PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 1, January 1996

REAL HYPERSURFACES OF CP^n WITH NON-NEGATIVE RICCI CURVATURE

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(Communicated by Christopher B. Croke)

ABSTRACT. We prove the non-existence of Levi flat compact real hypersurfaces without boundary in $CP^n, n > 1$, with non-negative totally real Ricci curvature.

1. INTRODUCTION

Let CP^n be the n-dimensional complex projective space with complex structure J and Kaehler metric g. It is well known (cf. A. Bejancu [1], p. 21) that a real hypersurface M of CP^n is a CR-submanifold of CP^n. More precisely, the CR-structure on M is defined as follows. Denote by $TM^{\perp}$ the normal bundle of M and consider the *totally real distribution* $RM = J(TM^{\perp})$ on M. Then the complementary orthogonal distribution HM to RM in TM is of rank $2(n-1)$ and it is invariant under J, that is, $J(HM) = HM$. That is why HM is called the *holomorphic distribution* on M. For any $x \in M$ denote by $(HM)_x$ the fibre of HM over x and define the complex vector space

$$H_x^{1,0}(M) = \{W_x = X_x - iJ_x(X_x)\colon X_x \in (HM)_x\}.$$

Consider the complexified tangent bundle $T^cM = TM \otimes C$ of M and note that

$$H^{1,0}(M) = \bigcup_{x \in M} H_x^{1,0}(M)$$

is an involutive complex vector subbundle of T^cM such that

$$H^{1,0}(M) \cap \overline{H^{1,0}(M)} = \{0\}.$$

Hence $H^{1,0}(M)$ defines a CR-structure on M (cf. A. Boggess [3], p. 121). Besides, we have $HM = H^{1,0}(M) \oplus \overline{H^{1,0}(M)}$.

Though $H^{1,0}(M)$ is involutive, it is not necessary that HM be integrable. Thus M. Okumura [6] and Y. Maeda [5] found different geometric characterizations of a class of real hypersurfaces of CP^n with non-integrable HM.

Received by the editors October 12, 1993 and, in revised form, July 19, 1994.
1991 *Mathematics Subject Classification.* Primary 53C40, 53C55; Secondary 53C12, 53C42, 53C16.
Key words and phrases. Real hypersurface, CR-submanifold, complex projective space, Levi flat, totally real Ricci curvature, holomorphic distribution.

For any $x \in M$ consider a unit vector $N_x \in T_xM^\perp$ and let $\xi_x = -J_x(N_x)$. Then following A. Boggess [3], define the Levi form at the point x as the map

$$L_x \colon H_x^{1,0}(M) \to \mathfrak{R}, \qquad L_x(W_x) = -g_x([X, JX]_x, \xi_x),$$

where X is the HM-vector field extension of $X_x = \frac{1}{2}(W_x + \overline{W}_x)$. We say that M is a *Levi flat CR-submanifold* if L_x vanishes for any $x \in M$. Taking into account that $\xi_x \in (RM)_x$, from the definition of L_x it follows that in case HM is integrable, M is Levi flat. The converse of this assertion is a consequence of Theorem 1 in A. Boggess [3], p. 158. Hence *M is Levi flat iff HM is integrable.*

Next, suppose M is an orientable real hypersurface in CP^n. Then there exists a globally defined unit normal vector field N on M. Thus the totally real distribution RM is globally spanned by $\xi = -JN$. The Ricci curvature of M in the direction ξ is called the *totally real Ricci curvature* and it is denoted by $\mathrm{Ric}(\xi, \xi)$. Our paper has the origin in the remark that the totally real Ricci curvature of K-contact manifold is a positive number (cf. D. Blair [2], p. 65) and that HM is not integrable. So it is natural to ask whether HM is not integrable in general for a real hypersurface of CP^n with non-negative totally real Ricci curvature. In this respect we prove the following result.

Theorem. *Let M be a compact orientable real hypersurface without boundary of $CP^n, n > 1$, such that $\mathrm{Ric}(\xi, \xi) \geq 0$ everywhere on M. Then HM is not integrable on M.*

The authors would like to express their sincere gratitude to the referee for many valuable suggestions to improve the paper.

