{n} - \|\nabla u\|^2 \)

\[ = 2u\lambda_u - \|\nabla u\|^2 + Ku^2 = \langle \langle u, u \rangle \rangle_K \]

Now from the results of the first section of this chapter. If \( M \) has constant curvature \( K \), then it is locally isometric to a Möbius metric in \( S^n \) which has the form

\[
\left( a|x|^2 + (b\cdot x) + c \right)^{-2} \text{(euc)}, \ a, c \in R, \ b \in R^n \text{ and } 4ac - |b|^2 = K
\]

if \( M \) is simply connected, it admits a global isometry into this model. From the equation

\[
\phi (x) = - \log \left( \frac{A|x|^2 + B\cdot x + C}{a|x|^2 + b\cdot x + c} \right), \ A, C \in R, \ B \in R^n
\]

in the first section, so the general solution to equation (3.4.1) in this model as

\[
u (x) = \left( \frac{A|x|^2 + B\cdot x + C}{a|x|^2 + b\cdot x + c} \right), \ A, C \in R, \ B \in R^n
\]

Now, as in this equation \( A, C \in R \) and \( B \in R^n \), we see that \( U(M) \) has dimension \( n + 2 \). In terms of these coefficients, \( \langle \langle u, u \rangle \rangle_K = 4AC - |B|^2 \). Also from the formula of \( u \) the product has signature \((+, - , ...., -)\).

Now, we will show the relationship between the existence of non-constant solutions to equation (3.4.1) and the local warped products.
Theorem 3.4.3 Let $M$ be a Riemannian manifold of dimension $n \geq 2$.

(a) Let $x$ be a regular point of $u \in U(M)$. Then near $x$ the metric can be expressed as a warped product $I \times fP$, where $I$ is an interval, the slices $\{t\} \times P$ are the level sets of $u$, and $\tilde{f}$ is a constant multiple of $\|\text{grad} \ u\|$.

(b) Let $x$ be a regular point of $u = (u_1, \ldots, u_m)$, where $u_1, \ldots, u_m \in U(M)$ and $2 \leq m < n$. Then near $x$ the metric can be expressed as a warped product $Q \times fP$, where $Q$ is a space of constant curvature $K \in \mathbb{R}$, the slices $\{q\} \times P$ are the level sets of $u$, and $\tilde{f}$ is a constant multiple of $\|\text{grad} \ u_1 \wedge \ldots \wedge \text{grad} \ u_m\|$.

Furthermore, $f \in U(Q)$ with $\langle (f, 1) \rangle_k = 0$.

(c) Let $x$ be a regular point of $u = (u_1, \ldots, u_n)$, where $u_1, \ldots, u_n \in U(M)$. Then $M$ has constant curvature near $x$.

Proof. We proof part $(a)$ and $(b)$ together. Here $u_1, u_2, \ldots, u_m \in U(M)$, $1 \leq m \leq n$, and the gradients $G_j = \nabla u_j$ are linearly independent at $x$. If $\lambda_j = \lambda_{u_j}$, then from (3.1.12) we have

$$[G_i, G_j] = \nabla_{G_i} G_j - \nabla_{G_j} G_i = \lambda_j G_i - \lambda_i G_j$$

Thus, the field of $m$-planes spanned by these gradients give us an integrable distribution, and so a foliation in a neighborhood of $x$.

Let $Q$ be a small, connected neighborhood in a leaf through $x$. And so the dimension of $Q$ is $m$, and let $P$ be small, connected neighborhood in the level set $u = u(x)$ (this mean that $\text{dim} \ P = n - m$). We parametrize a neighborhood
of $x$ by $Q \times P$ in the obvious way, using the leaves of the foliation and the level sets of $u$ to define the projections. Consider $Q \times P$ with the metric $g$ obtained from $M$. The horizontal slices $Q \times \{p\}$ and the vertical slices $\{q\} \times P$ are orthogonal in this metric, indeed for any $X \in \chi(P)$ we have

$$g(X, G_i) = g(X, \nabla u_i) = X(u_i) = 0, \quad i = 1, \ldots, m$$

where the last equality comes from the fact that $u_i$ is a constant on $P$. Moreover, $u$ is constant on each fiber $\{q\} \times P$.

Now we will show that:

(i) $g(G_i, G_j)$ is constant on each fiber $\{q\} \times P$.

(ii) $G_i$ is the horizontal lift of its restriction to $Q$.

