GEOMETRY OF SLANT SUBMANIFOLDS

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To my wonderful parents and my husband who set me on my path

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PREFACE

For a given Riemannian manifold (M, g), to study its geometry, some times it is convenient to immerse it in some Riemannian manifold with known geometry and then analyze the induced geometry. This lead to the study of geometry of submanifold. In that, most interesting is the geometry of submanifold of a complex manifold owing to the existence of complex structure which sends each vector to a vector which is orthogonal to the original vector. This leads to the study of two important classes of submanifolds of a complex manifold namely invariant and anti-invariant submanifolds. These submanifolds are generalized by B.Y. Chen by introducing what are known as Slant submanifolds [3]. This thesis is devoted to the study of Slant submanifolds and all the results are taken from the papers cited in [4], [5], [6], [8] and [9].

The thesis is divided into four chapters and each chapter is divided into subsections and the results in each section are numbered as $(a.b.c)$, for instance Theorem $a.b.c$, means Theorem number c in the section b of chapter a.

The first chapter is introductory and is basically intended to make the thesis as self-contained as possible. In this chapter we gave basic definitions and summarized the basic formulae and results on Riemannian manifolds, complex manifolds and submanifolds, which are essential for the other three chapters.

The second chapter is devoted to the study of the slant submanifolds In the first preliminary section, we introduce the notion of a slant submanifold of an almost Hermitian manifold and the operators P and F which arise naturally from the definition of a slant manifold as well as prepare the basic fundamental equations of submanifold theory in terms of moving frame. In section two we give some examples of slant submanifolds and in the last section we study properties of the operators P and F and the results on the geometry of slant submanifolds with restrictions on these operators.

In chapter three we are interested in slant submanifolds of a complex space form (A Kaehler manifold of constant holomorphic sectional curvature c). In first section we prepare for basic equations of submanifold theory for slant submanifolds of complex space form and in then in second section we study the existence theorem and uniqueness theorem for slant submanifolds in these spaces.

In last chapter, we are interested in deriving an inequality satisfied by the mean curvature vector of a slant submanifold of a complex space form. First we prepare some Lemmas in the first section for slant submanifolds of a complex space form. In second section we use the Lemmas in first section to prove a classification theorem for H-umbilical slant submanifolds of complex space forms. In last section we prove and inequality for the length of mean curvature vector of a slant submanifold of a complex space form. This inequality is an import result in it self as it gives an estimate for the length of mean curvature vector, however it is used in classifying cylindrical slant submanifolds of a complex space form which we do not consider here as it requires some more preparation not intended in this thesis.

The thesis ends with a list of references, which by no means is exhaustive on the subject, but lists only those references which have either been directly used in the thesis or have relevance to our work.

CHAPTER I

INTRODUCTION

In this introductory chapter, we will describe basic definitions, results and formulas which are related to our subsequent chapters. Throughout this thesis, we will denote by T_pM the tangent space to M at $p \in M$, by $\mathfrak{X}(M)$ the lie-algebra of the smooth vector field on M and by $C^{\infty}(M)$ the ring of smooth functions on M. The differential of the map $f : M \to N$ at $p \in M$ is denoted by df_p which is the linear map $df_p : T_pM \to T_{f(p)}N$.

1.1 RIEMANNIAN MANIFOLDS

In this section we will discuss the connection on a manifold, a Riemannian metric, the Riemannian connection on a Riemannian manifold and the properties of curvature tensor, Ricci tensor and scalar curvature. Moreover we will discuss some special types of spaces such as space of constant curvature, followed by some examples.

Definition 1.1.1 Let $f : M \to N$ be a smooth map. Then

1) f is called an immersion if $df_p : T_pM \to T_{f(p)}N$ is one-to-one map for all $p \in M$.

2) f is called imbedding if f is one-to-one immersion.

3) f is called diffeomorphism if f is a bijection and f^{-1} is smooth.

Now we state the Implicit function theorem.

Theorem 1.1.2 Suppose $f : M \to N$ is a smooth map and that $df_q : T_qM \to T_pN$ is onto for all $q \in M$ with $f(q) = p$. Then the set $F = \{q \in M : f(q) = p\}$ is a smooth manifold and $\dim F = \dim M - \dim N$, moreover the inclusion $i : F \to M$ is an imbedding.

Definition 1.1.3 Let M be an n-dimensional smooth manifold. A Riemannian metric on M is a tensor field g of type $(0, 2)$ which satisfies:

(i) g is symmetric that is $g(X, Y) = g(Y, X), \forall\ X, Y \in \mathfrak{X}(M)$.

(ii) for each $p \in M$, g_p is positive definite non-degenerate bilinear form on T_pM. That is $g_p(X_p, Y_p) \geq 0, \forall\ X_p \in T_pM$, $g_p(Y_p, Y_p) = 0$ implies $Y_p = 0$ and if $g_p(X_p, Y_p) = 0, \forall Y_p \in T_pM$ then $X_p = 0$.

A smooth manifold M together with a given Riemannian metric g is called a Riemannian manifold, and denoted by (M, g).

Example 1.1.4 Let $h : R^n \to R$ be map given by $h(x_1, \cdots, x_n) = \sum\limits_{i=1}^{n}(x_i^2 - 1)$. Then 0 is a regular value of h and $h^{-1}(0) = \left\{x_i \in R^n : x_1^2 + \cdots + x_n^2 = 1\right\} = S^{n-1}$ is the unit sphere of R^n. The metric induced from R^n on S^{n-1} is called the canonical metric of S^{n-1}, which is a Riemannian metric.

Definition1.1.5 (i) Let M be a Riemannian manifold of dimension $m = n + k$, and let us suppose that to each $p \in M$ is assigned an n-dimensional subspace D_p of T_pM. Suppose moreover that in a neighborhood U of each $p \in M$ there are n-linearly independent smooth vector fields $X_1, \cdots, X_n$, whose values at the point q form a basis of D_q for every $q \in U$. Then we shall say that D is a smooth distribution of dimension n on M, and $X_1, \cdots, X_n$ is a local basis of D, and for a vector field $X \in \mathfrak{X}(M)$, if $X_p \in D_p$ for each $p \in M$, we say $X \in D$.

(ii) We shall say that the distribution D is involutive if $[X, Y] \in D$,$\forall$ $X, Y \in D$.

(iii) Finally, if D is a smooth distribution on M, we say that D is integrable, if for each $p \in M$, there exists an n-dimensional submanifold N containing p of M such that $T_q N = D_q, \forall q \in N$, and then N is called a leaf of D passing through the point p.

Now we state the following theorem of Frobenius.

Theorem 1.1.6 A distribution D on a manifold M is integrable if and only if it is involutive.

Next we introduce a connection on a smooth manifold.

Definition 1.1.7 A connection ∇ on a smooth manifold M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \to \nabla_X Y$, which satisfy the following:

(i) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

(ii) $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$

(iii)$\nabla_{fX} Y = f \nabla_X Y$

(iv)$\nabla_X (fY) = X(f)Y + f\nabla_X Y,\ X, Y, Z \in \mathfrak{X}(M),\ f \in C^\infty (M)$.

Example 1.1.8 Let $(x_1, \cdots, x_n)$ be the coordinate system on R^n. Then for X,$Y \in \mathfrak{X}(R^n)$, we have $X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^{n} g^i \frac{\partial}{\partial x_i}$, $f^i, g^i \in C^\infty (R^n)$. Define $\nabla : \mathfrak{X}(R^n) \times \mathfrak{X}(R^n) \to \mathfrak{X}(R^n)$ by $\nabla_X Y = \sum_{i=1}^{n} X\left(g^i\right) \frac{\partial}{\partial x_i}$, then it can be easily verified that ∇ satisfies the requirement for a connection. This connection ∇ on R^n is known as the Euclidean connection.

Remark 1.1.9 On a smooth manifold M there could be several Riemannian metrics (once existence of one is known), for instance if g is a Riemannian metric on M and $f : M \to M$ is a non-singular smooth map (that is the matrix (df_p) is non-singular of each $p \in M$), then $g(X, Y) = f^* g(X, Y) = g(df(X), df(Y))$ is also a Riemannian metric on M.

Definition 1.1.10 (i) A connection ∇ on Riemannian manifold (M, g) with $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \forall X, Y, Z \in \mathfrak{X}(M)$ is called connection compatible to the metric g.

(ii) A connection ∇ on a smooth manifold M is said to be symmetric, if $\nabla_X Y - \nabla_Y X = [X, Y],\ X, Y \in \mathfrak{X}(M)$.

Theorem(Levi-civita) 1.1.11 Given a Riemannian manifold (M, g) there exists a unique connection ∇ on M satisfying:

(i) ∇ is symmetric

(ii) ∇ is compatible with g.

The unique connection ∇ on the Riemannian manifold (M, g) satisfying (i) & (ii) in (1.1.11) is called the Riemannian connection.

Definition 1.1.12 For a connection ∇ on a smooth manifold M, there is associated a tensor field R of type $(1,3)$ called the curvature tensor field of the connection ∇ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M)$$

where $[X, Y]$ is the Lie-bracket of the vector fields X, Y.

The properties of the curvature tensor R of the Riemannian connection ∇ on (M, g) are summarized in the following :

Theorem 1.1.13 The curvature tensor R of the Riemannian connection ∇ on a Riemannian manifold (M, g) satisfies:

(i) $R(X,Y)Z + R(Y,X)Z = 0$.

(ii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

(iii) $R(X,Y;Z,W) = R(Z,W;X,Y), \quad X, Y, Z, W \in \mathfrak{X}(M)$,

where $R(X,Y;Z,W) = g(R(X,Y)Z, W)$.

Definition 1.1.14 Let P be a plane section on (M, g). The sectional curvature of the plane P is defined by

$$K(P) = \frac{R(X,Y;Y,X)}{X \wedge Y}$$

where $X, Y \in \mathfrak{X}(M)$ are vector fields which span P, and $X \wedge Y = g(X,X)\, g(Y,Y) - g(X,Y)^2$.

If $K(P)$ is a constant c for all plane sections P on M, then M is said to have constant sectional curvature c.

For constant sectional curvature manifolds, we will have a simple formula for R given in the following theorem.

Theorem 1.1.15 If the Riemannian manifold (M, g) is of constant sectional curvature c, then its curvature tensor field is given by

$$R(X,Y)Z = c\left\{g(Y,Z)X - g(X,Z)Y\right\}, \quad X, Y, Z \in \mathfrak{X}(M)$$

Definition 1.1.16 Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame on a Riemannian manifold (M, g) i.e. it satisfies $g(e_i, e_j) = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$. If R is the tensor of type $(0,4)$ described in 1.1.13, then the Ricci tensor field Ric of M is defined by:

$$Ric(X,Y) = \sum_{i=1}^{n} R(e_i, X; Y, e_i), X, Y \in \mathfrak{X}(M)$$

and the scaler curvature S of M is the trace of the Ricci tensor, that is , S is defined by : $S = \sum\limits_{i=1}^{n} Ric(e_i, e_i)$.

Example 1.1.17 Consider (R^n, g) where g is the Euclidean inner product on R^n defined by $g(X,Y) = \sum\limits_{i=1}^{n} g^i f^i$, where $X = \sum\limits_{i=1}^{n} f^i \frac{\partial}{\partial x^i}$, $Y = \sum\limits_{i=1}^{n} g^i \frac{\partial}{\partial x^i}$, $f^i, g^i \in C^\infty(R^n)$. Then according to the Euclidean connection defined in example 1.1.8 it can be easily verified that ∇ is a Riemannian connection with respect to g. The curvature tensor field R will given by $R(X,Y)Z = \sum_{i=1}^{n} XY\left(h^i\right)\frac{\partial}{\partial x^i} - \sum_{i=1}^{n} YX\left(h^i\right)\frac{\partial}{\partial x^i} - \sum_{i=1}^{n}[X,Y]\left(h^i\right)\frac{\partial}{\partial x^i} = 0$, where $Z = \sum_{i=1}^{n} h^i \frac{\partial}{\partial x^i}$. Thus we get that $g(R(X,Y)Z,W) = 0,\ X,Y,Z,W \in \mathfrak{X}(R^n)$, which means that $K(P) = 0$, for any plane section P of R^n and thus R^n is of constant sectional curvature zero.

1.2 COMPLEX MANIFOLDS

In this section we discuss different classes of almost complex manifolds.

Definition 1.2.1 An almost complex structure on a smooth manifold M is a tensor field J of type $(1,1)$ which is, at every point p of M, an endomorphism of T_pM such that $J_p^2 = -I_p$, where I_p denotes the identity transformation of T_pM. A smooth manifold M together with an almost complex structure J is called an almost complex manifold (M,J). A complex manifold M is an analytic manifold where in the definition of analytic structure we replace the real Euclidean space R^n with the complex Euclidean space C^n.

Remark 1.2.2 Clearly from definition 1.2.1, it follows that an almost complex manifold is of even dimension and that an n-dimensional complex manifold is a $2n$-dimensional real analytic manifold.

Example 1.2.3 R^{2n}, S^2, CP^n and TM the tangent bundle of a smooth manifold M with a given connection are almost complex manifolds, of which $R^{2n} = C^n$, S^2 and CP^n are complex manifolds. The natural almost complex structure J of R^{2n} is defined by $J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$, $J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$, where

$x^1, ..., x^n, y^1, ..., y^n$ are Euclidean coordinates on R^{2n}.

Definition 1.2.4 A Hermitian metric on an almost complex manifold (M, J) is a Riemannian metric g such that $g(JX, JY) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$, that is, the Riemannian metric g and J are compatible.

An almost complex manifold (respectively a complex manifold) with Hermitian metric is called an almost Hermitian manifold (M, J, g) (respectively a Hermitian manifold). Notice that every almost complex manifold M with a Riemannian metric g admits an almost Hermitian metric. Indeed, for any almost complex structure J on M with a Riemannian metric g, putting

$$h(X, Y) = g(X, Y) + g(JX, JY), X, Y \in \mathfrak{X}(M)$$

we obtain an almost Hermitian metric h.

Definition 1.2.5 An almost Hermitian manifold (M, J, g) is called a Kaehler manifold if the almost complex structure J of M is parallel, that is,

$$(\nabla_X J)(Y) = 0, X, Y \in \mathfrak{X}(M)$$

It is easy to verify that the Euclidean metric on R^{2n} is Hermitian metric with respect to the natural almost complex structure J defined above and that R^{2n} is a Kaehler manifold.

Let (M, J, g) be a Kaehler manifold. Let $K(P)$ be the sectional curvature of M for a plane $P \subset T_q M$ spanned by orthonormal vectors X, Y . If P is invariant under J; that is, $J_q P = P$, then $K(P)$ is called holomorphic sectional curvature. If P is invariant under J and X is a unit vector in P, then $\{X, JX\}$ is orthonormal basis for P and hence $K(P) = R(X, JX; JX, X)$, we denote it by $H(X)$. If $K(P)$ is constant for all J-invariant planes $P \subset T_q M$, for all points q of M, then M is called space of constant holomorphic sectional curvature.

We have the following characterization for the spaces of constant holomorphic sectional curvature.

A Kaehler manifold (M, J, g) of constant holomorphic sectional curvature c is called a complex space form and is denoted by $M(c)$. The curvature tensor R of $M(c)$ is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ &+ 2g(X,JY)JZ\} \end{aligned}$$

We denoted by C^n the complex space, CP^n the complex projective space and D^n the open unit ball in C^n of complex dimension n. It is known that a simply connected complete Kaehler manifold of constant holomorphic sectional curvature c can be identified with CP^n, D^n or C^n according as $c > 0$, $c < 0$ or $c = 0$.