2. Preliminaries

Let M be an orientable real hypersurface of CP^n. Denote by ∇ and $\overline{\nabla}$ the Levi Civita connection on M and CP^n respectively. Then we have the well-known formulae

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{2.1}$$

$$\overline{\nabla}_X N = -AX, \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M, N is the unit normal vector field to M and A is the shape operator of M. Let η be the 1-form dual to ξ, that is, $\eta(X) = g(X, \xi)$ for any $X \in \mathfrak{X}(M)$. Then we have $JX = JPX + \eta(X)N$, where P is the projection morphism of TM on HM. On using (2.1) and (2.2) and taking into account that J is parallel with respect to $\overline{\nabla}$ one obtains

$$\nabla_X \xi = JPAX, \qquad \forall X \in \mathfrak{X}(M). \tag{2.3}$$

Consider CP^n as a complex space form of the constant holomorphic sectional curvature $c = 4$. Then using the formulae of curvature tensor field of CP^n (cf. Kobayashi-Nomizu [5], p. 167), the equations of Gauss and Codazzi for the immersion of M in CP^n become

$$\begin{aligned} &g(R(X,Y)Z,W) \\ &\quad = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(Z,JY)g(JX,W) \\ &\qquad - g(Z,JX)g(JY,W) + 2g(X,JY)g(JZ,W) \\ &\qquad + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W) \end{aligned} \tag{2.4}$$

and

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = g(X,\xi)JY - g(Y,\xi)JX + 2g(X,JY)\xi, \tag{2.5}$$

respectively, for any $X, Y, Z, W \in \mathfrak{X}(M)$, where R is the curvature tensor corresponding to ∇.

Finally denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(HM)$ and $\Gamma(RM)$ the $\mathcal{F}(M)$-modules of smooth vector fields on M which belong to HM and RM respectively. We may always consider a local orthonormal field of frames $\{E_1, \dots, E_{n-1}, JE_1, \dots, JE_{n-1}, \xi\}$ on a coordinate neighborhood $U \subset M$, where $\{E_i, JE_i\} \subset \Gamma(HM), i \in \{1, \dots, n-1\}$. Such a frame field is said to be a *local CR-frame field* on M.

3. Proof of the Theorem

First we prove

Lemma. *Let M be a compact orientable real hypersurface without boundary of CP^n. Suppose $\mathrm{Ric}(\xi,\xi) \geq 0$ everywhere on M and HM is an integrable distribution on M. Then we have*

(i) $\mathrm{Ric}(\xi,\xi) = 0$,
(ii) $\nabla_X \xi = 0, \forall X \in \Gamma(HM)$,
(iii) $\nabla_X Y \in \Gamma(HM), \forall X, Y \in \Gamma(HM)$,
(iv) $AX \in \Gamma(RM), \forall X \in \Gamma(HM)$,
(v) $\|A\xi\|^2 = 2(n-1) + g(A\xi,\xi)^2$.

Proof. Choose a local CR-frame field $\{E_i, JE_i, \xi\}$ on M. Then taking into account that A and J are symmetric and antisymmetric with respect to g, that is, $g(AX,Y) = g(X,AY)$ and $g(JX,Y) = -g(X,JY)$ holds for $X, Y \in \mathfrak{X}(M)$, and using (2.3) obtain

$$\delta\eta = \operatorname{div}\xi = 0. \tag{3.1}$$

On the other hand, by direct calculations, using the integrability of HM obtain $d\eta(X,Y) = 0$ and $d\eta(\xi,X) = g(\nabla_\xi \xi, X)$, for any $X, Y \in \Gamma(HM)$. Hence locally on U we have

$$\|d\eta\|^2 = 2\sum_{i=1}^{n-1}\{g(\nabla_\xi \xi, E_i)^2 + g(\nabla_\xi \xi, JE_i)^2\} = 2\|\nabla_\xi \xi\|^2. \tag{3.2}$$