(iii) If $X$ is the vertical lift of a vector field on $P$, then $\|X(q, p)\| = \tilde{f}(q, p) \|X(p)\|$, where

$$\tilde{f}(q, p) = \frac{\|(G_1 \wedge \ldots \wedge G_m)(q, p)\|}{\|(G_1 \wedge \ldots \wedge G_m)(p)\|}$$

Assertion (i): It comes from the observation

$$X\left(g(G_i, G_j)\right) = g(\lambda_i X, G_j) + g(G_i, \lambda_j X)$$

$$= \lambda_i g(X, G_j) + \lambda_j g(G_i, X) = 0$$

for any vertical vector field $X$.

Assertion (ii): if $\pi_1 : M \to Q$ and $\pi_2 : M \to P$ are the projections, then

$$d\pi_2(G_j) = d\pi_2(\nabla u_j) = 0$$
and $G_j$ is $\pi_1$-related to $G_j$, that is $d\, \pi_1 \circ G_j = G_j \circ \pi_1$. As for any $q \in Q$, $d\pi_1 \mid_q: T_qM \to T_qQ$, and $G_j = \nabla u_j \in \chi(M)$. But, $u_j$ is a constant on $P$ gives us that $\nabla u_j (p) = 0$, $\forall\, p \in P$ and this proves $G_j (p) = 0$, $\forall\, p \in P$ or equivalently $G_j (q, p) = G_j (q)$, $\forall\, p \in P$. Thus, $G_j$ is the horizontal lift of the restriction of $G_j$ to $Q$.

Assertion (iii): Let $X$ be a lifted vertical vector field, then we have to show that

$$\frac{\|X\|}{\|(G_1 \wedge \ldots \wedge G_m)\|}$$

is constant on each leaf $Q \times \{p\}$, or equivalently that the derivative of this expression in the direction of any of the vectors $G_j$ is zero. Now, using the multilinearity of the wedge product we get

$$\nabla G_j (G_1 \wedge \ldots \wedge G_m) = (\nabla G_j G_1) \wedge G_2 \wedge \ldots \wedge G_m + G_1 \wedge \nabla G_j G_2$$

$$\wedge G_3 \wedge \ldots \wedge G_m + \ldots + G_1 \wedge \ldots \wedge \nabla G_j G_m$$

$$= (\lambda_1 G_j) \wedge G_2 \wedge \ldots \wedge G_m + G_1 \wedge (\lambda_2 G_j)$$

$$\wedge G_3 \wedge \ldots \wedge G_m + \ldots + G_1 \wedge \ldots \wedge (\lambda_m G_j)$$

$$= \lambda_1 (G_j \wedge G_2 \wedge \ldots \wedge G_j \wedge \ldots \wedge G_m)$$

$$+ \lambda_2 (G_1 \wedge G_j \wedge \ldots \wedge G_j \wedge G_m) + \ldots$$

$$+ \lambda_j (G_1 \wedge \ldots \wedge G_m)$$

$$+ \ldots + \lambda_m (G_1 \wedge \ldots \wedge G_j \wedge \ldots \wedge G_j)$$

but as $G_j \wedge G_j = 0$, we get

$$\nabla G_j (G_1 \wedge \ldots \wedge G_m) = 0 + \ldots + 0 + \lambda_j (G_1 \wedge \ldots \wedge G_m) + 0 + \ldots + 0$$

$$= \lambda_j (G_1 \wedge \ldots \wedge G_m)$$
It follows that

\[ G_j (\| G_1 \wedge \ldots \wedge G_m \|) = G_j \sqrt{g (G_1 \wedge \ldots \wedge G_m, G_1 \wedge \ldots \wedge G_m)} = \frac{2g (\nabla G_j (G_1 \wedge \ldots \wedge G_m), G_1 \wedge \ldots \wedge G_m)}{2\sqrt{g (G_1 \wedge \ldots \wedge G_m, G_1 \wedge \ldots \wedge G_m)}} = \frac{g (\lambda_j (G_1 \wedge \ldots \wedge G_m), G_1 \wedge \ldots \wedge G_m)}{\sqrt{g (G_1 \wedge \ldots \wedge G_m, G_1 \wedge \ldots \wedge G_m)}} \]

\[ = \frac{\lambda_j G_1 \wedge \ldots \wedge G_m}{\| G_1 \wedge \ldots \wedge G_m \|^2} = \lambda_j \| G_1 \wedge \ldots \wedge G_m \| \] (3.4.4)