1.3 SUBMANIFOLDS

Given two smooth manifolds M and $\overline{M}$, if there exists a smooth immersion $\psi : M \to \overline{M}$, then we say that M is a submanifold of $\overline{M}$. If ψ is imbedding, then M is said to be an imbedded submanifold of $\overline{M}$. The geometry of submanifolds is a very useful branch in geometry, for some times it is convenient to immerse the given manifold into a known Riemannian manifold (if it is possible[1]) and then study the induced geometry on this manifold as submanifold of the known Riemannian. In fact the study of Differential Geometry started with the geometry of the submanifolds. For instance the regular smooth curves in R^3 are 1-dimensional submanifolds of R^3 as well as the regular surfaces in R^3 are its 2-dimensional submanifolds. Let $\left(\overline{M}, g\right)$ be a Riemannian manifold and $\psi : M \to \overline{M}$ be a smooth immersion of a smooth manifold M into $\overline{M}$. Then the submanifold M receives the induced Riemannian metric g^* defined by

$$g_p^*\left(X_p, Y_p\right) = g\left(d\psi_p X_p, d\psi_p Y_p\right), \; X, Y \in \mathfrak{X}\left(M\right), \; p \in M$$

Since an immersion is a local imbedding, when we are dealing with local expressions on a submanifold M of $\overline{M}$ we shall identify $d\psi_p X_p$ by X_p , $X \in \mathfrak{X}\left(M\right)$ and $\psi\left(p\right)$ by p, $p \in M$, and distinguish the entries on the manifolds M and $\overline{M}$ from the context. Thus we shall also denote by g the induced Riemannian metric on M; and we shall say M is a submanifold of a Riemannian manifold $\overline{M}$ without referring to the immersion.

[1] It is known that the real projective space RP^3 dose not admit any smooth immersion in R^4 which induces the usual metric on RP^3 of constant positive curvature.

Let M be a submanifold of a Riemannian manifold $\left(\overline{M}, g\right)$. Then for each $p \in M$, the tangent space $T_p\overline{M}$ of $\overline{M}$ is the direct sum $T_p\overline{M} = T_pM \oplus T_p^{\perp}M$, where T_pM is the tangent space of M and

$$T_p^{\perp}M = \left\{X_p \in T_p\overline{M} \ / \ g_p\left(X_p, Y_p\right) = 0,\ Y_p \in T_pM\right\}$$

is the orthogonal complement of T_pM in $T_p\overline{M}$. the subspace $T_p^{\perp}M$ of $T_p\overline{M}$ is said to be the normal space of M at $p \in M$. the bundle $T^{\perp}M = \underset{p\in M}{\cup} T_p^{\perp}M$ over M is called the normal bundle of M. We denote by υ the space of smooth sections of the normal bundle $T^{\perp}M$. Thus $N \in \upsilon$ implies N is a smooth vector field in $\mathfrak{X}\left(\overline{M}\right)$ and satisfies $g\left(X, N\right) = 0,\ X \in \mathfrak{X}\left(M\right)$. Though it is abuse of term, but υ it self is referred to as normal bundle (instead of space of sections on the normal bundle) in the literature on the geometry of submanifolds and so in this thesis.

Let $\overline{\nabla}$ be the Riemannian connection on the Riemannian manifold $\left(\overline{M}, g\right)$ and M be a submanifold of $\overline{M}$. Then using $\mathfrak{X}\left(\overline{M}\right)|_M = \mathfrak{X}\left(M\right) \oplus \upsilon$, where $\mathfrak{X}\left(\overline{M}\right)|_M$is the restriction of $\mathfrak{X}\left(\overline{M}\right)$ to M , for $X, Y \in \mathfrak{X}\left(M\right)$; as $\overline{\nabla}_XY \in \mathfrak{X}\left(\overline{M}\right)$, we express $\overline{\nabla}_XY$ as

$$\overline{\nabla}_XY = \nabla_XY + h\left(X, Y\right) \tag{1.3.1}$$

where $\nabla_XY \in \mathfrak{X}\left(M\right)$ and $h\left(X, Y\right) \in \upsilon$ are respectively the tangential and normal components of $\overline{\nabla}_XY$. At this stage the terms ∇_XY and $h\left(X, Y\right)$ are merely the notations till their properties are revealed to distinguish them geometrically . Utilizing the properties of the Riemannian connection $\overline{\nabla}$ as given in definition 1.1.7, it easily follows that the symbol ∇ appearing in equation

(1.3.1) satisfies the properties required by the Riemannian connection on M. In this verification it also turns out that the normal component in equation (1.3.1) gives rise to the bilinear symmetric mapping $h : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \upsilon$ called the second fundamental form of the submanifold M. Similarly, for $X \in \mathfrak{X}(M), N \in \upsilon$ as $\overline{\nabla}_X N \in \mathfrak{X}(\overline{M})$, we can express it as

$$\overline{\nabla}_X N = -A_N X + D_X N \tag{1.3.2}$$

where $-A_N X \in \mathfrak{X}(M)$ and $D_X N \in \upsilon$ are respectively the tangential and normal components of $\overline{\nabla}_X N$. Using the properties of $\overline{\nabla}$, it turns out that the tangential component $-A_N X$ in equation (1.3.2) gives rise to the linear mapping $A_N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying $g(A_N X, Y) = g(X, A_N Y)$; and $D : \mathfrak{X}(M) \times \upsilon \to \upsilon$ satisfies the properties similar to those for a connection. The mapping A_N is called the Weingarten map with respect to the normal $N \in \upsilon$ and D is called the connection in the normal bundle υ.

Taking inner product by $Y \in \mathfrak{X}(M)$ in the equation (1.3.2) and using $g(Y, N) = 0$, that is, $g\left(\overline{\nabla}_X Y, N\right) = -g\left(Y, \overline{\nabla}_X N\right)$, and equation (1.3.1), we obtain

$$g(h(X, Y), N) = g(A_N X, Y), X, Y \in \mathfrak{X}(M), N \in \upsilon$$

The fundamental equations (1.3.1) and (1.3.2) for a submanifold M of a Riemannian manifold $\overline{M}$ are called the Gauss and Weingarten formulae.

Let $\overline{R}$ and R be the curvature tensors of $\overline{M}$ and M respectively. then using the Gauss and Weingarten formulae we obtain the following equation relating the curvature tensors R and $\overline{R}$.

$$\begin{aligned}\overline{R}(X,Y;Z,W) &= R(X,Y;Z,W)+g(h(X,Z),h(Y,W)) \\ &\quad -g(h(Y,Z),h(X,W)) \qquad (1.3.3)\end{aligned}$$

$X,Y,Z,W\in\mathfrak{X}(M)$.

Example 1.3.1: Let S^n be a unit sphere centered at origin in R^{n+1}. Then the inclusion $i:S^n\to R^{n+1}$ is an immersion, and thus S^n is a submanifold of R^{n+1} with each normal space is 1-dimensional and thus the normal bundle υ is spanned by a single unit normal vector field N. Since N represents the position vector field of each point on S^n and the Riemannian connection $\overline{\nabla}$ on R^{n+1} is defined in example 1.1.8, we have, for each $X\in\mathfrak{X}(S^n)$, that

$$\overline{\nabla}_X N=\sum_{i=1}^{n}X(x_i)\frac{\partial}{\partial x_i}=X$$

that is, $A_N X=-X$ and $D_X N=0$, where $N=\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}$ is the position vector field on S^n.

Also, $h(X,Y)=g(h(X,Y),N)N=g(A_N X,Y)N=-g(X,Y)N$, $X,Y\in\mathfrak{X}(S^n)$. Thus the Gauss and Weingarten formulae for S^n are

$$\overline{\nabla}_X Y=\nabla_X Y-g(X,Y)N$$

and $\overline{\nabla}_X N=X$. Since the curvature tensor of R^{n+1}, $\overline{R}=0$, the curvature tensor R of S^n as given in equation 1.3.2 is

$$R(X,Y;Z,W) = g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\,,\ X,Y,Z,W \in \mathfrak{X}(S^n)\,.$$

CHAPTER 2

In this chapter we introduce the notion of a slant submanifold in an almost Hermitian manifold. In section 2.1 we formulate the basic equations of submanifold in terms of local frame and state the fundamental equations. In section 2.2 we give some examples of slant submanifolds and in the last section 2.3 we obtain some fundamental properties of the tensors P and F for the submanifolds of an almost Hermitian manifolds and use them to study the geometry of slant submanifolds. Section 2.3 is taken mostly from [3] and [4].

2.1 PRILIMINARIES

let N be an n-dimensional Riemannian manifold isometrically immersed in an almost Hermitian manifold M with almost complex structure J and almost Hermitian metric g . We denote by $\langle , \rangle$ the inner product for N as well as for M. For any vector X tangent to N we put

$$JX = PX + FX \tag{2.1.1}$$

where PX and FX are the tangential and the normal components of JX respectively . Thus, P is an endomorphism of the tangent bundle TN and F a normal-bundle-valued 1-form on TN. The submanifold N is called a complex submanifold if $F = 0$ and is called a totally real submanifold if $P = 0$, and called proper if it is neither a complex submanifold nor a totally real submanifold.

Definition 2.1.1 Let $\left(\overline{M}, J, g\right)$ be an almost Hermitian manifold and M be a submanifold of $\overline{M}$, if for each $p \in M$ and $X_p \in T_pM$ the angle $\theta\left(X_p\right)$ between JX_p and the tangent space T_pM is constant α then M is called a slant submanifold.

Remark 2.1.2 If M is a slant submanifold of $\left(\overline{M}, J, g\right)$ and $p \in M$, then for a local orthonormal frame $\{e_1, ..., e_n\}$ of M we should have $|\langle Je_i, e_j\rangle| = \cos\alpha$ =constant.

Example 2.1.3 For any $\alpha > 0$, consider $f : R^2 \to R^4$ defined by

$$f\left(u, v\right) = \left(u\cos\alpha, u\sin\alpha, v, 0\right)$$

then at any point p of R^2, we have

$$df_p = \begin{bmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Let $\{e_1, e_2\}$ be a local orthonormal frame on R^2, then we can choose it as

$$e_1 = \frac{df_p\left(\frac{\partial}{\partial u}\right)}{\left\|df_p\left(\frac{\partial}{\partial u}\right)\right\|} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \end{pmatrix}$$

and

$$e_2 = \frac{df_p\left(\frac{\partial}{\partial v}\right)}{\left\|df_p\left(\frac{\partial}{\partial v}\right)\right\|} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

where $\left\|df_p\left(\frac{\partial}{\partial u}\right)\right\| = \left\|df_p\left(\frac{\partial}{\partial v}\right)\right\| = 1$. Let J_0 be the natural almost complex structure of R^4 as defined in chapter 1. Then we have

$$J_0e_1 = \begin{pmatrix} 0 & 0 & \cos\alpha & \sin\alpha \end{pmatrix}$$

$$J_0e_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}$$

Then we see that $\langle J_0e_1, e_2\rangle = \cos\alpha$, $\langle J_0e_2, e_1\rangle = -\cos\alpha$, $|\langle J_0e_i.e_j\rangle| = \cos\alpha > 0$, which is a constant. This implies that R^2 is slant submanifold of R^4, so f is defines a slant plane with slant angle α in R^4.

Let M be an n-dimensional submanifold in an m-dimensional Riemannian manifold $\overline{M}$. Choose a local orthonormal frames $e_1, ..., e_n, e_{n+1}, ...e_m$ such that, restricted to N, the vectors $e_1, ..., e_n$ are tangent to M and $e_{n+1}, ..., e_m$ are normal to M. Let $1 \leq A, B, C, ... \leq m$ and $1 \leq i, j, k, ... \leq n$. Suppose that $w^1, ..., w^n, w^{n+1}, ..., w^m$ be the dual frame. Then we have $w_B^A = -w_A^B \Rightarrow w_B^A + w_A^B = 0$ where $w^A(X) = \langle X, e_A\rangle$, $X \in \mathfrak{X}\left(\overline{M}\right)$ and $1 \leq A \leq m$. Define the connection forms w_B^A by

$$\overline{\nabla}_X e_B = \sum_{C=1}^{m} w_B^C(X)\, e_C$$

where $\overline{\nabla}$ is Riemannian connection on $\overline{M}$.

For any vector field $X \in \mathfrak{X}(M)$ these forms are also given by

$$\overline{\nabla}_X e_i = T + N'$$

where $T \in \mathfrak{X}(M)$ is the tangential component and $N' \in \Gamma(\upsilon)$ is the normal component. If we write $\overline{\nabla}_X e_i = \sum_j \lambda^j e_j + \sum_r \mu^r e_r$, then we obtain $\lambda^j = \left\langle \overline{\nabla}_X e_i, e_j \right\rangle = w_j^i(X)$ and $\mu^r = \left\langle \overline{\nabla}_X e_i, e_r \right\rangle = w_r^i(X)$. Thus we have

$$\overline{\nabla}_X e_i = \sum_{j=1}^{n} w_j^i(X)\, e_j + \sum_{r=n+1}^{m} w_r^i(X)\, e_r$$

and

$$\overline{\nabla}_X e_r = \sum_{j=1} w_j^r (X) e_j + \sum_{s=n+1} w_s^r (X) e_s$$

These 1-forms w_j^i, w_r^i and w_s^r are called the connection forms of M in $\overline{M}$.

Lemma 2.1.4

$$dw^A = -\sum_B w_B^A \wedge w^B$$

Proof: For $X, Y \in \mathfrak{X}\left(\overline{M}\right)$ we have

$$dw^A (X, Y) = X\left(w^A (Y)\right) - Y\left(w^A (X)\right) - w^A ([X, Y])$$

and

$$\left(w^B \wedge w_B^A\right)(X, Y) = w^B (X) w_B^A (Y) - w^B (Y) w_B^A (X)$$

Thus we arrive at

$$\sum_B \left(w^B \wedge w_B^A\right)(X, Y) = -\sum_B \left(w_B^A \wedge w^B\right)(X, Y)$$

and

$$\sum_B \left(w^B \wedge w_B^A\right)(X, Y) = dw^A (X, Y)$$

which proves

$$dw^A = -\sum_B w_B^A \wedge w^B$$

Now, suppose that ϕ_B^A be the curvature 2-form on $\overline{M}$, then the structure equations of $\overline{M}$ are given by

$$\phi_B^A (X, Y) = \frac{1}{2} \sum_{C,D} K_{BCD}^A w^C \wedge w^D (X, Y)$$

where, $K^A_{BCD} = R\left(e_A, e_B, e_C, e_D\right)$. Thus we have

$$\begin{aligned}\phi^A_B\left(X,Y\right) &= \frac{1}{2}\sum_{C,D} R\left(e_A, e_B, e_C, e_D\right)\left[w^C\left(X\right)w^D\left(Y\right) - w^C\left(Y\right)w^D\left(X\right)\right] \\ &= R\left(X, Y, e_A, e_B\right)\end{aligned}$$

Next, we prove

Lemma 2.1.5

$$dw^A_B = \sum_C w^A_C \wedge w^C_B + \phi^A_B$$

Proof: For $X, Y \in \mathfrak{X}(\overline{M})$, we compute

$$\begin{aligned}\sum_C w^A_C \wedge w^C_B\left(X,Y\right) + \phi^A_B\left(X,Y\right) &= \sum_C w^A_C\left(X\right)w^C_B\left(Y\right) - \sum_C w^A_C\left(Y\right)w^C_B\left(X\right) \\ &\quad + R\left(X, Y, e_A, e_B\right) \\ &= \sum_C \left\langle \overline{\nabla}_X e_A, e_C\right\rangle\left\langle \overline{\nabla}_Y e_C, e_B\right\rangle \\ &\quad - \sum_C \left\langle \overline{\nabla}_Y e_A, e_C\right\rangle\left\langle \overline{\nabla}_X e_C, e_B\right\rangle \\ &\quad + R\left(X, Y, e_A, e_B\right) \\ &= \sum_C \left\langle \overline{\nabla}_X e_A, e_C\right\rangle\left[Y\left\langle e_C, e_B\right\rangle - \left\langle e_C, \overline{\nabla}_Y e_B\right\rangle\right] \\ &\quad - \sum_C \left\langle \overline{\nabla}_Y e_A, e_C\right\rangle\left[X\left\langle e_C, e_B\right\rangle - \left\langle e_C, \overline{\nabla}_X e_B\right\rangle\right] \\ &\quad + R\left(X, Y, e_A, e_B\right)\end{aligned}$$

As $\langle e_C, e_B \rangle$ =constant, we get $Y \langle e_C, e_B \rangle = 0$, consequently that

$$\begin{aligned} \sum_C w_C^A \wedge w_B^C (X,Y) + \phi_B^A (X,Y) &= -\langle \nabla_X e_A, \nabla_Y e_B \rangle + \langle \nabla_Y e_A, \nabla_X e_B \rangle \\ &\quad + R(X,Y,e_A,e_B) \\ &= -\langle \nabla_X \nabla_Y e_A - \nabla_Y \nabla_X e_A, e_B \rangle - Y w_B^A (X) \\ &\quad + X w_B^A (Y) + R(X,Y,e_A,e_B) \\ &= -\left\langle R(X,Y) e_A + \nabla_{[X,Y]} e_A, e_B \right\rangle - Y w_B^A (X) \\ &\quad + X w_B^A (Y) + R(X,Y,e_A,e_B) \\ &= X w_B^A (Y) - Y w_B^A (X) - w_B^A ([X,Y]) \\ &= dw_B^A [X,Y] \end{aligned}$$

This proves that

$$dw_B^A = \sum_C w_C^A \wedge w_B^C + \phi_B^A$$

where $K_{BCD}^A + K_{BDC}^A = 0$.