Since for each $x \in M$, we have a coordinate neighborhood U and a local CR-frame field $\{E_i, JE_i, \xi\}$ on U, (3.2) holds for each $x \in M$ and hence globally on M. Next, we recall that on any compact orientable Riemannian manifold M without boundary we have (cf. K. Yano [7], p. 41)

$$\int_M \left\{\mathrm{Ric}(X,X) - \frac{1}{2}\|d\alpha\|^2 + \|\nabla X\|^2 - (\delta\alpha)^2\right\} dv = 0,$$

where α is a 1-form dual to X on M. Replace X by ξ and α by η in the above integral formula and use (3.1) to obtain

$$\int_M \left\{\mathrm{Ric}(\xi,\xi) - \frac{1}{2}\|d\eta\|^2 + \|\nabla\xi\|^2\right\} dv = 0. \tag{3.3}$$

Since on each coordinate neighborhood U, we have

$$\|\nabla\xi\|^2 = \sum_{i=1}^{n-1}\{\|\nabla_{E_i}\xi\|^2 + \|\nabla_{JE_i}\xi\|^2\} + \|\nabla_\xi\xi\|^2,$$

on account of (3.2) we thus find

$$\|\nabla\xi\|^2 - \frac{1}{2}\|d\eta\|^2 = \sum_{i=1}^{n-1}\{\|\nabla_{E_i}\xi\|^2 + \|\nabla_{JE_i}\xi\|^2\} \geq 0 \tag{3.4}$$

on each U and consequently on M. As $\mathrm{Ric}(\xi,\xi) \geq 0$, it follows that the integrand in (3.3) is non-negative. Hence we must have $\mathrm{Ric}(\xi,\xi) = 0$ and $\|\nabla\xi\|^2 - \frac{1}{2}\|d\eta\|^2 = 0$. The second equation together with (3.4) gives $\nabla_{E_i}\xi = \nabla_{JE_i}\xi = 0$ for each local CR-frame field $\{E_i, JE_i, \xi\}$ and consequently we obtain (i) and (ii). Taking into account that ∇ is a Riemannian connection from (ii) we infer (iii). Moreover (iv) follows from (ii) on using (2.3). Finally using (iv), from (2.4) obtain

$$g(R(E_i,\xi)\xi, E_i) = 1 - g(A\xi, E_i)^2$$

and

$$g(R(JE_i,\xi)\xi, JE_i) = 1 - g(A\xi, JE_i)^2.$$

Hence (i) gives

$$\begin{aligned} 0 = \mathrm{Ric}(\xi,\xi) &= \sum_{i=1}^{n-1}\{g(R(E_i,\xi)\xi,E_i) + g(R(JE_i,\xi)\xi,JE_i)\} \\ &= 2(n-1) - \sum_{i=1}^{n-1}\{g(A\xi,E_i)^2 + g(A\xi,JE_i)^2\} \\ &= 2(n-1) - \{\|A\xi\|^2 - g(A\xi,\xi)^2\}, \end{aligned}$$

on any coordinate neighborhood U, which gives (v). This completes the proof of the Lemma. □

Remark. The assertions (i)–(iv) of the Lemma hold in a more general setting, namely, in case the complex projective space is replaced by an arbitrary Kaehler manifold.

Now we proceed with the proof of the Theorem. Suppose HM is integrable. Then using (iv) and (2.1) and taking into account that J is parallel with respect to $\overline{\nabla}$ obtain

$$\nabla_X JY = J\nabla_X Y, \quad \forall X, Y \in \Gamma(HM). \tag{3.5}$$

Thus on a coordinate neighborhood U, (iii) and (3.5) imply

$$\nabla_{E_i}E_j = \sum_{k=1}^{n-1}\{a_{ijk}E_k + b_{ijk}JE_k\}, \qquad \nabla_{E_i}JE_j = \sum_{k=1}^{n-1}\{a_{ijk}JE_k - b_{ijk}E_k\}, \tag{3.6}$$

and

(3.7)

$$\nabla_{JE_i}E_j = \sum_{k=1}^{n-1}\{c_{ijk}E_k + d_{ijk}JE_k\}, \qquad \nabla_{JE_i}JE_j = \sum_{k=1}^{n-1}\{c_{ijk}JE_k - d_{ijk}E_k\},$$

where $\{a_{ijk}, b_{ijk}, c_{ijk}, d_{ijk}\}$ are smooth functions on U satisfying

$$a_{ijk} + a_{ikj} = 0, \quad c_{ijk} + c_{ikj} = 0, \quad b_{ijk} = b_{ikj}, \quad d_{ijk} = d_{ikj}. \tag{3.8}$$