Since \( \nabla G_j X = \nabla X G_j = \lambda_j X \), we also have (using the same arguments as above) that \( G_j (\| X \|) = \lambda_j X \). Thus,

\[ G_j \left( \frac{\| X \|}{\| (G_1 \wedge \ldots \wedge G_m) \|} \right) = G_j (\| X \|) \cdot \frac{\| (G_1 \wedge \ldots \wedge G_m) \|}{\| (G_1 \wedge \ldots \wedge G_m) \|} = -G_j (\| (G_1 \wedge \ldots \wedge G_m) \|) \cdot \| X \| = 0 \]

By (i) and (ii), \( \tilde{f} \) is constant on each fiber \( \{ q \} \times P \). Then the three assertions show that the metric \( g \) can be expressed as a warped product \( Q \times_f P \), where \( f : Q \rightarrow R \) is defined by \( f (q) = \tilde{f} (q, p) \). This establishes part (a) of this theorem.

For part (b), suppose that \( m \geq 2 \). Since \( Q \) is totally geodesic, the Hessian operator in \( Q \) is just the restriction of the Hessian in \( M \). Therefore, the restriction of \( u_j \) to \( Q \) is an element of \( U (Q) \). So by theorem 3.1.8, \( Q \) is of constant curvature \( K \), for some \( K \). Now as \( \lambda_j + Ku_j = \) constant, we have

\[ G_i (\lambda_j) = -KG_i (u_j) = -Kg (G_i, G_j) \] (3.4.5)
Using that \(\|G_1 \wedge \ldots \wedge G_m\|\) is constant on \(Q\) and the equation (3.4.5) we get that

\[
G_j (f) = \lambda_j f
\]

(3.4.6)

Moreover, using (3.1.12), (3.4.5) and (3.4.6) we compute

\[
Hess_f^Q (G_i, G_j) = Hess_f (G_i, G_j) = G_i (G_j (f)) - (\nabla G_i G_j) (f)
\]

\[
= G_i (\lambda_j f) - \lambda_j G_i (f) = f G_i (\lambda_j)
\]

Consequently we have \(\nabla G_i \nabla f + K f G_i = 0\) which gives \(\lambda_f G_i + K f G_i = 0\), that is \(\lambda_f + K f = 0\), or \(\langle \langle f, 1 \rangle \rangle = 0\).

Bart (c) has already been established in theorem 3.1.8. Thus the proof of this theorem completed.

**Theorem 3.4.4.** Let \(M = Q \times_f P\), where \(Q\) and \(P\) are connected Riemannian manifolds of dimension at least one.

(a) Suppose that \(Hess^Q(f)/f\) is not a constant multiple of \(g^Q\). Then \(U(M)\) consists of the functions \(u(q, p) = h(q)\) such that \(h \in U(Q)\) and \(f \lambda_h = \langle grad^Q h, grad^Q f \rangle\).

(b) Suppose that \(Hess(f) + K f g^Q = 0\), where \(K\) is a constant. Let \(L = -\langle \langle f, f \rangle \rangle_K\). Then \(U(M)\) consists of the functions \(u(q, p) = h(q) + f(q)v(p)\) such that \(h \in U_K(Q)\), \(v \in U_L(P)\), and \(\langle \langle v, 1 \rangle \rangle_L = \langle \langle h, f \rangle \rangle_K\). In this case, \(U(M) = U_K(M)_K \langle \langle u, 1 \rangle \rangle_K = \langle \langle h, 1 \rangle \rangle_K\), and \(\langle \langle u, u \rangle \rangle_K = \langle \langle h, h \rangle \rangle_K + \langle \langle v, v \rangle \rangle_L\).
Proof. Let $u \in U(M)$, where $M = Q \times_P P$. If $A$ is any lifted horizontal vector field on $M$, and $X$ is any lifted vertical vector field, then using, $Hess(u)(A, X) = A(X(u)) - (\nabla_A X)(u)$ and proposition 1.5.4 we get

$$A(X(u)) = Hess(u)(A, X) + (\nabla_A X)(u)$$

$$= g(\nabla_A \nabla u, X) + (\nabla_A X)(u)$$

$$= \lambda g(A, X) + \frac{A(f)}{f} X(u) = 0 + \frac{A(f)}{f} X(u)$$

$$= \frac{A(f)}{f} X(u)$$

which implies

$$A\left( \frac{X(u)}{f} \right) = \frac{f A(X(u)) - X(u) A(f)}{f^2} = 0$$

that is $\frac{X(u)}{f}$ is constant on the horizontal slice $Q \times \{p\}$, and thus $\frac{X(u)}{f} |_{(q, p)} = v(p)$, where $v : P \to R$. Thus we get,