Now, we restrict these forms to the submanifold M . Then we have $w^r = 0$, which gives

$$0 = dw^r = -\sum_i w_i^r \wedge w^i$$

Let $\{N_1, ...N_r\}$be the local frame for the normals. Then $\forall \eta \in \Gamma(\upsilon)$, $\eta = \sum_\alpha \langle \eta, N_\alpha \rangle N_\alpha$ as $e_{n+1}, ..., e_m$ are normal to M, so for any $\eta \in \Gamma(\upsilon)$, $\eta = \sum_r \langle \eta, e_r \rangle e_r$, $n+1 \leq r \leq m$ where $e_{n+1}, ..., e_m$ are normal to M. For the second fundamental form $h : \mathfrak{X}(\overline{M}) \times \mathfrak{X}(\overline{M}) \to \Gamma(\upsilon)$, we have $h(e_i, e_j) \in \Gamma(\upsilon)$, and consequently $h(e_i, e_j) = \sum_r h_{ij}^r e_r$, which gives $\langle h(e_i, e_j), e_r \rangle = h_{ij}^r$. Also we have

$$\sum_j h_{ij}^r w^j (X) = \sum_j \langle h(e_i, e_j), e_r \rangle \langle X, e_j \rangle$$

$$= \left\langle h\left(e_i, \sum_j \langle X, e_j \rangle e_j\right), e_r \right\rangle = \langle h(X, e_i), e_r \rangle = \langle e_i, A_{e_r} X \rangle,$$

where A_{e_r} is Weingarten map and as $D_X e_r \in \Gamma(v)$, $\overline{\nabla}_X e_r = -A_{e_r} X + D_X e_r$

So $\langle e_i, D_X e_r \rangle = 0$, where $\overline{\nabla}$ be connection on $\overline{M}$. Thus we have

$$\begin{aligned} \sum_j h_{ij}^r w^j (X) &= \langle e_i, A_{e_r} X \rangle - \langle e_i, D_X e_r \rangle = -\left\langle e_i, \overline{\nabla}_X e_r \right\rangle \\ &= \left\langle \overline{\nabla}_X e_i, e_r \right\rangle - X \langle e_i, e_r \rangle = \left\langle \overline{\nabla}_X e_i, e_r \right\rangle \\ &= w_i^r (X) \end{aligned}$$

This proves $w_i^r = \sum_j h_{ij}^r w^j$, where h is symmetric $h_{ij}^r = h_{ji}^r$. Also we have

$$\begin{aligned} w_i^r (X) &= \left\langle \overline{\nabla}_X e_i, e_r \right\rangle = X \langle e_i, e_r \rangle - \left\langle e_i, \overline{\nabla}_X e_r \right\rangle \\ &= -\langle e_i, -A_{e_r} X + D_X e_r \rangle = \langle e_i, A_{e_r} X \rangle = \langle A_{e_r} e_i, X \rangle \end{aligned}$$

where we have used the fact that the Weingarten maps are symmetric.

Now , for any $X \in \mathfrak{X}(M)$, if $\nabla, \overline{\nabla}$ are the Riemannian connections on M and $\overline{M}$ respectively. Then the components of curvature tensor fields R and K corresponding to ∇ and $\overline{\nabla}$ satisfy the relation given in the following

Lemma 2.1.6

$$R_{ijkl} = K_{ijkl} - \sum_r \left(h_{ik}^r h_{jl}^r - h_{il}^r h_{jk}^r \right)$$

Proof: We have with the choice of above local orthonormal frame

$$\begin{aligned}\sum_r \left(h^r_{ik}h^r_{jl} - h^r_{il}h^r_{jk}\right) &= \sum_r \langle h(e_i,e_k), e_r\rangle \langle h(e_j,e_l), e_r\rangle \\ &\quad -\sum_r \langle h(e_i,e_l), e_r\rangle \langle h(e_j,e_k), e_r\rangle \\ &= \langle h(e_i,e_k), h(e_j,e_l)\rangle - \langle h(e_i,e_l), h(e_j,e_k)\rangle\end{aligned}$$

Also,

$$K(e_i,e_j)\,e_k = \overline{\nabla}_{e_i}\overline{\nabla}_{e_j}e_k - \overline{\nabla}_{e_j}\overline{\nabla}_{e_i}e_k - \overline{\nabla}_{[e_i,e_j]}e_k$$

Then for any $X, Y \in X(M)$ and $N \in \Gamma(\upsilon)$, using the equations (1.3.1) and (1.3.2),

we compute

$$\begin{aligned}K(e_i,e_j)\,e_k &= \nabla_{e_i}\nabla_{e_j}e_k - \nabla_{e_j}\nabla_{e_i}e_k - \nabla_{[e_i,e_j]}e_k + A_{h(e_i,e_k)}e_j - A_{h(e_j,e_k)}e_i \\ &\quad + D_{e_i}h(e_j,e_k) - D_{e_j}h(e_i,e_k) + h\left(e_i, \nabla_{e_j}e_k\right) \\ &\quad - h(e_j, \nabla_{e_i}e_k) - h([e_i,e_j], e_k)\end{aligned}$$

Thus the components of K are given by

$$\begin{aligned}K_{ijkl} &= K(e_i,e_j,e_k,e_l) = \langle K(e_i,e_j)\,e_k, e_l\rangle = \langle R(e_i,e_j)\,e_k, e_l\rangle \\ &\quad + \left\langle A_{h(e_i,e_k)}e_j, e_l\right\rangle - \left\langle A_{h(e_j,e_k)}e_i, e_l\right\rangle \\ &= R(e_i,e_j,e_k,e_l) + \langle h(e_i,e_k), h(e_j,e_l)\rangle - \langle h(e_i,e_l), h(e_j,e_k)\rangle \\ &= R_{ijkl} + \sum_r \left(h^r_{ik}h^r_{jl} - h^r_{il}h^r_{jk}\right)\end{aligned}$$

Lemma 2.1.7

$$K^r_{skl} = R^r_{skl} + \sum_i \left(h^r_{ik}h^s_{il} - h^r_{il}h^s_{ik}\right)$$

Proof: Continuing with the same local orthonormal frame we have $w_i^r = \sum_j h_{ij}^r w^j$, and $dw_B^A = \sum_C w_C^A \wedge w_B^C + \phi_B^A$, $1 \leq C \leq m$. Thus $dw_i^r = \sum_j w_j^r \wedge w_i^j + \sum_s w_s^r \wedge w_j^s + \phi_i^r$

$$= \sum_j \left(\sum_k h_{jk}^r w^k \right) \wedge w_i^j + \sum_s w_s^r \wedge \left(\sum_j h_{ij}^s w^j \right) + \phi_i^r$$

$$= \sum_{j,k} h_{jk}^r \left(w^k \wedge w_i^j \right) + \sum_{j,s} h_{ij}^s \left(w_s^r \wedge w^j \right) + \phi_i^r$$

$$dw_i^r = \sum_{j,k} h_{jk}^r \left(w^k \wedge w_i^j \right) + \sum_{j,s} h_{ij}^s \left(w_s^r \wedge w^j \right) + \tfrac{1}{2} \sum_{j,k} K_{ijk}^r w^j \wedge w^k.$$

and $dw_s^r = \sum_t w_t^r \wedge w_s^t + \Omega_s^r$, where $\Omega_s^r = \frac{1}{2} \sum_{k,l} R_{skl}^r w^k \wedge w^t$.

Now

$$\begin{aligned} K(e_r, e_s, e_k, e_l) &= K(e_k, e_l, e_r, e_s) \\ &= \left\langle \overline{\nabla}_{e_k} \overline{\nabla}_{e_l} e_r - \overline{\nabla}_{e_l} \overline{\nabla}_{e_k} e_r - \overline{\nabla}_{[e_k, e_l]} e_r, e_s \right\rangle \end{aligned}$$

and

$$\begin{aligned} \overline{\nabla}_{e_k} \overline{\nabla}_{e_l} e_r &= \overline{\nabla}_{e_k} \left(-A_{e_r} e_l + D_{e_l} e_r \right) = -h\left(e_k, A_{e_r} e_l \right) + D_{e_k} D_{e_l} e_r \\ &\quad + (\text{tangential part}) \end{aligned}$$

$$\overline{\nabla}_{e_l} \overline{\nabla}_{e_k} e_r = -h\left(e_l, A_{e_r} e_k \right) + D_{e_l} D_{e_k} e_r + (\text{tangential part})$$

$$\overline{\nabla}_{[e_k, e_l]} e_r = D_{[e_k, e_l]} e_r + (\text{tangential part})$$

Thus

$$\begin{aligned} K(e_r, e_s, e_k, e_l) &= R^{\perp}(e_k, e_l, e_r, e_s) + \langle h(e_l, A_{e_r} e_k), e_s \rangle - \langle h(e_k, A_{e_r} e_l), e_s \rangle \\ &= R^{\perp}(e_r, e_s, e_k, e_l) + \langle A_{e_s} e_l, A_{e_r} e_k \rangle - \langle A_{e_s} e_k, A_{e_r} e_l \rangle \end{aligned}$$

where $R^{\perp}$ is the curvature tensor of the normal connection. Since, $A_{e_r}e_k$ is tangential,

$$A_{e_r}e_k = \sum_i \langle A_{e_r}e_k, e_i\rangle e_i = \sum_i \langle h(e_i, e_k), e_r\rangle e_i = \sum_i h_{ik}^r e_i$$

we get

$$\begin{aligned} K(e_r, e_s, e_k, e_l) &= R^{\perp}(e_r, e_s, e_k, e_l) + \sum_i h_{ik}^r \langle A_{e_s}e_l, e_i\rangle - \sum_i h_{il}^r \langle A_{e_s}e_k, e_i\rangle \\ &= R^{\perp}(e_r, e_s, e_k, e_l) + \sum_i h_{ik}^r h_{il}^s - \sum_i h_{il}^r h_{ik}^s \end{aligned}$$

Thus we have

$$K_{skl}^r = R_{skl}^r + \sum_i (h_{ik}^r h_{il}^s - h_{il}^r h_{ik}^s)$$

Finally we have the following Lemma which related the curvature tensors $\overline{R}, R$ and $R^{\perp}$ of the connections $\overline{\nabla}, \nabla$ and D, the first equation is called the equation of Gauss the second is called the equation of Codazzi and the third is called equation of Ricci

Lemma 2.1.8

$$\overline{R}(X, Y; Z, W) = R(X, Y; Z, W) + \langle h(X, Z), h(Y, W)\rangle - \langle h(X, W), h(Y, Z)\rangle \tag{i}$$

$$\left[\overline{R}(X, Y)Z\right]^{\perp} = \left(\overline{\nabla}_X h\right)(Y, Z) - \left(\overline{\nabla}_Y h\right)(X, Z) \tag{ii}$$

$$R^D(X, Y; \xi, \eta) = \overline{R}(X, Y; \xi, \eta) + \langle [A_\xi, A_\eta](X), Y\rangle \tag{iii}$$

where $\left(\overline{\nabla}_X h\right)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, $\left[\overline{R}(X, Y)Z\right]^{\perp}$ denotes the normal component of $\overline{R}(X, Y)Z$, $X, Y, Z, W \in \mathfrak{X}(\overline{M})$, ξ,$\eta \in \Gamma(\upsilon)$.

Definition 2.1.9 A submanifold M of a Riemannian manifold $\overline{M}$ is called a parallel submanifold if the second fundamental form h is parallel , that is $\overline{\nabla}h = 0$ identically.

2.2 EXAMPLES

In the following , R^{2m} denotes the Euclidean $2m$-space with the standard metric . An almost complex structure J on R^{2m} is said to be compatible if $\left(R^{2m}, J\right)$ is complex analytically isometric to the complex number space C^m with standard flat Kaehlerian metric. We denote by J_0 (the almost complex structure introduced in chapter 1) and J_1^- (when m is even) the compatible almost complex structure on R^{2m} defined respectively by

$$J_0\left(a_1, ...a_m, b_1, ..., b_m\right) = \left(-b_1, ..., -b_m, a_1, ..., a_m\right)$$

$$J_1^-\left(a_1, ..., a_m, b_1, ..., b_m\right) = \left(-a_2, a_1, ..., -a_m, a_{m-1}, b_2, -b_1, ..., b_m, -b_{m-1}\right)$$

In this section we give some examples of proper slant submanifolds in $C^2 = \left(R^4, J_0\right)$.

Example 2.2.1 Let M be a complex surface in C^2. Then for any constant α , $0 < \alpha \leq \frac{\pi}{2}$, M is slant surface in $\left(R^4, J_\alpha\right)$ with slant angle α, where J_α is the compatible almost complex structure on R^4 defined by

$$J_\alpha\left(a, b, c, d\right) = \left(\cos\alpha\right)\left(-c, -d, a, b\right) + \left(\sin\alpha\right)\left(-b, a, d. - c\right)$$

Since complex submanifolds in a Kaehler manifold are minimal. This example shows that there exist infinitely many proper slant minimal surfaces in C^2. It is clear that

$$J_\alpha (a, b, c, d) = (\cos \alpha) J_0 (a, b, c, d) + (\sin \alpha) J_1^- (a, b, c, d)$$

and it is easy to show that J_α is an almost complex structure on C^2.

The following example provides us non-minimal proper slant surfaces in C^2.

Example 2.2.2 For any positive constant k, define $f : R^2 \to R^4$, by $f(u, v) = (u, k \cos v, v, k \sin v)$. Then for any point $p = (a, b) \in R^2$, we get

$$df_p = \begin{bmatrix} 1 & 0 \\ 0 & -k \sin b \\ 0 & 1 \\ 0 & k \cos b \end{bmatrix}$$

We compute $\{e_1, e_2\}$ a local orthonormal frame on R^2 as follows: We have

$$\left\| df_p \left(\tfrac{\partial}{\partial u}\right)_p \right\| = 1 \ , \ \left\| df_p \left(\tfrac{\partial}{\partial v}\right)_p \right\| = \sqrt{k^2 \sin^2 b + k^2 \cos^2 b + 1} = \sqrt{k^2 + 1}$$

Thus we choose $e_1 = \frac{df\left(\frac{\partial}{\partial u}\right)}{\left\|df\left(\frac{\partial}{\partial u}\right)\right\|} = (1, 0, 0, 0)$ and

$e_2 = \frac{df\left(\frac{\partial}{\partial v}\right)}{\left\|df\left(\frac{\partial}{\partial v}\right)\right\|} = \frac{1}{\sqrt{k^2+1}} (0, -k \sin v, 1, k \cos v)$.

