Here are model answers for the questions in M-374 I-midterm examination

Q. 1. Consider the function $\alpha : R \to R^3$ defined by $\alpha(t) = \left(\sqrt{2}\tan^{-1} t, \frac{1}{2}\ln(1+t^2), \frac{1}{2}\ln\left(\frac{1}{1+t^2}\right)\right)$.

(a) Is α a regular curve?

(b) Find the expression for the arc-length parameter $s = f(t)$ for this curve α.

Answer: (a) Each component function $\sqrt{2}\tan^{-1} t$, $\frac{1}{2}\ln(1+t^2)$, $\frac{1}{2}\ln\left(\frac{1}{1+t^2}\right)$ of α is of class C^∞, therefore α is of class C^∞. Moreover, we have

$$\alpha^{'}(t) = \frac{d\alpha}{dt} = \left(\frac{\sqrt{2}}{1+t^2}, \frac{t}{1+t^2}, -\frac{t}{1+t^2}\right)$$

Consequently we have $\left\|\alpha^{'}(t)\right\| = \sqrt{\frac{2}{1+t^2}} \neq 0$ for each $t \in R$. Hence α is a regular curve.

(b) We know that the arc-length parameter $s = f(t)$ is given by

$$\begin{aligned} s &= f(t) = \int_0^t \left\|\alpha^{'}(t)\right\| dt = \int_0^t \sqrt{\frac{2}{1+t^2}} dt \\ &= \sqrt{2}\ln\left[t + \sqrt{1+t^2}\right]_0^t \\ &= \sqrt{2}\ln\left[t + \sqrt{1+t^2}\right] \end{aligned}$$

this gives the arc-length parameter.

Q.2 If a unit speed curve $\alpha(s)$ of class C^k, $k \geq 3$ has tangent vector field

$$T(s) = \left(\frac{1}{2}\sqrt{1+s}, -\frac{1}{2}\sqrt{1-s}, \frac{1}{\sqrt{2}}\right), \quad |s| < 1 \tag{1}$$

Find the Frenet-Serret apparatus of α. Is this curve α a helix?

Answer: Recall that $\alpha^{'}(s) = T(s) = \left(\frac{1}{2}\sqrt{1+s}, -\frac{1}{2}\sqrt{1-s}, \frac{1}{\sqrt{2}}\right)$. Thus we find $T^{'}(s) = \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right)$. The curvature

$$\kappa(s) = \left\|T^{'}(s)\right\| = \frac{1}{4}\sqrt{\frac{2}{1-s^2}} = \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{1-s^2}} \neq 0 \tag{2}$$

The principal normal vector field N is given by $N(s) = \frac{T^{'}(s)}{k(s)}$, which together with $T^{'}(s) = \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right)$ and equation (2) gives

$$N(s) = \frac{1}{\sqrt{2}}\left(\sqrt{1-s}, \sqrt{1+s}, 0\right) \tag{3}$$

Now the binormal vector field B is given by $B(S) = T(s) \times N(s)$, which together with equations (1) and (3) give

$$B(s) = \left(-\frac{1}{2}\sqrt{1+s}, \frac{1}{2}\sqrt{1-s}, \frac{1}{\sqrt{2}} \right) \tag{4}$$

Finally, to find the torsion τ, we differentiate equation (3) to get

$$N^{'}(s) = \frac{1}{2\sqrt{2}} \left(-\frac{1}{\sqrt{1-s}}, \frac{1}{\sqrt{1+s}}, 0 \right)$$

and thus the torsion of the curve α is given by

$$\tau(s) = \left\langle N^{'}(s), B(s) \right\rangle = \frac{1}{4\sqrt{2}}\sqrt{\frac{1+s}{1-s}} + \frac{1}{4\sqrt{2}}\sqrt{\frac{1-s}{1+s}} = \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{1-s^2}} \tag{5}$$

The equations (1)-(5) give the Frenet-Serret apparatus $\{\kappa, \tau, T, N, B\}$ of the unit speed curve α.