$$X(u) |_{(q, p)} = f(q) v(p)$$

(we can do that as $f > 0$) but as $u$ is constant at each $p$ on $P$, for any $p \in P$ we can write the last equation as

$$X(u) |_{q} = f(q) v(p)$$

This implies that $u$ has the form $u(q, p) = h(q) + f(q) v(p)$, where $h : Q \to R$.

Now, if $g^Q$, $g^P$ and $g$ denotes the Riemannian metric on $Q$, $P$ and $M$ respectively, then for any function $u$ of this form we have

$$(\nabla^Q u)(q, p) = (\nabla u)(q, p)$$

(we can do that as $f > 0$) but as $u$ is constant at each $p$ on $P$, for any $p \in P$ we can write the last equation as

$$X(u) |_{q} = f(q) v(p)$$

This implies that $u$ has the form $u(q, p) = h(q) + f(q) v(p)$, where $h : Q \to R$. 

Now, if $g^Q$, $g^P$ and $g$ denotes the Riemannian metric on $Q$, $P$ and $M$ respectively, then for any function $u$ of this form we have

$$(\nabla^Q u)(q, p) = (\nabla u)(q, p)$$

(we can do that as $f > 0$) but as $u$ is constant at each $p$ on $P$, for any $p \in P$ we can write the last equation as

$$X(u) |_{q} = f(q) v(p)$$

This implies that $u$ has the form $u(q, p) = h(q) + f(q) v(p)$, where $h : Q \to R$. 

Now, if $g^Q$, $g^P$ and $g$ denotes the Riemannian metric on $Q$, $P$ and $M$ respectively, then for any function $u$ of this form we have

$$(\nabla^Q u)(q, p) = (\nabla u)(q, p)$$
and

\[ (\nabla^P u) (q, p) = \frac{1}{f^2} (\nabla u) (q, p) \]

Note that: \( X (u) = g(\nabla u, X) = g^Q (\nabla^Q u, X) \) implies \( g(\nabla u, X) = g^Q (\nabla^Q u, X) \), or

\[ \nabla u = \nabla^Q u \] (3.4.7)

Also as \( X (u) = g(\nabla u, X) = g^P (\nabla^P u, X) \), that is \( f^2 g^P(\nabla u, X) = g^P (\nabla^P u, X) \),
or \( f^2(\nabla u) = (\nabla^P u) \), we conclude

\[ (\nabla u) = \frac{1}{f^2} (\nabla^P u) \] (3.4.8)

Using (3.4.7) and (3.4.8) we get

\[
(\nabla u) (q, p) = (\nabla^Q u) (q, p) + \frac{1}{f^2} (\nabla^P u) (q, p)
\]

\[ = \nabla^Q (h + v (p) f) (q) + \frac{1}{f (q)} (\nabla^P v) (p) \]

Thus, by using proposition 1.5.4 we have

\[ \nabla_A (\nabla u) (q, p) = \nabla^Q_A (\nabla^Q (h + v (p) f)) (q) \] (3.4.9)

and that

\[
\nabla_X (\nabla u) (q, p) = \frac{\nabla^Q (h + v (p) f) (f)}{f} |_{q} X + \frac{1}{f (q)} \{ \nabla^P_X (\nabla^P v) (p)
\]

\[ - \frac{\langle X, \nabla^P v \rangle |_{p} \nabla f \} + X \left( \frac{1}{f} \right) (\nabla^P v) (p) \]

However as \( f : Q \rightarrow R \Rightarrow X \left( \frac{1}{f} \right) = 0 \), for all \( X \in \chi (P) \). For fixed \( p \in P \), we
have \( \langle X, \nabla^P v \rangle (p) = X(v(p)) = 0 \) and consequently

\[
\nabla_X (\nabla u) (q, p) = \frac{\nabla^Q (h + v(p) f) (f)}{f} \nabla^P_X (\nabla^P v) (p) \tag{3.4.10}
\]

\[
= \frac{1}{f(q)} \left\{ \langle \nabla^Q (h + v(p) f), \nabla^Q f \rangle |q \nabla_X + \frac{1}{f(q)} \nabla^P_X (\nabla^P v) |_p \right\}
\]