As $J_0 e_1 = (0, 0, 1, 0)$, $J_0 e_2 = \frac{1}{\sqrt{k^2+1}} (-1, -k \cos v, 0, -k \sin v)$ we have

$$\langle J_0 e_1, e_2 \rangle = \frac{1}{\sqrt{k^2 + 1}} (0 + 0 + 1 + 0) = \frac{1}{\sqrt{k^2 + 1}}$$

and

$$\langle J_0 e_2, e_1 \rangle = \frac{-1}{\sqrt{k^2 + 1}}$$

thus

$$|\langle J_0 e_i, e_j \rangle| = \frac{1}{\sqrt{k^2+1}}\ ,\ 1 \leq i \neq j \leq 2$$

Choosing $e_2 = J_0 e_1$, we conclude that f defines a slant surface with slant angle $\cos^{-1}\left(\frac{1}{\sqrt{k^2+1}}\right)$. It is easy to verify that this surface is also flat and non-minimal.

Example2.2.3 Let $\alpha : (a,b) \to R^2$, $\alpha(s) = (g(s), h(s))$ be a unit speed curve in R^2 and k any positive number. Then it can be verified that $f : R \times (a,b) \to R^4$ defined by $f(u,s) = (-ks\sin u, g(s), ks\cos u, h(s))$ is a non-minimal, flat, proper slant surface in R^4.

2.3 PROPERTIES OF OPERATORS P AND F

Let $f : N \to M$ be isometric immersion of an n-dimensional Riemannian manifold into an almost Hermitian manifold. Let P and F be the endomorphism and the normal-bundle-valued 1-form on the tangent bundle defined in equation 2.1.1. Since M is almost Hermitian, we have $\langle PX, Y\rangle = -\langle X, PY\rangle$, $X, Y \in \mathfrak{X}(N)$. Hence if we put $Q = P^2$ then Q is a symmetric (self-adjoint) endomorphism of $\mathfrak{X}(N)$.

Therefore, each tangent space T_xN of N at $x \in N$ admits an orthogonal direct decomposition of eigenspaces of Q

$$T_xN = D_x^1 \oplus ... \oplus D_x^{k(x)}$$

Since P is skew-symmetric and $J^2 = -I$, each eigenvalue λ_i of Q lies in $[-1, 0]$ and; moreover , if $\lambda_i \neq 0$,then corresponding eigenspace D_x^i is of even dimension and it invariant under the endomorphism P , that is $P\left(D_x^i\right) = D_x^i$. Furthermore, for each $\lambda_i \neq -1$, $\dim F\left(D_x^i\right) = \dim D_x^i$ and the normal subspaces $F\left(D_x^i\right), i = 1, ..., k\left(x\right)$, are mutually perpendicular. From these arguments, we have $\dim M \geq 2\dim N - \dim \mu_x$, where μ_x denotes the eigenspace of Q corresponding to eigenvalue -1.

The following lemma follows from the definition of ∇Q which is defined by

$$\left(\nabla_X Q\right) Y = \nabla_X \left(QY\right) - Q\left(\nabla_X Y\right)$$

for $X, Y \in \mathfrak{X}(N)$.

Lemma 2.3.1 Let N be a submanifold of an almost Hermitian manifold M. Then the symmetric endomorphism Q is parallel, that is $\nabla Q = 0$, if and only if

(i) each eigenvalue λ_i of Q is constant on N.

(ii) each distribution D^i (associated with the eigenvalue λ_i) is completely integrable and

(iii) N is locally the Riemannian product $N_1 \times ... \times N_k$ of the leaves of the distributions.

Proof. Since Q is a self-adjoint endomorphism of the tangent bundle TN, there exist n continuous functions $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ such that λ_i , $i = 1, ..., n$, are the eigenvalues of Q at each point $p \in N$. Let $e_1, ..., e_n$ be a local orthonormal frame given by eigenvectors of Q . If Q is parallel $\nabla_X QY = Q(\nabla_X Y)$, $\forall X, Y \in \mathfrak{X}(N)$, we get $\nabla_X Qe_i = Q(\nabla_X e_i)$, that is $X(\lambda_i)e_i + \lambda_i \nabla_X e_i = Q(\nabla_X e_i)$ take inner product with e_i, we arrive at $X(\lambda_i) + \lambda_i \langle \nabla_X e_i, e_i \rangle = \langle Q(\nabla_X e_i), e_i \rangle$. Thus we conclude that $X(\lambda_i) + \lambda_i \langle \nabla_X e_i, e_i \rangle = \langle \nabla_X e_i, Qe_i \rangle = \lambda_i \langle \nabla_X e_i, e_i \rangle$, that is $X(\lambda_i) = 0$, $X \in \mathfrak{X}(N)$ and this proves that λ_i is a constant.

For statements (ii) and (iii) we let $\lambda_1, ..., \lambda_k$ denote the distinct eigenvalues of Q. For each $i = 1, ..., k$ let D^i be the distribution given by the eigenspaces of Q corresponding to the eigenvalue λ_i. Then any two vector fields $X, Y \in D^i$, $Q(X) = \lambda_i X$, $Q(Y) = \lambda_i Y$ and consequently we get $\nabla_X QY = Q(\nabla_X Y)$ and $\nabla_Y QX = Q(\nabla_Y X)$, that is $\nabla_X(\lambda_i Y) - \nabla_Y(\lambda_i X) = Q([X, Y])$ or $\lambda_i [X, Y] =$

$Q([X,Y])$, which proves that $[X,Y] \in D^i$, so D^i is completely integrable. Also we can similarly show that $\nabla_X Y \in D^i$. Therefore each maximal integrable submanifold N_i of D^i is totally geodesic in N . Consequently , $N = N_1 \times ... \times N_k$

For the converse of this statement, let $X, Y \in \mathfrak{X}(N)$ and $Y \in D^i$ $1 \leq i \leq k$. Thus we have $QY = \lambda_i Y$ and $\nabla_X (QY) = \nabla_X (\lambda_i Y) = X(\lambda_i) Y + \lambda_i \nabla_X Y = 0 + Q(\nabla_X Y)$, that is $(\nabla_X Q) Y = 0$, which proves Q is parallel.

Lemma 2.3.2 Let N be a submanifold of an almost Hermitian manifold M. Then $\nabla P = 0$ if and only if N is locally the Riemannian product $N_1 \times ... \times N_k$ where each N_i is either a complex submanifold, a totally real submanifold ,or a Kaehlerian slant submanifold of M.

Proof. If P is parallel then Q is parallel. Thus, by applying lemma 2.3.1, we see that N is locally the Riemannian product $N_1 \times ... \times N_k$ of leaves N_i of distributions D_i corresponding to eigenvalues λ_i. Moreover if each λ_i is a constant. If $\lambda_i = 0$, then $QX = 0$, $X \in \mathfrak{X}(N)$ would imply $PX = 0$, that is N is totally real submanifold. If $\lambda_i = -1$, then $P : \mathfrak{X}(N_i) \to \mathfrak{X}(N_i)$, satisfies $P^2 = -I$, that is P is almost complex structure on N_i and this would imply N_i is almost complex manifold .

If $\lambda_i \neq 0, -1$, then because D^i is invariant under the endomorphism P and $\langle PX, PY \rangle = -\lambda_i \langle X, Y \rangle$ for any $X, Y \in D^i$ we would have$|PX| = \sqrt{-\lambda_i}\,|X|$. Thus the Wirtinger angle $\theta(X)$ satisfies

$$\cos\theta(X) = \sqrt{-\lambda_i}$$

which is a constant $\neq 0, -1$, therefore ,N_i is a proper slant submanifold.

Assume $\lambda_i \neq 0$, we put $P_i = P\mid_{TN_i}$. Then P_i is nothing but the endomorphism of TN_i induced from the almost complex structure on totally geodesic submanifold in N, we have $\left(\nabla^i_X P_i\right) Y = (\nabla_X P) Y = 0$ for any $X, Y \in \mathfrak{X}(N_i)$, this shows that if N_i is a complex submanifold. Thus N_i is a Kaehlerian slant submanifold of M by definition.

From above two lemmas we conclude that

Proposition 2.3.3 Let N be an irreducible submanifold of an almost Hermitian manifold M . If N is neither complex nor totally real, then N is a Kaehlerian slant submanifold if and only if the endomorphism P is parallel, that is $\nabla P = 0$.

Next we prove the following theorem for surfaces in an almost Hermitian manifold.

Theorem 2.3.4 Let N be a surface in an almost Hermitian manifold M . Then the following three statement are equivalent

(i) N is neither totally real nor complex in M and $\nabla P = 0$ that is P is parallel.

(ii) N is a Kaehlerian slant surface.

(iii) N is a proper slant surface.

Proof. Since every proper slant submanifold is of even dimension, lemma 2.3.2 implies that if the endomorphism P is parallel then N is a Kaehlerian surface, or a totally real surface, or a Kaehlerian slant surface. Thus if N is neither totally real nor complex, then statement (i) and (ii) are equivalent by definition.

It obvious that (ii) implies (iii). Now we prove that (iii) implies (ii):

Let N be a proper slant surface in M with slant angle θ. If we choose an orthonormal frame e_1, e_2 tangent to N such that, $Pe_1 = (\cos\theta)\, e_2$, $Pe_2 = -(\cos\theta)\, e_1$. Then we know that $\nabla_X e_1 = w_1^1(X)\, e_1 + w_1^2(X)\, e_2$ and $\nabla_X e_2 = w_2^1(X)\, e_1 + w_2^2(X)\, e_2$, which implies $(\nabla_X P)\, e_1 = 0$ and $(\nabla_X P)\, e_2 = 0$. Thus $\nabla P = 0$, that is P is parallel and this implies N is a Kaehlerian slant surface.

For submanifolds of a Kaehlerian manifold we have the following general lemma.

Lemma 2.3.5 Let N be a submanifold of a Kaehlerian manifold M. Then

(i) For vectors $X, Y \in \mathfrak{X}(N)$, we have

$$(\nabla_X P)\, Y = th\,(X, Y) + A_{FY} X$$

and hence $\nabla P = 0$ if and only if $A_{FX} Y = A_{FY} X, \;\; X, Y \in \mathfrak{X}(N)$

(ii) For any $X, Y \in \mathfrak{X}(N)$, we have

$$(\nabla_X F)\, Y = fh\,(X, Y) - h\,(X, PY)$$

and hence $\nabla F = 0$ if and only if $A_{f\xi} X = -A_\xi PX$, for any $\xi \in \Gamma(\upsilon)$ and $X \in \mathfrak{X}(N)$, such that

$$Jh(X,Y) = th(X,Y) + fh(X,Y)$$

where $th(X,Y)$ and $fh(X,Y)$ is the tangential and the normal components of $Jh(X,Y)$

Proof. Since M is a Kaehlerian, J is parallel. Thus for $X, Y \in \mathfrak{X}(N)$

$$\begin{aligned} 0 &= \overline{\nabla}_X JY - J\overline{\nabla}_X Y = \overline{\nabla}_X (PY + FY) - J(\nabla_X Y + h(X,Y)) \\ &= \nabla_X PY + h(X,PY) - A_{FY}X + D_X FY - P(\nabla_X Y) \\ &\quad -F(\nabla_X Y) - th(X,Y) - fh(X,Y) \end{aligned}$$

Equating tangential and normal components we arrive at $(\nabla_X P)Y = th(X,Y) + A_{FY}X$ and $(\nabla_X F)Y = fh(X,Y) - h(X,PY)$. Thus P is parallel if and only if $\langle th(X,Y) + A_{FY}X, Z\rangle = 0$, $X, Y, Z \in \mathfrak{X}(N)$, which is equivalent to $\langle A_{FY}X, Z\rangle = -\langle th(X,Y), Z\rangle = \langle A_{FX}Y, Z\rangle$.

Also $\nabla F = 0$ if and only if $\langle fh(X,Y) - h(X,PY), \xi\rangle = 0 \Leftrightarrow \langle h(X,PY), \xi\rangle = \langle fh(X,Y), \xi\rangle = -\langle A_{f\xi}Y, X\rangle$

$\Leftrightarrow \langle h(PY,X), \xi\rangle = -\langle A_{f\xi}Y, X\rangle \Leftrightarrow \langle A_\xi PY, X\rangle = -\langle A_{f\xi}Y, X\rangle \Leftrightarrow -A_\xi PY = A_{f\xi}Y$.

Remark 2.3.1 If N is either a totally real or complex submanifold of a Kaehlerian manifold, then $\nabla P = \nabla F = 0$, automatically.

Corollary 2.3.6 Let N be a surface in a Kaehlerian manifold M. Then N is slant if and only if $A_{FY}X = A_{FX}Y$, $X, Y \in \mathfrak{X}(N)$.

Let N be a slant surface in the complex number space C^2 with slant angle θ. For a unit tangent vector field e_1 to N we put

$$e_2 = (\sec\theta)\, Pe_1, e_3 = (\csc\theta)\, Fe_1, e_4 = (\csc\theta)\, Fe_2$$

Then we get $e_1 = -(\sec\theta)\, Pe_2$, and e_1, e_2, e_3, e_4 is an orthonormal frame such that $e_1, e_2 \in \mathfrak{X}(N)$ and $e_3, e_4 \in \Gamma(\upsilon)$. As before we put $h^r_{ij} = \langle h(e_i, e_j), e_r\rangle$, $i, j = 1, 2$; $r = 3, 4$. Let R and R^D denote the Gauss and normal curvature of N in C^2 respectively. Then we have

$$R = h^3_{11}h^3_{22} - \left(h^3_{12}\right)^2 + h^4_{11}h^4_{22} - \left(h^4_{12}\right)^2$$

$$R^D = h^3_{11}h^4_{12} + h^3_{12}h^4_{22} - h^3_{12}h^4_{11} - h^3_{22}h^4_{12}$$

Theorem 2.3.7 If N is a slant surface in C^2, then $R = R^D$, identically.

Proof. Let N be a slant surface in C^2. Then corollary 2.3.6 implies $A_{FY}X = A_{FX}Y$, $X, Y \in \mathfrak{X}(N)$. Let e_1, e_2, e_3, e_4 be an orthonormal frame as in the proof of theorem 2.3.4. Then we have

$$\begin{aligned} h^3_{12} &= \langle h(e_1, e_2), e_3\rangle = \langle h(e_1, e_2), (\csc\theta)Fe_1\rangle \\ &= (\csc\theta)\langle h(e_1, e_2), Fe_1\rangle = (\csc\theta)\langle A_{Fe_1}e_1, e_2\rangle \\ &= (\csc\theta)\langle A_{Fe_1}e_2, e_1\rangle = (\csc\theta)\langle A_{Fe_2}e_1, e_1\rangle \\ &= (\csc\theta)\langle h(e_1, e_1), Fe_2\rangle = \langle h(e_1, e_1), e_4\rangle = h^4_{11} \end{aligned}$$

Similarly, we can prove that $h^3_{22} = h^4_{12}$. Therefore, by (3.6) and (3.7), we obtain $R = R^D$.

In the remaining part of this section we mention some properties of the normal-bundle valued 1-form F. In order to do so, we recall the following definition .

Definition 2.3.1 Let N be a submanifold of a Riemannian manifold M . Then N is called a minimal submanifold if $trh = 0$. And it is called auster if for each normal vector ξ the set of eigenvalues of A_ξ is invariant under multiplication by -1 this is equivalent to the condition that all the invariants of odd order of the Weingarten map at each normal vector of N vanish identically. Of course every auster submanifold is a minimal submanifold.

Theorem 2.3.8 Let N be a proper slant submanifold of a Kaehlerian manifold M. If $\nabla F = 0$, then N is auster.