As $\nabla_\xi \xi$ is orthogonal to ξ, there exist smooth functions $\{a_i, b_i\}, i \in \{1, \ldots, n-1\}$, such that

$$\nabla_\xi \xi = \sum_{k=1}^{n-1} \{a_k E_k + b_k JE_k\}. \tag{3.9}$$

Finally using (ii) and (3.9) and taking into account that ∇ is a Riemannian connection, obtain

$$\nabla_\xi E_i = \sum_{k=1}^{n-1} \{a_{ik} E_k + b_{ik} JE_k\} - a_i \xi, \qquad \nabla_{E_i} \xi = 0, \tag{3.10}$$

and

$$\nabla_\xi JE_i = \sum_{k=1}^{n-1} \{c_{ik} E_k + d_{ik} JE_k\} - b_i \xi, \qquad \nabla_{JE_i} \xi = 0, \tag{3.11}$$

where $\{a_{ik}, b_{ik}, c_{ik}, d_{ik}\}$ are smooth functions on U. Now using (2.1), (2.2) and (3.9) and taking into account that $\overline{\nabla}$ is a Riemannian connection we infer that

$$a_i = -g(A\xi, JE_i) \quad \text{and} \quad b_i = g(A\xi, E_i).$$

Thus on account of (iv), we have

$$AE_i = b_i \xi, \quad AJE_i = -a_i \xi, \quad A\xi = \sum_{k=1}^{n-1} \{b_k E_k - a_k JE_k\} + c\xi, \tag{3.12}$$

where c is a smooth function on U. Then (v) in the Lemma and the last equation in (3.12) imply

$$\sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} = 2(n-1). \tag{3.13}$$

Further, take $X = E_i$ and $Y = \xi$ in (2.5), use (3.6), (3.9), (3.10) and (3.12) and equate the components with respect to the holomorphic frame $\{E_k, JE_k\}$ of $\Gamma(HM)$ to obtain

$$\begin{cases} E_i(a_k) = a_i a_k - b_i b_k + \delta_{ik} + \sum_{j=1}^{n-1} \{b_j b_{ijk} - a_j a_{ijk}\}, \\ E_i(b_k) = a_i b_k + b_i a_k - \sum_{j=1}^{n-1} \{b_j a_{ijk} + a_j b_{ijk}\}, \qquad i, k \in \{1, \ldots, n-1\}. \end{cases} \tag{3.14}$$

In a similar way, take $X = JE_i$ and $Y = \xi$ in (2.5) and use (3.7), (3.9), (3.11) and (3.12) to infer

$$\begin{cases} JE_i(b_k) = b_i b_k - a_i a_k + \delta_{ik} - \sum_{j=1}^{n-1} \{b_j c_{ijk} + a_j \, d_{ijk}\}, \\ JE_i(a_k) = a_i b_k + b_i a_k + \sum_{j=1}^{n-1} \{b_j \, d_{ijk} - a_j c_{ijk}\}. \end{cases} \tag{3.15}$$

From (3.14) and (3.15) on using (3.8) it is easy to obtain

$$\frac{1}{2} E_i \left(\sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} \right) = a_i \left(1 + \sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} \right) \tag{3.16}$$

and

$$\frac{1}{2}JE_i\left(\sum_{k=1}^{n-1}\{(a_k)^2+(b_k)^2\}\right)=b_i\left(1+\sum_{k=1}^{n-1}\{(a_k)^2+(b_k)^2\}\right), \tag{3.17}$$

respectively. On using (3.13) in (3.16) and (3.17) obtain $a_i = 0$ and $b_i = 0$ for any $i \in \{1, \dots, n-1\}$. Thus we get a contradiction to (3.13) and this completes the proof of the theorem. As a direct consequence of the Theorem we have

Corollary. *Let M be a Levi flat compact real hypersurface without boundary of $CP^n, n > 1$. Then there exists an open subset U of M such that the totally real Ricci curvature of M is negative everywhere on U.*

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