Now by (3.4.9) and (3.4.10), for such a function to be in \( U(M) \) it is necessary and sufficient that

1. \( h + v(p) f \in U(Q), \quad \forall p \in P. \)
2. \( v \in U(P). \)
3. \( \lambda_{(h+v(p) f)} (q) = \frac{1}{f(q)} \left\{ \langle \nabla^Q (h + v(p) f), \nabla^Q f \rangle (q) + \lambda_v(p) \right\}, \forall (q, p) \in M. \)

Now, assume the hypotheses of part (a), and consider a function \( u(q, p) = h(q) + f(q) v(p) \) in \( U(M) \), and suppose on the contradictory that \( v \) is not a constant, say \( v(p_1) \neq v(p_2) \), then by condition (1), \( h \) and \( f \) are in \( U(Q) \), and by condition (3) we have

\[
\lambda_{(h+v(p_1) f)} (q) = \frac{1}{f(q)} \left\{ \langle \nabla^Q (h + v(p_1) f), \nabla^Q f \rangle (q) + \lambda_v(p_1) \right\} \quad \tag{3.4.11}
\]

and,

\[
\lambda_{(h+v(p_2) f)} (q) = \frac{1}{f(q)} \left\{ \langle \nabla^Q (h + v(p_2) f), \nabla^Q f \rangle (q) + \lambda_v(p_2) \right\} \quad \tag{3.4.12}
\]

Then, the equations (3.4.11) and (3.4.12) can be written as

\[
\lambda_h(q) + v(p_1) \lambda_f(q) = \frac{1}{f(q)} \left\{ \langle \nabla^Q (h + v(p_1) f), \nabla^Q f \rangle (q) + \lambda_v(p_1) \right\} \quad \tag{3.4.13}
\]
and,

\[ \lambda_h(q) + v(p_2) \lambda_f(q) + \frac{1}{f(q)} \left\{ \langle \nabla^Q (h + v(p_2)f), \nabla^Q f \rangle(q) + \lambda_v(p_2) \right\} \]  

(3.4.14)

Subtracting (3.4.13) and (3.4.14) we arrive at

\[
(v(p_1) - v(p_2)) \lambda_f(q) = \frac{1}{f(q)} \left\{ \langle \nabla^Q (v(p_1) - v(p_2)f), \nabla^Q f \rangle(q) + (\lambda_v(p_1) - \lambda_v(p_2)) \right\} 
\]

\[
= \frac{1}{f(q)} \left\{ (v(p_1) - v(p_2)) \langle \nabla^Q f, \nabla^Q f \rangle(q) + (\lambda_v(p_1) - \lambda_v(p_2)) \right\} 
\]

\[
= \frac{1}{f(q)} \left\{ (v(p_1) - v(p_2)) \| \nabla^Q f \| ^2(q) + (\lambda_v(p_1) - \lambda_v(p_2)) \right\} 
\]

which implies

\[
(v(p_1) - v(p_2)) f \lambda_f(q) = \left\{ (v(p_1) - v(p_2)) \| \nabla^Q f \| ^2(q) + (\lambda_v(p_1) - \lambda_v(p_2)) \right\} 
\]

that is,

\[
(v(p_1) - v(p_2)) \left\{ f \lambda_f - \| \nabla^Q f \| ^2 \right\}(q) = \lambda_v(p_1) - \lambda_v(p_2) 
\]

or

\[
\left\{ f \lambda_f - \| \nabla^Q f \| ^2 \right\}(q) = \frac{\lambda_v(p_1) - \lambda_v(p_2)}{(v(p_1) - v(p_2))} 
\]
Thus we have

\[
0 = A \left( f\lambda_f - \|\nabla^Q f\|^2 \right) \\
= fA(\lambda_f) + \lambda_f A(f) - 2 \left( \nabla^Q f, \nabla^Q f \right) \\
= fA(\lambda_f) + \lambda_f A(f) - 2 \left( \lambda_f A, \nabla^Q f \right) \\
= fA(\lambda_f) + \lambda_f A(f) - 2\lambda_f A(f) \\
= fA(\lambda_f) - \lambda_f A(f) = f^2 A \left( \frac{\lambda_f}{f} \right)
\]

that is, \( A \left( \frac{\lambda_f}{f} \right) = 0 \). Hence \( \frac{\lambda_f}{f} \) is a constant, which contradicts our hypotheses that \( \frac{\lambda_f}{f} \) is not a constant multiple of \( g^Q \). Therefore, \( v \) must be a constant, and so \( u \) is a function of \( q \) only, that is \( v(p) = 0, \forall p \in P \). Substituting this in the form of \( u \) one gets \( u(q,p) = h(q) \), and condition (2) is then trivial, so the conclusion of part (a) is hold by condition (1) and (3).