Proof. Let N be a proper slant submanifold of a Kaehlerian manifold M. If $\nabla F = 0$, then we have from Lemma 2.3.5 that $fh\left(X,Y\right) = h\left(X,PY\right)$. Let X be any unit eigenvector of $Q = P^2$ with eigenvalue $\lambda \neq 0$. Then $X_* = \frac{PX}{\sqrt{-\lambda}}$ is a unit vector perpendicular to X. Thus, we have

$$h\left(X_*,X_*\right) = h\left(\frac{PX}{\sqrt{-\lambda}},\frac{PX}{\sqrt{-\lambda}}\right) = \frac{1}{-\lambda}h\left(P^2X,X\right) = \frac{-1}{\lambda}h\left(\lambda X,X\right) = -h\left(X,X\right)$$

which gives for any normal vector ξ

$$\left\langle A_\xi X,X\right\rangle = \left\langle h\left(X,X\right),\xi\right\rangle = -\left\langle h\left(X_*,X_*\right),\xi\right\rangle = -\left\langle A_\xi X_*,X_*\right\rangle$$

Now, suppose that μ is an eigenvalue of A_ξ, then

$$\begin{aligned} \mu &= \mu\left\langle X,X\right\rangle = \left\langle \mu X,X\right\rangle = \left\langle A_\xi X,X\right\rangle = -\left\langle A_\xi X_*,X_*\right\rangle \\ &= -\left\langle \mu X_*,X_*\right\rangle = -\mu\left\langle X_*,X_*\right\rangle = -\mu \end{aligned}$$

Thus μ is invariant under multiplication by -1, which implies that N is auster.

If $M(c)$ is a complex-space-form, then we have the following reduction theorem.

Theorem 2.3.9 Let N be an n-dimensional proper slant submanifold of a complex m-dimensional complex space form $M^m(c)$ with constant holomorphic sectional curvature c. If $\nabla F = 0$, then N is contained in a complex n-dimensional complex totally geodesic submanifold of $M^m(c)$ as an auster submanifold.

Proof. Let N be an n-dimensional proper slant submanifold of C^m. Assume that $\nabla F = 0$ Then the normal bundle υ of N has the following orthogonal direct decomposition

$$\upsilon = F(TN) \oplus \mu$$

such that $\mu_p \perp F(T_pN)$, for any point $p \in N$. For any vector field $\xi \in \mu$, and any $N \in F(TN)$ there exists $X \in \mathfrak{X}(N)$, such that $N = FX$ that is $JN = J(FX)$. Thus $tN + fN = t(FX) + f(FX)$, that is $tN = t(FX)$ and $fN = 0$, since $f \circ F = 0$. This proves $JN = tN$. Since N and ξ are orthogonal, $0 = \langle N, \xi\rangle = \langle JN, J\xi\rangle = \langle tN, t\xi\rangle$, and as $tN \neq 0$, we get $t\xi = 0$ that is $J\xi = f\xi$ proving that $J\xi \in \mu$. Thus we have shown that $\xi \in \mu \Rightarrow J\xi \in \upsilon$.

Now for $\xi \in \mu$, and $X, Y \in \mathfrak{X}(N)$

$$\begin{aligned}
\langle A_{J\xi}X, Y\rangle &= \langle h(X,Y), J\xi\rangle = \left\langle \overline{\nabla}_X Y, J\xi\right\rangle = -\left\langle J\overline{\nabla}_X Y, \xi\right\rangle = -\left\langle \overline{\nabla}_X JY, \xi\right\rangle \\
&= \left\langle JY, \overline{\nabla}_X \xi\right\rangle = \langle JY, -A_\xi X + D_X\xi\rangle \\
&= -\langle PY, A_\xi X\rangle + \langle FY, D_X\xi\rangle \\
&= -\langle PY, A_\xi X\rangle + \langle D_X FY, \xi\rangle
\end{aligned}$$

Thus we have $\langle D_X FY, \xi\rangle = -\langle A_\xi PY + A_{J\xi}Y, X\rangle$. On the other hand, for any ξ normal to N

$$J\xi = t\xi + f\xi$$

and the lemma 2.3.5 gives $A_{f\xi}Y + A_\xi PY = 0$. Since $f = J$ on the normal subbundle μ, formulas in Lemma 2.3.5 imply $\langle D_X FY, \xi\rangle = 0$, $\xi \in \mu$. From this we conclude that the normal subbundle $F(TN)$ is a parallel normal subbundle.

Next, we claim the first normal subbundle Imh is contained in $F(TN)$ .This can be proved as follows:

Since $\nabla F = 0$, statement (ii) of lemma 2.3.5 implies $\langle h(X,Y), J\xi\rangle = -\langle h(X, PY), \xi\rangle$ for any normal vector $\xi \in \mu$. Thus, for any eigenvector Y of the self-adjoint endomorphism Q with eigenvalue λ and any normal vector $\xi \in \mu$, we have

$$\left\langle h(X,Y), J^2\xi\right\rangle = -\langle h(X, PY), J\xi\rangle = \left\langle h\left(X, P^2Y\right), \xi\right\rangle = \lambda\langle h(X,Y), \xi\rangle$$

since $QY = P^2Y = \lambda Y$. Thus $-\langle h(X,Y), \xi\rangle = \lambda\langle h(X,Y), \xi\rangle$ that is $\langle h(X,Y), \xi\rangle = -\lambda\langle h(X,Y), \xi\rangle$. Since N is a proper slant submanifold, $-1 < \lambda \leq 0$. Thus $\lambda \neq -1$ and this gives $\langle h(X,Y), \xi\rangle = 0$, $\in \mu$ that is $h(X,Y) \in F(TN)$ and consequently $Imh \subset F(TN)$.

CHAPTER 3

In this chapter we are interested in proving an existence theorem [5] for slant submanifold in a complex projective space and prove that these submanifolds are unique up to an isometry required by the data of the slant submanifold. There are two sections in this chapter. In section 3.1, we collect basic facts about slant submanifold which are required to prove the existence theorem in section 3.2. Here we would like to point out that we use the notations of [5], where some times the smooth vector fields are taken as elements of tangent bundle but it is understood from the context that they are smooth vector fields.

3.1 PRILIMNAIRIES

We denote by $\overline{M}^m(4c)$ the complete simply -connected Kaehlerian m-manifold with constant holomorphic sectional curvature $4c$. The curvature tensor $\overline{R}$ of $\overline{M}^m(4c)$ is given by

$$\begin{aligned}\overline{R}\left(\overline{X},\overline{Y}\right)\overline{Z} \quad &= c\left\{\left\langle\overline{Y},\overline{Z}\right\rangle\overline{X}-\left\langle\overline{X},\overline{Z}\right\rangle\overline{Y}+\left\langle J\overline{Y},\overline{Z}\right\rangle J\overline{X}\right.\\ &\left.-\left\langle J\overline{X},\overline{Z}\right\rangle J\overline{Y}+2\left\langle\overline{X},J\overline{Y}\right\rangle J\overline{Z}\right\},\end{aligned} \tag{3.1.1}$$

for $\overline{X},\overline{Y},\overline{Z}$ tangent to $\overline{M}^m(4c)$.

Let $f:M\to\overline{M}^m(4c)$ be an isometric immersion of a Riemannian n-manifold into $\overline{M}^m(4c)$. We denote by h and A the second fundamental form and the Weingarten map of f and by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\overline{M}$.

For $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma(\upsilon)$, the second fundamental form h and the Weingarten map A are related by

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle . \tag{3.1.2}$$

The mean curvature vector H of the immersion is

$$H = \frac{1}{n} trh = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) \tag{3.1.3}$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame field on M.

Denote by R the curvature tensor of M and by R^D the curvature tensor of the normal connection D . Then the equation of Gauss and the equation of Ricci are given respectively by

$$\begin{aligned} \overline{R}(X, Y; Z, W) &= R(X, Y; Z, W) + \langle h(X, Z), h(Y, W) \rangle \\ &\quad - \langle h(X, W), h(Y, Z) \rangle , \end{aligned} \tag{3.1.4}$$

$$R^D(X, Y; \xi, \eta) = \overline{R}(X, Y; \xi, \eta) + \langle [A_\xi, A_\eta](X), Y \rangle \tag{3.1.5}$$

for vectors $X, Y, Z, W \in \mathfrak{X}(M)$ and $\xi, \eta \in \Gamma(\upsilon)$.

For the second fundamental form h, we define the covariant derivative $\overline{\nabla}h$ of h with respect to the connection on $TM \oplus T^{\perp}M$ by

$$\left(\overline{\nabla}_X h\right)(Y, Z) = D_X\left(h(Y, Z)\right) - h\left(\nabla_X Y, Z\right) - h\left(Y, \nabla_X Z\right). \tag{3.1.6}$$

The equation of Codazzi is given by

$$\left(\overline{R}(X, Y) Z\right)^{\perp} = \left(\overline{\nabla}_X h\right)(Y, Z) - \left(\overline{\nabla}_Y h\right)(X, Z), \tag{3.1.7}$$

where $\left(\overline{R}(X, Y) Z\right)^{\perp}$ denotes the normal component of $\overline{R}(X, Y) Z$.

For an endomorphism Q on the tangent bundle of the submanifold, we define ∇Q by

$$\left(\nabla_X Q\right) Y = \nabla_X (QY) - Q\left(\nabla_X Y\right). \tag{3.1.8}$$

Now , suppose that M is θ-slant in $\overline{M}^n(4c)$, then we have

$$P^2 = -\left(\cos^2\theta\right) I, \langle PX, Y\rangle + \langle X, PY\rangle = 0, \tag{3.1.9}$$

$$\left(\nabla_X P\right) Y = th(X, Y) + A_{FY}X, \tag{3.1.10}$$

$$D_X (FY) - F (\nabla_X Y) = fh (X, Y) - h (X, PY), \tag{3.1.11}$$

where I is the identity map. For simplicity for each $X \in \mathfrak{X}(M)$, we put

$$X^* = \frac{1}{\sin \theta} FX. \tag{3.1.12}$$

We define a symmetric bilinear TM-valued form α on M by

$$\alpha (X, Y) = th (X, Y) \tag{3.1.13}$$

$$\Rightarrow \alpha^* (X, Y) = \tfrac{1}{\sin \theta} F\alpha (X, Y)$$

and

$$J\alpha (X, Y) = P\alpha (X, Y) + F\alpha (X, Y)$$

$$\Rightarrow J\alpha (X, Y) = P\alpha (X, Y) + (\sin \theta)\alpha^* (X, Y). \tag{3.1.14}$$

Since $Jh (X, Y) = th (X, Y) + fh (X, Y)$, put

$$Jh (X, Y) = \alpha (X, Y) + \beta^* (X, Y), \tag{3.1.15}$$

where β is also a symmetric bilinear TM-valued on M . From (3.1.12),(3.1.14) and (3.1.15) we have

$$\begin{aligned}
J^2h(X,Y) &= -h(X,Y) = J\alpha(X,Y) + J\beta^*(X,Y) \\
-h(X,Y) &= P\alpha(X,Y) + (\sin\theta)\,\alpha^*(X,Y) + \tfrac{1}{\sin\theta}JF\beta(X,Y) \\
&= P\alpha(X,Y) + (\sin\theta)\alpha^*(X,Y) \\
&\quad -(\csc\theta)\beta(X,Y) - (\csc\theta)\,J\,(P\beta(X,Y)) \\
&= P\alpha(X,Y) + (\sin\theta)\,\alpha^*(X,Y) - (\csc\theta)\,\beta(X,Y) \\
&\quad -(\csc\theta)\,P^2\beta(X,Y) - (\csc\theta)\,(F\,(P\beta(X,Y))) \\
&= P\alpha(X,Y) + (\sin\theta)\,\alpha^*(X,Y) \\
&\quad -\left(\tfrac{1}{\sin\theta}\right)\beta(X,Y) + \tfrac{\cos^2\theta}{\sin\theta}\beta(X,Y) - (P\beta(X,Y))^* \\
&= P\alpha(X,Y) + (\sin\theta)\,\alpha^*(X,Y) \\
&\quad -\tfrac{1-\cos^2\theta}{\sin\theta}\beta(X,Y) - (P\beta(X,Y))^* \\
\Rightarrow -h(X,Y) &= P\alpha(X,Y) - (\sin\theta)\,\beta(X,Y) \\
&\quad +(\sin\theta)\,\alpha^*(X,Y) - (P\beta(X,Y))^*
\end{aligned}$$

$$\begin{aligned}
-h(X,Y) &= P\alpha(X,Y) - (\sin\theta)\,\beta(X,Y) \\
&\quad +(\sin\theta)\,\alpha^*(X,Y) - (P\beta(X,Y))^*
\end{aligned} \tag{3.1.16}$$

Thus

$$P\alpha(X,Y) = (\sin\theta)\,\beta(X,Y),$$

and

$$-h(X,Y) = (\sin\theta)\,\alpha^*(X,Y) - (P\beta(X,Y))^*$$

Thus

$$\beta(X,Y) = (\csc\theta)\,P\alpha(X,Y)$$

$$\Rightarrow P\beta(X,Y) = (\csc\theta)\,P^2\alpha(X,Y) = -(\csc\theta)\left(\cos^2\theta\right)\alpha(X,Y)$$

and

$$h(X,Y) = (P\beta(X,Y))^* - (\sin\theta)\,\alpha^*(X,Y) = -(\csc\theta)\,\alpha^*(X,Y)$$

So

$$\begin{aligned} h(X,Y) &= -(\csc\theta)\,\alpha^*(X,Y) \\ &= -(\csc\theta)\left[\tfrac{1}{\sin\theta}\left(J\alpha(X,Y) - P\alpha(X,Y)\right)\right] \\ &= \csc^2\theta\left[P\alpha(X,Y) - J\alpha(X,Y)\right] \end{aligned}$$

$$\Rightarrow h(X,Y) = \csc^2\theta\left[P\alpha(X,Y) - J\alpha(X,Y)\right].$$

By (3.1.10)

$$(\nabla_X P)\,Y = th(X,Y) + A_{FY}X = \alpha(X,Y) + A_{FY}X,$$

so

$$\langle(\nabla_X P)\,Y, Z\rangle = \langle\alpha(X,Y), Z\rangle - \langle\alpha(X,Z), Y\rangle \tag{3.1.17}$$

For an n-dimensional θ-slant submanifold in $\overline{M}^n(4c)$ with $\theta \neq 0$ (3.1.1), (3.1.7), (3.1.8), (3.1.10) and (3.1.17) imply that the equation of Gauss and Codazzi of M in $\overline{M}^n(4c)$ are

$$\begin{aligned} R(X,Y;Z,W) = {}& c\{\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle PY,Z\rangle\langle PX,W\rangle \\ & - \langle PX,Z\rangle\langle PY,W\rangle + 2\langle X,PY\rangle\langle PZ,W\rangle\} \\ & + \csc^2\theta\{\langle\alpha(X,W),\alpha(Y,Z)\rangle \\ & - \langle\alpha(X,Z),\alpha(Y,W)\rangle\} \end{aligned} \tag{3.1.18}$$

$$\begin{aligned} &(\nabla_X\alpha)(Y,Z) + \csc^2\theta\{P\alpha(X,\alpha(Y,Z)) + \alpha(X,P\alpha(Y,Z))\} \\ &+(\sin^2\theta)c\{\langle X,PY\rangle Z + \langle X,PZ\rangle Y\} \\ &= (\nabla_Y\alpha)(X,Z) + \csc^2\theta\{P\alpha(Y,\alpha(X,Z)) + \alpha(Y,P\alpha(X,Z))\} \\ &+\left(\sin^2\theta\right)c\{\langle Y,PX\rangle Z + \langle Y,PZ\rangle X\}. \end{aligned} \tag{3.1.19}$$

3.2 EXISTANCE THEOREM

Theorem(Existence) 3.2.1 Let c, θ be two constants with $0 < \theta \leq \frac{\pi}{2}$ and M a simply connected Riemannian n-manifold with inner product $\langle , \rangle$. Suppose there exist an endomorphism P of the tangent bundle TM and a symmetric bilinear TM-valued form α on M such that for $X, Y, Z, W \in TM$, we have

$$P^2 = -\left(\cos^2\theta\right) I, \tag{3.2.1}$$

$$\langle PX, Y\rangle + \langle X, PY\rangle = 0, \tag{3.2.2}$$

$$\langle (\nabla_X P) Y, Z\rangle = \langle \alpha(X, Y), Z\rangle - \langle \alpha(X, Z), Y\rangle, \tag{3.2.3}$$

$$\begin{aligned} R(X, Y, Z, W) \quad = \quad & \csc^2\theta \{\langle \alpha(X, W), \alpha(Y, Z)\rangle - \langle \alpha(X, Z), \alpha(Y, W)\rangle\} \\ & + c\{\langle X, W\rangle \langle Y, Z\rangle - \langle X, Z\rangle \langle Y, W\rangle + \langle PX, W\rangle \langle PY, Z\rangle \\ & - \langle PX, Z\rangle \langle PY, W\rangle + 2\langle X, PY\rangle \langle PZ, W\rangle\} \end{aligned} \tag{3.2.4}$$

$$\begin{gathered} (\nabla_X \alpha)(Y, Z) = \csc^2\theta \{P\alpha(X, \alpha(Y, Z)) + \alpha(X, P\alpha(Y, Z))\} \\ + \left(\sin^2\theta\right) c\{\langle X, PZ\rangle Y + \langle X, PY\rangle Z\} \end{gathered} \tag{3.2.5}$$

is totally symmetric. Then there exists a θ-slant isometric immersion from M into $\overline{M}^n(4c)$ whose second fundamental form h is given by

$$h(X,Y) = \csc^2\theta\left(P\alpha(X,Y) - J\alpha(X,Y)\right). \tag{3.2.6}$$

Proof : Let c, θ be two constants with $0 < \theta \leq \frac{\pi}{2}$ and M a simply-connected Riemannian n-manifold equipped with an endomorphism P and a symmetric bilinear TM-valued form α satisfying the five conditions stated in the theorem.