Since the equations (2) and (5) give $\tau(s) = \kappa(s)$, the curve is a helix.

Q.3 Let $\alpha(s)$ be a sphere curve with Frenet-Serret apparatus $\{\varkappa, \tau, T, N, B\}$. If $\varkappa \neq 0$, $\tau \neq 0$ and

$$\alpha(s) = -\rho N - (\rho^{'} \sigma) B$$

where $\rho = \frac{1}{\varkappa}$, and $\sigma = \frac{1}{\tau}$. Prove that $\rho^{''}\sigma + \rho^{'}\sigma^{'} + \frac{\rho}{\sigma} = 0$. Assuming that σ is a nonzero constant and the function ρ satisfies $\rho(0) = 1$, $\rho^{'}(0) = 0$ for this spherical curve, show that $\alpha(s) = -\cos(\tau s)N + \sin(\tau s)B$.

Answer: From the given expression of the curve $\alpha(s) = -\rho N - (\rho^{'}\sigma)B$, we have

$$T = \alpha^{'}(s) = -\rho^{'} N - \rho N^{'} - (\rho^{'}\sigma)^{'} B - (\rho^{'}\sigma) B^{'}$$

Using the formulas $N^{'} = -\kappa T + \tau B$ and $B^{'} = -\tau N$ in above equation, we get

$$\left(\rho^{''}\sigma + \rho^{'}\sigma^{'} + \frac{\rho}{\sigma} \right) B = 0$$

Since B is a unit vector we get

$$\rho^{''}\sigma + \rho^{'}\sigma^{'} + \frac{\rho}{\sigma} = 0 \tag{A}$$

Now assume that σ is a constant and $\sigma \neq 0$. Then the equation (A) takes the form $\rho^{''} + \frac{1}{\sigma^2}\rho = 0$, which has the solution

$$\rho(s) = c_1 \cos\frac{s}{\sigma} + c_2 \sin\frac{s}{\sigma}$$

where the constants c_1, c_2 are to be determined by the conditions $\rho(0) = 1$, $\rho^{'}(0) = 0$. This gives $c_1 = 1$ and $c_2 = 0$. Thus we have $\rho(s) = \cos\frac{s}{\sigma}$ and σ is a constant.Using this in the equation of the curve $\alpha(s) = -\rho N - (\rho^{'}\sigma)B$, we get

$$\alpha(s) = -\cos(\tau s)N + \sin(\tau s)B$$

Q.4 Show that the tangent spherical image of the unit speed curve

$$\alpha(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

is a plane curve.

Answer: The tangent spherical image of α is the curve $\beta(s) = T(s)$, and it is a non-unit speed curve

$$\beta(s) = \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right)$$

from which we calculate

$$\dot{\beta} = \frac{1}{2}\left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right), \ddot{\beta} = \frac{1}{2\sqrt{2}}\left(\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 0\right)$$

$$\dddot{\beta} = \frac{1}{4}\left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right)$$

and thus we find the product

$$\left[\dot{\beta}, \ddot{\beta}, \dddot{\beta}\right] = \frac{1}{16\sqrt{2}}\begin{vmatrix} -\cos\frac{s}{\sqrt{2}} & -\sin\frac{s}{\sqrt{2}} & 0 \\ \sin\frac{s}{\sqrt{2}} & -\cos\frac{s}{\sqrt{2}} & 0 \\ \cos\frac{s}{\sqrt{2}} & \sin\frac{s}{\sqrt{2}} & 0 \end{vmatrix} = 0$$

consequently the torsion $\tau = \frac{\left[\dot{\beta}, \ddot{\beta}, \dddot{\beta}\right]}{\left\|\dot{\beta}\times\ddot{\beta}\right\|^2} = 0$. Therefore the tangent spherical image $\beta(s) = T(s)$ is a plane curve.