Now assume the hypotheses for part (b), where \( f \in U_K(Q) \) and \( \langle (f, 1) \rangle_K = 0 \). Consider a function \( u(q,p) = h(q) + f(q)v(p) \) in \( U(M) \). Since \( f \) is in \( U(Q) \), condition (1) shows \( h \) is too. Indeed,

\[
h + v(p)f \in U(Q) \text{ and } f \in U(Q), \text{ this gives that both } h + v(p)f \text{ and } f
\]
satisfy

\[
Hess\left( h + v(p)f \right) = \lambda_{h+v(p)f}g, \quad \lambda_{h+v(p)f} = \frac{\triangle (h + v(p)f)}{\dim(M)}
\]

and

\[
Hess(f) = \lambda_fg, \quad \lambda_f = \frac{\triangle f}{\dim(M)}
\]
Thus
\[
\text{Hess} (h + v(p)f) - v(p) \text{Hess} (f) = (\lambda_{h+v(p)f} - v(p)\lambda_f)g
\]
\[
= (\lambda_h + v(p)\lambda_f - v(p)\lambda_f)g = \lambda_h g
\]

However, from definition of Hessian we have,
\[
\text{Hess} (h + v(p)f) = \text{Hess} (h) + v(p) \text{Hess}(f)
\]

Substitute this in last equation to get \( \text{Hess} (h) = \lambda_h g \), that is \( h \in U(Q) \).

Using condition (3), for any \((q,p) \in M\), we have
\[
\lambda_h (q) + v(p)\lambda_f (q) = \frac{1}{f(q)} \{ \langle \nabla^Q (h + v(p)f), \nabla^Q f \rangle (q) + \lambda_v (p) \}
\]
that is,
\[
f (\lambda_h + v(p)\lambda_f) (q) = \langle \nabla^Q h, \nabla^Q f \rangle (q) + v(p) \| \nabla^Q f \| ^2 (q) + \lambda_v (p)
\]
or
\[
(f\lambda_h - \langle \nabla^Q h, \nabla^Q f \rangle) (q) = v(p) \{ \| \nabla^Q f \| ^2 - f\lambda_f \} (q) + \lambda_v (p)
\]
(3.4.15)

Now, as
\[
\langle \langle f,f \rangle \rangle_K = 2f\lambda_f - \langle \nabla^Q f, \nabla^Q f \rangle + Kf^2
\]
\[
= f\lambda_f + f (\lambda_f + Kf) - \langle \nabla^Q f, \nabla^Q f \rangle
\]
\[
= f\lambda_f + f \langle \langle f,1 \rangle \rangle_K - \langle \nabla^Q f, \nabla^Q f \rangle
\]
and, \( \langle \langle f,1 \rangle \rangle_K = 0 \), it follows that \( \langle \langle f,f \rangle \rangle_K = f\lambda_f - \| \nabla^Q f \| ^2 \). This together with (3.4.15), for each \((q,p) \in M\) gives
\[
(f\lambda_h - \langle \nabla^Q h, \nabla^Q f \rangle) (q) = \lambda_v (p) - v(p) \langle \langle f,f \rangle \rangle_K (q)
\]
But, in this equation the right hand side is function on $q$ only as will as the left hand side is function on $p$ only, and thus we conclude that both sides of equation must be constant and therefore,

$$0 = A \left( f \lambda_h - \langle \nabla^Q h, \nabla^Q f \rangle \right) (q) = f A (\lambda_h + K h), \quad \forall A \in \chi (Q)$$

Hence, $0 = A (\lambda_h + K h), \forall A \in \chi (Q)$, which implies $h \in U_K (Q)$, and substituting this in condition (3), we get $v \in U_L (P)$, where $L = - \langle \langle f, f \rangle \rangle$, with $\langle \langle v, 1 \rangle \rangle = \langle \langle h, f \rangle \rangle$. 
REFERENCES


[10] Osgood, B. and Stowe, D. TheSchwarzian Derivative And Conformal