Consider the Witney sum $TM \oplus TM$. For each $X \in TM$, we identify $(X, 0)$ with X; and also we denote $(0, X)$ by X^*. We define the inner product $\langle , \rangle$ on $TM \oplus TM$ by using the product metric. Let $\widehat{J}$ be the endomorphism on $TM \oplus TM$ defined by

$$\widehat{J}X = PX + (\sin\theta)X^*,\ \widehat{J}X^* = -(\sin\theta)X - PX^*, \tag{3.2.7}$$

for $X \in TM$, then we have

$$\begin{aligned}\widehat{J}^2 X &= \widehat{J}^2((X,0)) = \widehat{J}(PX,0) + \widehat{J}(0,(\sin\theta)X)\\ &= \left(P^2X, (\sin\theta)PX\right) - \left(\left(\sin^2\theta\right)X, (\sin\theta)PX\right)\\ &= -(X,0) = -X\end{aligned}$$

which gives

$$\widehat{J}^2 X = -X.$$

Similarly, we have

$$\widehat{J}^2 X^* = -X^*.$$

Thus,

$$\widehat{J}^2 = -I.$$

Now as

$$\langle (X, Y), \langle Z, W \rangle \rangle = \langle X, Z \rangle + \langle Y, W \rangle$$

we have

$$\begin{aligned}
\left\langle \widehat{J}X, \widehat{J}Y \right\rangle &= \left\langle \widehat{J}(X, 0), \widehat{J}(Y, 0) \right\rangle \\
&= \langle (PX, (\sin\theta) X), (PY, (\sin\theta) Y) \rangle \\
&= \langle PX, PY \rangle + \langle (\sin\theta) X, (\sin\theta) Y \rangle \\
&= \cos^2\theta \langle X, Y \rangle + \sin^2\theta \langle X, Y \rangle = \langle X, Y \rangle
\end{aligned}$$

Thus, $\left(\widehat{J}, \langle , \rangle\right)$is a Hermitian structure on $TM \oplus TM$.

Now , we define A, h and D by

$$A_{Y^*} X = \csc\theta \{(\nabla_X P) Y - \alpha(X, Y)\}, \tag{3.2.8}$$

$$h(X, Y) = -(\csc\theta)\alpha^*(X, Y), \tag{3.2.9}$$

$$D_X Y^* = (\nabla_X Y)^* + \csc^2\theta \{P\alpha^*(X, Y) + \alpha^*(X, PY)\}, \tag{3.2.10}$$

for vector fields $X, Y \in TM$. It is easy to verify that each A_{Y^*} is an endomorphism on TM.

Now,

$$h(X,Y) = -(\csc\theta)\,\alpha^*(X,Y),$$

as we see

$$\widehat{J}X = PX + (\sin\theta)\,X^* = PX + FX$$

$$\Rightarrow (\sin\theta)\,X^* = FX \Rightarrow X^* = \frac{1}{\sin\theta}FX$$

thus we conclude

$$\alpha^*(X,Y) = \frac{1}{\sin\theta}F\alpha(X,Y),$$

and as α is a symmetric bilinear TM-valued, α^* is also a symmetric bilinear $(TM)^*$-valued form, that is, h is also a $(TM)^*$-valued symmetric bilinear form on TM, and D is a metric connection of the vector bundle $(TM)^*$ over M.

Let $\widehat{\nabla}$ denote the canonical connection on $TM \oplus TM$ induced from the Levi-Civita connection on TM. Then from (3.2.7)-(3.2.10), we have on using Kozul's formula that :

$$\left(\widehat{\nabla}_X \widehat{J}\right)Y = \left(\widehat{\nabla}_X \widehat{J}\right)Y^* = 0 \tag{3.2.11}$$

for vector fields X, Y tangent to M.

Let R^D denote the curvature tensor associated with the connection D on $(TM)^*$, that is,

$$R^D(X,Y)Z^* = D_X D_Y Z^* - D_Y D_X Z^* - D_{[X,Y]} Z^*, \tag{3.2.12}$$

for X, Y tangent to M. Then by (3.2.1),(3.2.5),(3.2.10),(3.2.12) and a simple computation, we obtain

$$\begin{aligned} R^D(X,Y)Z^* = \quad & (R(X,Y)Z)^* + \{cP[\langle Y, PZ\rangle X - \langle X, PZ\rangle Y - 2\langle X, PY\rangle Z] \\ & c\left[\left\langle Y, P^2 Z\right\rangle X - \left\langle X, P^2 Z\right\rangle Y - 2\langle X, PY\rangle PZ\right] \\ & + \csc^2\theta\,[(\nabla_X P)\alpha(Y,Z) - (\nabla_Y P)\alpha(X,Z) \\ & - \alpha(X, (\nabla_Y P)Z) + \alpha(Y, (\nabla_X P)Z)]\}^* \end{aligned} \tag{3.2.13}$$

Also, (3.2.8) yields

$$\begin{aligned} \sin^2\theta\,\langle [A_{Z^*}, A_{W^*}]X, Y\rangle \quad &= \langle (\nabla_Y P)Z, (\nabla_X P)W\rangle - \langle (\nabla_X P)Z, (\nabla_Y P)W\rangle \\ & + \langle (\nabla_X P)Z, \alpha(Y,W)\rangle + \langle (\nabla_Y P)W, \alpha(X,Z)\rangle \\ & - \langle (\nabla_Y P)Z, \alpha(X,W)\rangle - \langle (\nabla_X P)W, \alpha(Y,Z)\rangle \\ & + \langle \alpha(X,W), \alpha(Y,Z)\rangle - \langle \alpha(X,Z), \alpha(Y,W)\rangle . \end{aligned} \tag{3.2.14}$$

From (3.2.2) we have

$$\langle \alpha(Y,Z), PW\rangle + \langle P\alpha(Y,Z), W\rangle = 0 \tag{3.2.15}$$

By taking the derivative of (3.2.15) with respect to X and using (3.1.8) and

(3.2.2), we find

$$X\left\langle \alpha\left(Y,Z\right),PW\right\rangle +X\left\langle P\alpha\left(Y,Z\right),W\right\rangle =0$$

$$\left\langle \nabla_{X}\alpha\left(Y,Z\right),PW\right\rangle +\left\langle \alpha\left(Y,Z\right),\nabla_{X}PW\right\rangle +\left\langle \nabla_{X}P\alpha\left(Y,Z\right),W\right\rangle + \left\langle P\alpha\left(Y,Z\right),\nabla_{X}W\right\rangle =0$$

$$\Rightarrow\left\langle \alpha\left(Y,Z\right),\left(\nabla_{X}P\right)W\right\rangle +\left\langle \left(\nabla_{X}P\right)\alpha\left(Y,Z\right),W\right\rangle =0 \tag{3.2.16}$$

Also by (3.2.3) we obtain

$$\left\langle \left(\nabla_{X}P\right)Z,\left(\nabla_{Y}P\right)W\right\rangle =\left\langle \left(\nabla_{X}P\right)Z,\alpha\left(Y,W\right)\right\rangle -\left\langle \alpha\left(Y,\left(\nabla_{X}P\right)Z\right),W\right\rangle \tag{3.2.17}$$

Hence on using (3.2.13),(3.2.14),(3.2.16)-(3.2.17) and a direct computation, we arrive at

$$\begin{gathered}\left\langle R^{D}\left(X,Y\right)Z^{*},W^{*}\right\rangle -\left\langle \left[A_{Z^{*}},A_{W^{*}}\right]X,Y\right\rangle = \\ c\left\{ \sin^{2}\theta\left(\left\langle Y,Z\right\rangle \left\langle X,W\right\rangle -\left\langle X,Z\right\rangle \left\langle Y,W\right\rangle \right)-2\left\langle X,PY\right\rangle \left\langle PZ,W\right\rangle \right\} \end{gathered} \tag{3.2.18}$$

Equation (3.1.1),(3.2.1),(3.2.2) and (3.2.18) imply that (M,A,D) satisfies the equation of Ricci for an n-dimensional θ-slant submanifold in $\overline{M}^{n}(4c)$. Also, (3.2.4) and (3.2.5) imply that (M,h) satisfies the equation of Gauss and Codazzi for a θ-slant submanifold in $\overline{M}^{n}(4c)$. Hence, the vector bundle

$TM \oplus TM$ over M equipped with the product metric, the Weingarten map A, the second fundamental form h, and the connections D and $\overline{\nabla}$ satisfy the structure equations of n-dimensional θ-slant submanifolds in $\overline{M}^n(4c)$. This proves that there exists a θ-slant isometric immersion of M into $\overline{M}^n(4c)$ with $h = \csc^2\theta\,(P\alpha - J\alpha)$ as its second fundamental form A as its Weingarten map, and D as its normal connection.

Theorem(Uniqueness) 3.2.2 Let $x^1, x^2 : M \to \overline{M}^n(4c)$ be two θ-slant $\left(0 < \theta < \frac{\pi}{2}\right)$ isometric immersions of a connected Riemannian n-manifold M into the complex -spacef orm $\overline{M}^n(4c)$ with second fundamental form h^1 and h^2. If

$$\left\langle h^1(X,Y), Jx^1_* Z\right\rangle = \left\langle h^2(X,Y), Jx^2_*\right\rangle$$

for all vector fields X, Y, Z tangent to M, then there exists an isometry ϕ of $\overline{M}^n(4c)$ such that $x^1 = \phi \circ x^2$.

Proof : Let p any point of M. If necessary by applying an isometry of $\overline{M}^n(4c)$, we may assume that $x^1(p) = x^2(p)$ and $dx^1_p = dx^1_p$. Let us then take a geodesic γ through the point p that is $p = \gamma(0)$. It is sufficient to prove that $\gamma_1 = x^1(\gamma)$ and $\gamma_2 = x^2(\gamma)$ coincide. As $x^1(p) = x^2(p)$ and $dx^1_p = dx^2_p$ we have

$$\gamma_1'(0) = d\gamma_1|_0\left(\frac{d}{dt}|_0\right) = \left(dx^1 \circ d\gamma\right)_0\left(\frac{d}{dt}|_0\right) = dx^1_p\left(\gamma'(0)\right)$$

where $\gamma_1(0) = x^1(p)$ and $\gamma_2(0) = x^2(p)$, and $\gamma_2'(0) = dx^2_p\left(\gamma'(0)\right) \Rightarrow \gamma_1'(0) = \gamma_2'(0)$.

then $\gamma_1 = \gamma_2$.

CHAPTER 4

In this chapter we are interested in deriving an inequality satisfied by the mean curvature of a slant submanifold. First we prepare lemmas in section 4.1, then in section 4.2 we use these lemmas to classify H-umbilical slant submanifolds, and in the section 4.3, we obtain an inequality which can be used classify a cylindrical slant submanifold in C^n that requires some material which is out of realm of present thesis therefore not included in this thesis.

4.1 SOME LEMMAS

If M is an n-dimensional proper θ-slant submanifold of a complex space form $\overline{M}^n(4\varepsilon)$ of constant holomorphic sectional curvature 4ε, then n is even; say $n = 2m$. We choose a canonical orthonormal frame $e_1, ..., e_n, e_1, ..., e_{n^*}$ in such way that

$$\begin{aligned} e_2 &= (\sec\theta)\, Pe_1, ..., e_{2m} = (\sec\theta)\, Pe_{2m-1}, \\ e_{1^*} &= (\csc\theta)\, Fe_1, ..., e_{2m^*} = (\csc\theta)\, Fe_{2m}, \end{aligned} \tag{4.1.1}$$

where θ is the slant angle. We call such an orthonormal frame an adapted frame.

Now we claim

$$te_{i^*} = -(\sin\theta)\, e_i, i = 1, ..., 2m$$

To prove it observe that

$$\begin{aligned} Je_{i^*} &= J\left[(\csc\theta)\, Fe_i\right] = (\csc\theta)\, J\left[Je_i - Pe_i\right] \\ &= (\csc\theta)\left[-e_i - J\left((\cos\theta)\, e_{i+1}\right)\right] \\ &= -(\csc\theta)\, e_i - (\cot\theta)\, Je_{i+1} \\ &= -(\csc\theta)\, e_i - (\cot\theta)\, Pe_{i+1} - (\cot\theta)(\sin\theta)\, e_{(i+1)^*} \\ &= -(\csc\theta)\, e_i - (\cot\theta)\, Pe_{i+1} - (\cos\theta)\, e_{(i+1)^*} \end{aligned}$$

Thus we have

$$te_{i^*} = -(\csc\theta)\, e_i - (\cot\theta)\, Pe_{i+1}$$

and

$$fe_{i^*} = -(\cos\theta)\, e_{(i+1)^*}$$

that is

$$te_{i^*} = -(\csc\theta)\, e_i - (\cot\theta)(-\cos\theta)\, e_i = \left(\frac{\cos^2\theta}{\sin\theta} - \frac{1}{\sin\theta}\right) e_i = -(\sin\theta)\, e_i$$

Hence $te_{i^*} = -(\sin\theta)\, e_i$ and that $-(\cos\theta)\, e_{(i+1)^*} = fe_{i^*}$ put $i = 2j-1$ $\Rightarrow fe_{(2j-1)^*} = -(\cos\theta)\, e_{(2j)^*}$.

By direct computation we also have

$$fe_{(2j)^*} = (\cos\theta)\, e_{(2j-1)^*}, Pe_{2j} = -(\cos\theta)\, e_{2j-1}, j = 1, ..., m$$

For any vector X tangent to M we put

$$\overline{\nabla}_X e_i = \sum_{j=1}^{n} \omega_i^j(X)\, e_j + \sum_{j=1}^{n} \omega_i^{j^*}(X)\, e_{j^*},$$

$$\overline{\nabla}_X e_{i^*} = \sum_{j=1}^{n} \omega_{i^*}^j(X)\, e_j + \sum_{j=1}^{n} \omega_{i^*}^{j^*}(X)\, e_{j^*}\ , i, j = 1, ..., n.$$

Then we have $\omega_i^j = -\omega_j^i$, to prove this, we take inner product of (1.4) with e_j then with e_{j^*} to arrive at

$$\left\langle \overline{\nabla}_X e_i, e_j \right\rangle = \omega_i^j (X)$$

$$\left\langle \overline{\nabla}_X e_i, e_{j^*} \right\rangle = \omega_i^{j^*} (X)$$

$$\omega_i^j (X) = \left\langle \overline{\nabla}_X e_i, e_j \right\rangle = X \langle e_i, e_j \rangle - \left\langle e_i, \overline{\nabla}_X e_j \right\rangle = - \left\langle \overline{\nabla}_X e_j, e_i \right\rangle = -\omega_j^i (X) .$$

$$\Rightarrow \omega_i^j = -\omega_j^i.$$

Similarly we find $\omega_{i^*}^{j^*} = -\omega_{j^*}^{i^*}$, and taking inner product of (1.5) with e_j we find $\left\langle \overline{\nabla}_X e_{i^*}, e_j \right\rangle = \omega_{i*}^j (X) - \left\langle e_{i^*}, \overline{\nabla}_X e_j \right\rangle$

$= -\omega_j^{i^*} (X)$, which proves $\omega_{i^*}^j = -\omega_j^{i^*}$. Moreover , we also have

$$\omega_i^{j^*} = \sum_{k=1}^{n} h_{ik}^{j*} \omega^k, h_{ik}^{j^*} = \langle h (e_i, e_k) , e_{j^*} \rangle . \tag{4.1.2}$$

where $\omega^1, ..., \omega^n$ is the dual frame of $e_1, ..., e_n$.

We need the following lemmas for later use.

Lemma 4.1.1 Let M be an n-dimensional $(n = 2m)$ proper θ-slant submanifold of a Kahlerian n-manifold. Then with respect to an adapted frame, we have

$$
\begin{aligned}
\omega_{2j-1}^{(2i-1)^*} - \omega_{2i-1}^{(2j-1)^*} &= \cot\theta\left(\omega_{2i-1}^{2j} - \omega_{2j-1}^{2i}\right),\\
\omega_{2j}^{(2i-1)^*} - \omega_{2i-1}^{(2j)^*} &= \cot\theta\left(\omega_{2i}^{2j} - \omega_{2i-1}^{2j-1}\right),\\
\omega_{2i}^{(2j)^*} - \omega_{2j}^{(2i)^*} &= \cot\theta\left(\omega_{2i}^{2j-1} - \omega_{2j}^{2i-1}\right),\\
\omega_{2j-1}^{(2i-1)^*} - \omega_{2i-1}^{(2j-1)^*} &= \cot\theta\left(\omega_{(2i-1)^*}^{(2j)^*} - \omega_{(2j-1)^*}^{(2i)^*}\right),\\
\omega_{2i}^{(2j)^*} - \omega_{2j}^{(2i)^*} &= \cot\theta\left(\omega_{(2i)^*}^{(2j-1)^*} - \omega_{(2j)^*}^{(2i-1)^*}\right),\\
\omega_{(2i-1)^*}^{(2j-1)^*} - \omega_{2i-1}^{2j-1} &= \cot\theta\left(\omega_{2i-1}^{(2j)^*} - \omega_{2i}^{(2j-1)^*}\right),\\
\omega_{(2j)^*}^{(2i-1)^*} - \omega_{2j}^{2i-1} &= \cot\theta\left(\omega_{2i-1}^{(2j-1)^*} - \omega_{2i}^{(2j)^*}\right),\\
\omega_{(2i)^*}^{(2j)^*} - \omega_{2i}^{2j} &= \cot\theta\left(\omega_{2i-1}^{(2j)^*} - \omega_{2i}^{(2j-1)^*}\right),\\
\omega_{2j-1}^{(2i)^*} - \omega_{2i}^{(2j-1)^*} &= \cot\theta\left(\omega_{(2i)^*}^{(2j)^*} - \omega_{(2i-1)^*}^{(2j-1)^*}\right),
\end{aligned}
$$

for any $i, j = 1, ..., m$.

Lemma 4.1.2 Let M be an n-dimensional proper θ-slant submanifold of a complex space form $\overline{M}^m(4\varepsilon)$. Then the curvature tensor $\overline{R}$ of $\overline{M}^m(4\varepsilon)$ satisfies

$$
\left(\overline{R}(X,Y)Z\right)^{\perp} = \varepsilon\left\{\langle JY, Z\rangle FX - \langle JX, Z\rangle FY + 2\langle X, JY\rangle FZ\right\}
$$

for X, Y, Z tangent to M, where $\left(\overline{R}(X,Y)Z\right)^{\perp}$ denotes the normal component of $\overline{R}(X,Y)Z$

This lemma follows easily from the curvature formula of complex space form.

For Kahlerian slant submanifolds we have the following.

Lemma 4.1.3 Let M be an n-dimensional $(n = 2m)$ proper θ-slant submanifold of a Kahlerian slant submanifold. If M is Kahlerian slant, then, with respect to an adapted frame, we have

$$\omega_1^{2j-1} = \omega_2^{2j}, \omega_1^{2j} = -\omega_2^{2j-1}, j = 1, ..., m$$

Proof: Since M is Kahlerian slant , $\nabla_X (PY) = P(\nabla_X Y)$ for X, Y tangent to M. Thus we have

$$\begin{aligned}
\omega_1^{2j-1}(X) &= \left\langle \overline{\nabla}_X e_1, e_{2j-1} \right\rangle = \left\langle \overline{\nabla}_X e_1, -(\sec\theta) P e_{2j} \right\rangle = \sec\theta \left\langle e_1, \overline{\nabla}_X (P e_{2j}) \right\rangle \\
&= \sec\theta \left\langle e_1, P \overline{\nabla}_X e_{2j} \right\rangle = -\sec\theta \left\langle P e_1, \overline{\nabla}_X e_{2j} \right\rangle = -\left\langle e_2, \overline{\nabla}_X e_{2j} \right\rangle \\
&= \left\langle \overline{\nabla}_X e_2, e_{2j} \right\rangle = \omega_2^{2j}(X)
\end{aligned}$$

that is $\omega_1^{2j-1} = \omega_2^{2j}$. Also we have

$$\begin{aligned}
\omega_2^{2j-1}(X) &= \left\langle \overline{\nabla}_X e_2, e_{2j-1} \right\rangle = \left\langle \overline{\nabla}_X (\sec\theta) P e_1, -(\sec\theta) P e_{2j} \right\rangle \\
&= \sec^2\theta \left\langle P e_1, \overline{\nabla}_X (P e_{2j}) \right\rangle = \sec^2\theta \left\langle P e_1, P\left(\overline{\nabla}_X e_{2j}\right) \right\rangle \\
&= \sec^2\theta \left\langle P^2 e_1, \overline{\nabla}_X e_{2j} \right\rangle = \left\langle e_1, \overline{\nabla}_X e_{2j} \right\rangle = -\left\langle \overline{\nabla}_X e_1, e_{2j} \right\rangle = -\omega_1^{2j}(X)
\end{aligned}$$

that is $\omega_1^{2j} = -\omega_2^{2j-1}, j = 1, ..., m$.

Definition 4.1.4: An n-dimensional submanifold M of a Kahlerian manifold is called H-umbilical if its second fundamental form h takes the following simple form:

$$h(e_1, e_1) = \lambda e_{1^*} \;\; , h(e_2, e_2) = ... = h(e_n, e_n) = \mu e_{1^*}$$

$$h(e_1, e_j) = \mu e_{j^*} \; , h(e_j, e_k) = 0 \; , j \neq k \text{ and } j, k = 2, ..., n$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field $e_1, ..., e_n$ where $e_{1^*}, ..., e_{n^*}$ are unit vectors in the directions of $Fe_1, ..., Fe_n$ respectively.

Definition 4.1.5: An H-umbilical slant submanifold with $\mu = 0$ is simply called cylindrical slant.

Lemma 4.1.6 Every n-dimensional H-umbilical proper slant submanifolds of a Kahlerian n-manifold $\overline{M}^n$ is Kahlerian slant.

Proof: Follows from the definition of H-umbilical submanifolds and the fact that a proper slant submanifold M of a Kahlerian manifold is a Kahlerian slant if and only if the shape operator A of M satisfies $A_{FX}Y = A_{FY}X$ for X, Y tangent to M.

4.2 ClASSIFICATION OF H-UMBLICAL SLANT SUBMANIFOLDS

We introduce the notion of slant space forms as follows. We call a Riemannian manifold (N, g) a slant space form if there exist a $\theta \in \left(0, \frac{\pi}{2}\right)$ and an endomorphism P on the tangent bundle TN so that:

(1) $\nabla P = 0$, $P^2 = -\left(\cos^2\theta\right) I$ and $g\left(PX, Y\right) + g\left(X, PY\right) = 0$ for X, Y tangent to N.

(2) N has constant slant sectional curvature, i.e., the slant sectional curvature $K\left(X, PY\right)$ of the slant 2-plane spanned by X, PX is independent of the choice of the vector $X \in TN$, $X \neq 0$.

Clearly, a Kahlerian slant submanifold M of a Kahlerian manifold is a slant space form if $\left(M, g, \widetilde{J}\right)$ is a complex space form where $\widetilde{J} = \left(\sec\theta\right) P$.

The purpose of this section to prove the following classification theorem.

Theorem 4.2.1 Let M be an n-dimensional H-umbilical proper slant submanifold of a complete simply-connected complex space form $\overline{M}^n(4c)$, $n > 2$. Then one of the following three statements holds:

(1) M is flat and is immersed as an open part of a slant n-plane in the complex Euclidean n-space C^n;

(2) M is flat and is immersed as a cylindrical slant submanifold in C^n;

(3) $c < 0$, M is a slant space form of constant slant sectional curvature $4c\cos^2\theta$, and M is immersed as an H-umbilical submanifold in the complex hyperbolic n-space $CH^n(4c)$ satisfying 4.1.4 with $\lambda = 2\mu = \pm 2\sqrt{-c}\sin\theta$.

*Proof***.** If M is an H-umbilical proper slant submanifold of $\overline{M}^n(4c)$, then the second fundamental form of M takes the following form:

$$\begin{array}{c} h(e_1, e_1) = \lambda e_{1^*}, h(e_2, e_2) = ... = h(e_n, e_n) = \mu e_{1^*}, \\ h(e_1, e_j) = \mu e_{j^*}, h(e_j, e_k) = 0, j \neq k, j, k = 2, ..., n \end{array} \tag{4.2.1}$$

for some suitable functions λ and μ with respect to some adapted local frame field $e_1, ..., e_n, e_{1^*}, ..., e_{n^*}$. From lemma 4.1.6 we know that M is Kahlerian slant.

Using 4.1.2 and 4.2.1, we have

$$\begin{array}{c} \omega_1^{1^*} = \lambda\omega^1, \omega_1^{j^*} = \omega_j^{1^*} = \mu\omega^j, \omega_j^{j^*} = \mu\omega^1, 2 \leq j \leq n, \\ \omega_j^{k^*} = 0, 2 \leq j \neq k \leq n. \end{array} \tag{4.2.2}$$

$$\omega_1^{1^*} = \sum_{k=1}^{n} h_{1k}^{1^*} \omega^k$$

$$h_{1k}^{1^*} = g\left(h\left(e_1, e_k\right), e_{1^*}\right) = \begin{cases} \lambda g\left(e_{1^*}, e_{1^*}\right) = \lambda, \ k = 1 \\ \mu g\left(e_{k^*}, e_{1^*}\right) = 0, \ k = 2, 3, .. \end{cases}$$

since,

$$h\left(e_1, e_k\right) = \begin{cases} \lambda e_{1^*}, \ k = 1 \\ \mu e_{k^*,} \ k = 2, 3, .. \end{cases}$$

$$\Rightarrow \omega_1^{1^*} = \lambda \omega^1.$$

From Lemma 4.1.1 and 4.2.2 we find

$$\begin{array}{c} \omega_{1^*}^{2^*} = \omega_1^2 - (\lambda + \mu) \cot \theta \omega^1, \\ \omega_{2k-1^*}^{1^*} = \omega_{2k-1}^1 - \mu \cot \theta \omega^{2k}, \ k \geq 2, \\ \omega_{2k^*}^{1^*} = \omega_{2k}^1 + \mu \cot \theta \omega^{2k-1}, \ k \geq 2, \\ \omega_{j^*}^{2^*} = \omega_j^2 - \mu \cot \theta \omega^j, \ j \geq 3, \\ \omega_{l^*}^{j^*} = \omega_l^j, \ j, l \geq 3. \end{array} \tag{4.2.3}$$

If $\lambda = \mu = 0$, then M is a totally geodesic θ-slant submanifolds, since only totally geodesic submanifolds of a complex space form $\overline{M}^n(4c)$ with $c \neq 0$ are either complex submanifolds or totally real submanifolds, but M is proper so neither complex nor totally real $\Rightarrow c = 0$.

Now, we assume that M non totally geodesic. From Equation of Codazzi

$$\left(\overline{R}\left(X, Y\right) Z\right)^{\perp} = \left(\overline{\nabla}_X h\right)\left(Y, Z\right) - \left(\overline{\nabla}_Y h\right)\left(X, Z\right)$$

with $X = e_1, Y = Z = e_2$, and using (4.1.1), (4.2.1-3) and lemma (4.1.2) we get

$$e_1\mu = (\lambda - 2\mu)\,\omega_1^2(e_2)\,, \tag{4.2.4}$$

$$e_2\mu = 3\mu\omega_1^2(e_1) - (\lambda+\mu)\,\mu\cot\theta + 3c\sin\theta\cos\theta, \tag{4.2.5}$$

$$\mu\omega_1^j(e_1) = 0,\ \ j = 3, ..., n \tag{4.2.6}$$

to prove these using

$$\left(\overline{R}\,(X,Y)\,Z\right)^{\perp} = c\left\{\langle JY,Z\rangle\,FX - \langle JX,Z\rangle\,FY + 2\,\langle X,JY\rangle\,FZ\right\}$$

we have

$$\left(\overline{R}\,(e_1,e_2)\,e_2\right)^{\perp} = c\left\{\langle Je_2,e_2\rangle\,Fe_1 - \langle Je_1,e_2\rangle\,Fe_2 + 2\,\langle e_1,Je_2\rangle\,Fe_2\right\}$$

$$\langle Je_2,e_2\rangle = \langle Pe_2,e_2\rangle = \langle -\cos\theta e_1,e_2\rangle = 0$$

$$\langle Je_1,e_2\rangle = \langle Pe_1,e_2\rangle = \langle \cos\theta e_2,e_2\rangle = \cos\theta$$

and $\langle Je_2,e_1\rangle = -\cos\theta$

$$\begin{aligned}\left(\overline{R}\,(e_1,e_2)\,e_2\right)^{\perp} &= c\left\{0 - \cos\theta Fe_2 - 2\cos\theta Fe_2\right\} = -3c\cos\theta Fe_2\\ &= -3c\sin\theta\cos\theta e_{2^*}\end{aligned}$$

Now,

$$\begin{aligned}(D_{e_1}h)\,(e_2,e_2) - (D_{e_2}h)\,(e_1,e_2) &= D_{e_1}h\,(e_2,e_2) - 2h\,(\nabla_{e_1}e_2,e_2) - D_{e_2}h\,(e_1,e_2)\\ &\quad + h\,(\nabla_{e_2}e_1,e_2) + h\,(e_1,\nabla_{e_2}e_2)\end{aligned}$$

with

$$\nabla_{e_1} e_2 = \sum_{k=1}^{n} \omega_2^k (e_1) e_k = \omega_2^1 (e_1) e_1 + \omega_2^2 (e_1) e_2 + ...$$

$$\nabla_{e_2} e_1 = \sum_{k=1}^{n} \omega_1^k (e_2) e_k = \omega_1^1 (e_2) e_1 + \omega_1^2 (e_2) e_2 + ...$$

$$\nabla_{e_2} e_2 = \sum_{k=1}^{n} \omega_2^k (e_2) e_k = \omega_2^1 (e_2) e_1 + \omega_2^2 (e_2) e_2 + ...$$

since $\omega_i^i = 0$, $\Rightarrow h(e_2, \nabla_{e_1} e_2) = \omega_2^1 (e_1) \mu e_{2^*}$

$$h(\nabla_{e_2} e_1, e_2) = \omega_1^2 (e_2) \mu e_{1^*}$$

$$h(e_1, \nabla_{e_2} e_2) = \sum_{k=1}^{n} \omega_2^k (e_2) \mu e_{k^*}.$$

and

$$D_{e_2} \mu e_{2^*} = \mu D_{e_2} e_{2^*} + e_2(\mu) e_{2^*} = \mu \sum_{j=1}^{n} \omega_{2^*}^j (e_2) e_j + \mu \sum_{j=1}^{n} \omega_{2^*}^{j^*} (e_2) e_{j^*} + e_2(\mu) e_{2^*}$$

$$D_{e_1} \mu e_{1^*} = \mu D_{e_1} e_{1*} + e_1(\mu) e_{1^*} = \mu \sum_{j=1}^{n} \omega_{1^*}^j (e_1) e_j + \mu \sum_{j=1}^{n} \omega_{1^*}^{j^*} (e_1) e_{j^*} + e_1(\mu) e_{1^*}$$

we get

$$\begin{gathered}(D_{e_1} h)(e_2, e_2) - (D_{e_2} h)(e_1, e_2) \\ = D_{e_1} \mu e_{1^*} - 2\omega_2^1 (e_1) \mu e_{2^*} - D_{e_2} \mu e_{2^*} \\ + \omega_1^2 (e_2) \mu e_{1^*} + \sum_{k=1}^{n} \omega_2^k (e_2) \mu e_{k^*} = -3c \sin\theta \cos\theta e_{2^*}\end{gathered} \tag{4.2.7}$$

Now taking inner product in (4.2.7) with e_{1^*} we get $\langle D_{e_2} \mu e_{2^*}, e_{1^*} \rangle = \mu \omega_{2^*}^{1^*} (e_2)$ and $\langle D_{e_1} \mu e_{1^*}, e_{1^*} \rangle = e_1(\mu)$.

So,

$$e_1(\mu) - \mu\omega_{2^*}^{1^*}(e_2) + \mu\omega_1^2(e_2) + \mu\omega_2^1(e_2) = 0$$

$$\Rightarrow e_1(\mu) = \mu\left(-\omega_{1^*}^{2^*}(e_2)\right) = -\mu\left(\omega_1^2(e_2) - (\lambda+\mu)\cot\theta\omega^1(e_2)\right).$$

Next taking the inner product in (4.2.7) with e_{2^*}

$$\langle D_{e_2}\mu e_{2^*}, e_{2^*}\rangle = e_2(\mu)$$

$$\langle D_{e_1}\mu e_{1^*}, e_{2^*}\rangle = \mu\omega_{1^*}^{2^*}(e_1) = \mu\omega_1^2(e_1) - \mu(\lambda+\mu)\cot\theta,$$

and,

$$\mu\omega_1^2(e_1) - \mu(\lambda+\mu)\cot\theta - 2\mu\omega_2^1(e_1)\mu - e_2\mu = -3c\sin\theta\cos\theta$$

$$\Rightarrow e_2(\mu) = 3\mu\omega_1^2(e_1) - \mu(\lambda+\mu)\cot\theta + 3c\sin\theta\cos\theta.$$

Taking inner product in (4.2.7) with e_{j^*}, $j \geq 3$, we get

$$\langle D_{e_1}\mu e_{1^*}, e_{j^*}\rangle = \left\langle \mu\sum_{k=1}^{n}\omega_{1^*}^{j}(e_1)e_k + \mu\sum_{k=1}^{n}\omega_{1^*}^{k^*}(e_1)e_{k^*} + e_1(\mu)e_{1^*}, e_{j^*}\right\rangle$$

$$= \mu\omega_{1^*}^{j^*}(e_1),$$

$$\langle D_{e_2}\mu e_{2^*}, e_{j^*}\rangle = \mu\omega_{2^*}^{j^*}(e_2).$$

and,

$$\mu\omega_{1^*}^{j^*}(e_1) - 2(0) - \mu\omega_{2^*}^{j^*}(e_2) + \mu\omega_2^j(e_2) = 0$$

$$\Rightarrow \mu\omega_{1^*}^{j^*}(e_1) + \mu\omega_j^2(e_2) - \mu^2\cot\theta\omega^j(e_2) - \mu\omega_j^2(e_2) = 0$$

$$\Rightarrow \mu\omega_{1^*}^{j^*}(e_1) = 0.$$

Similarly, from Equation of Codazzi with $X = e_1, \{Y, Z\} = \{e_2, e_j\}$ for $j \geq 3$, and using (4.2.1), (4.2.2) and Lemmas (4.1.1) and (4.1.2), we find

$$e_2\mu = \mu\omega_1^2(e_1) + 2c\sin\theta\cos\theta, \tag{4.2.8}$$

$$\mu\omega_1^j(e_1) = 0, \tag{4.2.9}$$

$$\mu\omega_2^k(e_1) = (\lambda - 2\mu)\,\omega_1^k(e_2) = (\lambda - 2\mu)\,\omega_1^2(e_k)\,,\ \ k \geq 3. \tag{4.2.10}$$

Combining (4.2.5) and (4.2.8) we get

$$3\mu\omega_1^2(e_1) - (\lambda + \mu)\,\mu\cot\theta + 3c\sin\theta\cos\theta = \mu\omega_1^2(e_1) + 2c\sin\theta\cos\theta$$

$$\Rightarrow 2\mu\omega_1^2(e_1) = (\lambda + \mu)\,\mu\cot\theta - c\sin\theta\cos\theta \tag{4.2.11}$$

From Equation of Codazzi with $X = Z = e_1$, $Y = e_{2j-1}$ for $j > 1$, and using (4.1.1), (4.2.1-3) and Lemma (4.1.2), we find

$$e_1\mu = (\lambda - 2\mu)\,\omega_1^{2j-1}(e_{2j-1})\,, \tag{4.2.12}$$

$$e_{2j-1}\lambda = (\lambda - 2\mu)\,\omega_1^{2j-1}(e_1)\,, \tag{4.2.13}$$

$$(\lambda - 2\mu)\,\omega_1^2\,(e_{2j-1}) = (\lambda - 2\mu)\,\omega_1^k\,(e_{2j}) = 0,\;\; k \neq 2, 2j-1, 2j, \tag{4.2.14}$$

$$(\lambda - 2\mu)\omega_1^{2j}\,(e_{2j-1}) = (\lambda - 2\mu)\,\mu\cot\theta. \tag{4.2.15}$$

Similarly, from Equation of Codazzi with $X = e_{2j-1}$, $Y = e_1$, $Z = e_{2j}$ for $j > 1$, and using (4.1.1), (4.2.1-3) and Lemma (4.1.2), we find

$$(2\mu - \lambda)\,\omega_1^{2j}\,(e_{2j-1}) = c\sin\theta\cos\theta + \mu^2\cot\theta \tag{4.2.16}$$

by comparing the coefficients of e_{1^*}. Combining (4.2.15) and (4.2.16) yields

$$(2\mu - \lambda)\,\mu\cot\theta = c\sin\theta\cos\theta + \mu^2\cot\theta$$

$$\Rightarrow \left(2\mu^2 - \lambda\mu - \mu^2\right)\cot\theta = c\sin\theta\cos\theta$$

$$\Rightarrow \mu^2 - \lambda\mu = c\sin^2\theta$$

$$\Rightarrow (\mu - \lambda)\,\mu = c\sin^2\theta. \tag{4.2.17}$$

Divide (4.2.11) by $\cot\theta$ we find

$$2\mu\omega_1^2\,(e_1)\tan\theta = (\lambda + \mu)\,\mu - c\sin^2\theta$$

then using (4.2.17) we get

$2\mu\omega_1^2(e_1)\tan\theta = (\lambda+\mu)\mu - (\mu-\lambda)\mu = \lambda\mu + \mu^2 - \mu^2 + \lambda\mu = 2\lambda\mu$

$$\Rightarrow \mu\omega_1^2(e_1) = \lambda\mu\cot\theta. \tag{4.2.18}$$

On the other hand, from Equation of Codazzi with $X = e_2$, $Y = Z = e_j$ for $j \geq 3$, and using (4.1.1), (4.2.1-3) and Lemma (4.1.2), we find

$$e_2\mu = 0, \tag{4.2.19}$$

$$\mu\omega_1^2(e_2) = \mu\omega_1^j(e_j),\ j \geq 3, \tag{4.2.20}$$

$$\mu\omega_1^k(e_2) = 0,\ k \geq 3, \tag{4.2.21}$$

$$3\mu\omega_1^j(e_2) = \mu\omega_1^2(e_j). \tag{4.2.22}$$

From (4.2.19) and (4.2.9) we obtain $\mu\omega_1^2(e_1) = -2c\sin\theta\cos\theta$. Therefore, we find

$\lambda\mu = -2c\sin^2\theta$ by (4.2.18). By Combing this with (4.2.17) we get

$\lambda\mu = -2(\mu-\lambda)\mu \Rightarrow -\lambda\mu + 2\lambda\mu - 2\mu^2 = 0 \Rightarrow -2\mu^2 + \lambda\mu = 0$

$$\Rightarrow \mu(\lambda - 2\mu) = 0. \tag{4.2.23}$$

Consequently, either $\mu = 0$ or $\lambda = 2\mu$ at each point on the slant submanifold.

If $\mu = 0$ at some point on the proper slant submanifold, then (4.2.17) yields $c = 0$. Moreover, in this case, (4.2.17) and (4.2.23) imply that $\mu = 0$ identically. Hence, we obtain statement (2) in this case.

If $\lambda = 2\mu$ and $\mu \neq 0$, then (4.2.17) implies $\varepsilon < 0$ and $\mu = \pm\sqrt{-c}\sin\theta$. Hence, the curvature tensor R of M satisfies

$$\begin{aligned} R(X,Y;Z,W) = \quad & c\cos^2\theta\,\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\} + \\ & c\,\{\langle PX,W\rangle\langle PY,Z\rangle - \langle PX,Z\rangle\langle PY,W\rangle + \\ & 2\langle X,PY\rangle\langle PZ,W\rangle\}\,. \end{aligned} \tag{4.2.24}$$

Thus M is a slant space form with slant sectional curvature $4c\cos^2\theta$. Consequently, we obtain statement (3).

4.3 A GENERAL INEQUALITY FOR KAEHLERIAN SLANT SUBMANIFOLDS

Consider the complex number $(m+1)$-space C_1^{m+1} endowed with the pseudo-Euclidean metric

$$g_0 = -dz_0 d\overline{z}_0 + \sum_{j=1}^{n} dz_j d\overline{z}_j$$

Put

$$H_1^{2m+1} = \{z = (z_0, z_1, ..., z_m) : \langle z, z \rangle = -1\}$$

where $\langle , \rangle$ denotes the inner product on C_1^{m+1} induced from the metric g_0. Let $C^* = \left\{\lambda \in C : \lambda\overline{\lambda} = 1\right\}$. Then there is a C^*-action on H_1^{2m+1} defined by $z \longmapsto \lambda z$. At $z \in H_1^{2m+1}$, iz is tangent to the flow of the action. The orbit of this action is given by $z_t = e^{it}z$ with $dz_t/dt = iz_t$ which lies in the negative-definite plane spanned by z and iz. The quotient space $H_1^{2m+1}/\sim$ under the C^*-action is nothing but the complex hyperbolic space $CH^m(-4)$. The canonical projection $\pi : H_1^{2m+1} \to CH^m(-4)$ is called the hyperbolic Hopf fibration.

For each isometric immersion $f : M \to CH^m(-4)$, the preimage $\widehat{M} = \pi^{-1}(M)$ is a principal circle bundle over M with totally geodesic fibers and the lift $\widehat{f} : \widehat{M} \to H_1^{2m+1}$ of f is an isometric immersion such that the diagram

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{f}} & H_1^{2m+1} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & CH^m(-4) \end{array}$$

commutes.

Conversely, if $\psi : \widehat{M} \to H_1^{2m+1}$ is an isometric immersion which is invariant under the action of C^*, there is a unique isometric immersion $\psi_\pi : \pi\left(\widehat{M}\right) \to CH^m(-4)$, called the projection of ψ, such that the associated diagram commutes.

Theorem 4.3.1 Let $\chi : M \to \widetilde{M}^n(4c)$, $c \in \{-1, 0, 1\}$, be a Kaehlerian θ-slant submanifold of dimension n in a complete simply-connected complex space form $\widetilde{M}^n(4\varepsilon)$. Then we have

$$H^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau - \frac{n+2}{n}\left(1+\frac{3\cos^2\theta}{n-1}\right)c$$

where H^2 is the squared mean curvature and τ is the scalar curvature defined by $\tau = \sum_{i<j} K_{ij}$, K_{ij} is the sectional curvature of the 2-plane spanned by e_i and e_j for a local orthonormal frame $e_1, ...e_n$.

Proof: If M is a complex submanifold of $\overline{M}^n(4c)$, then $\theta = 0$ and M is a minimal submanifold. Thus, in this case the above inequality reduces to $\tau \leq \frac{n(n+2)c}{2}$, with the equality holding if and only if M is a totally geodesic complex submanifold.

Now, assume that M is a Kaehlerian θ-slant submanifold of $\overline{M}^n(4c)$ with $\theta \neq 0$. Then we know that the shape operator satisfies

$$A_{FX}Y = A_{FY}X, \text{ for } X, Y \in TM. \tag{4.3.1}$$

From the definition of mean curvature function we have

$$n^2H^2 = \sum_i\left(\sum_j\left(h^i_{jj}\right)^2 + 2\sum_{j<k}h^i_{jj}h^i_{kk}\right) \tag{4.3.2}$$

From Equation (3.2.4) of Gauss, we get

$$2\tau = n^2H^2 - \sum_{i,j,k=1} \left(h^i_{jk}\right)^2 + \left(n\left(n-1\right) + 3n\cos^2\theta\right)c, \tag{4.3.3}$$

where $h^i_{jk} = \langle h\left(e_j, e_k\right), e_{i^*}\rangle$. Thus, by applying (4.3.1), (4.3.2) and (4.3.3), we obtain

$$\tau = \frac{1}{2}\left(n\left(n-1\right) + 3n\cos^2\theta\right)c + \sum_i \sum_{j<k} h^i_{jj}h^i_{kk} - \sum_{i\neq j}\left(h^i_{jj}\right)^2 - 3\sum_{i<j<k}\left(h^i_{jk}\right)^2. \tag{4.3.4}$$

Let $m = \frac{(n+2)}{(n-1)}$. Then, from (4.3.2), (4.3.3) and (4.3.4), we get

$$\begin{aligned}
n^2H^2 - m\left(2\tau - n\left(n-1\right)c - 3nc\cos^2\theta\right) &= \sum_i\left(h^i_{jj}\right)^2 + (1+2m)\sum_{i\neq j}\left(h^i_{jj}\right)^2 \\
&= +6m\sum_{i<j<k}\left(h^i_{jk}\right)^2 - 2\left(m-1\right)\sum_i\sum_{j<k} h^i_{jj}h^i_{kk} \\
&= \sum_i\left(h^i_{ii}\right)^2 + 6m\sum_{i<j<k}\left(h^i_{jk}\right)^2 \\
&\quad + \left(m-1\right)\sum_{i\neq j}\sum_{,kj<k}\left(h^i_{jj} - h^i_{kk}\right)^2 \\
&\quad + \left(1 + 2m - \left(n-2\right)\left(m-1\right)\right)\sum_{j\neq i}\left(h^i_{jj}\right)^2 \\
&\quad -2\left(m-1\right)\sum_{j\neq i} h^i_{ii}h^i_{jj} \\
&= 6m\sum_{i<j<k}\left(h^i_{jk}\right)^2 + \left(m-1\right)\sum_{i\neq j,k}\sum_{j<k}\left(h^i_{jj} - h^i_{kk}\right)^2 \\
&\quad + \frac{1}{n-1}\sum_{j\neq i}\left(h^i_{ii} - \left(n-1\right)\left(m-1\right)h^i_{jj}\right)^2 \\
&\geq 0
\end{aligned} \tag{4.3.5}$$

which implies our inequality.

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