A Characterization for 3-Spheres

Sharief Deshmukh

Introduction

Let $M$ be an $n$-dimensional compact smooth manifold. Jacobowitz [3] proved that if $\psi: M \to R^{n+k}$ ($k \leq n-1$) is a smooth immersion with $\psi(M)$ contained in a closed ball of radius $\lambda$, then the sectional curvature of $M$ with respect to the induced metric satisfies $\sup K \geq \lambda^{-2}$. Coghlan and Itokawa [1] proved a pinching theorem for sectional curvature of a compact hypersurface $\psi: M \to R^{n+1}$. They proved that if $\psi(M)$ is contained in a closed ball of radius $\lambda$ and the sectional curvature satisfies $\sup K = \lambda^{-2}$, then $\psi(M)$ is the boundary of the closed ball.

The purpose of the present paper is to consider the pinching of the Ricci curvature for the hypersurface $\psi: M \to R^{n+1}$. In the case of the standard embedding $\psi: S^n(c) \to R^{n+1}$ of the sphere $S^n(c)$, for each unit vector field $X$ on $S^n(c)$ we have $\|\psi\|^2 \text{Ric}(X, X) = n-1$, where Ric is the Ricci tensor of $S^n(c)$. Suppose that $\psi: M \to R^{n+1}$ is an arbitrary compact connected immersed hypersurface such that $\|\psi\|^2 \text{Ric}(X, X) = n-1$ for all unit vector fields $X$ on $M$. We consider the question: Must $\psi(M)$ be a Euclidean sphere?

In the present paper we answer this question in the affirmative for $n = 3$. In fact, we prove the following theorem.

**Theorem.** Let $\psi: M \to R^4$ be an orientable, compact and connected hypersurface. If $0 < \|\psi\|^2 \text{Ric}(X, X) \leq 2$ for each unit vector field $X$ on $M$, then $\psi(M)$ is a Euclidean sphere in $R^4$.

Preliminaries

Let $\langle \cdot, \cdot \rangle$ be the inner product on $R^4$ and let $\bar{\nabla}$ be the Euclidean connection on $R^4$. We denote by $J_1$, $J_2$, and $J_3$ the complex structures on $R^4$ which define the quaternion structure on $R^4$. Then we have

\begin{align*}
(1.1) & \quad J_1 J_2 = -J_2 J_1 = J_3, \quad J_2 J_3 = -J_3 J_2 = J_1, \quad J_3 J_1 = -J_1 J_3 = J_2; \\
(1.2) & \quad \bar{\nabla} J_i = 0, \quad \langle J_i, J_i \rangle = \langle \cdot, \cdot \rangle, \quad i = 1, 2, 3.
\end{align*}

Received April 16, 1991. Revision received July 22, 1991.
Let $\psi : M \to R^4$ be an orientable hypersurface of $R^4$, and let $N$ be a unit normal vector field globally defined on $M$. Let $g$ and $\nabla$ be the induced metric and the Riemannian connection on $M$, respectively. Then we have

$$\vec{\nabla}_X Y = \nabla_X Y + g(A X, Y)N, \quad \vec{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M),$$

where $A$ is the shape operator of $M$ and $\mathfrak{X}(M)$ is the Lie algebra of vector fields on $M$.

Define the unit vector fields $\xi_1, \xi_2, \xi_3$ on $M$ by $J_i \xi_i = N$, $i = 1, 2, 3$. Also define three $(1, 1)$ tensor fields $\phi_i$ on $M$ by setting $J_i X = \phi_i X + \eta_i (X) N$, $i = 1, 2, 3$, $X \in \mathfrak{X}(M)$, where the 1-forms $\eta_i$ are respective duals of $\xi_i$. It can be verified that $\phi_i, \xi_i, \eta_i$ satisfy

$$\phi_i^2 = -I + \eta_i \otimes \xi_i, \quad \phi_i \xi_i = 0, \quad \eta_i \circ \phi_i = 0,$$

$$g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i (X) \eta_i (Y), \quad X, Y \in \mathfrak{X}(M), \quad i = 1, 2, 3.$$

Now, using (1.2) and (1.3), it is easy to obtain

$$\nabla_X \xi_i = \phi_i AX, \quad X \in \mathfrak{X}(M).$$

For a local unit vector field $e$ on $M$ satisfying $g(e, \xi_i) = 0$ for a fixed $i$, \{e, \phi_i e, \xi_i\} is a local orthonormal frame on $M$; such a frame will be referred to as an adapted frame. From the first equation in (1.6), using an adapted frame, we get $\div \xi_i = 0, i = 1, 2, 3$.

Using $\psi$ as the position vector field in $R^4$ of the hypersurface, we define a smooth function $\rho$ on $M$ by $\rho = \langle \psi, N \rangle$, which is commonly known as the support function of the hypersurface. Using the complex structures $J_i$, we define three smooth functions $\rho_i$ on $M$ by $\rho_i = \langle J_i \psi, N \rangle$. Also define three vector fields $t_i \in \mathfrak{X}(M)$ by setting $J_i \psi = t_i + \rho_i N$. Then, using (1.2), (1.3), and $\vec{\nabla} \psi = I$ in $J_i \psi = t_i + \rho_i N$, we obtain

$$\nabla_X t_i = \phi_i X + \rho_i AX, \quad d\rho_i (X) = -g(AX, t_i) + \eta_i (X), \quad X \in \mathfrak{X}(M).$$

We also have

$$g(t_i, \xi_i) = \langle J_i \psi, \xi_i \rangle = -\langle \psi, J_i \xi_i \rangle = -\langle \psi, N \rangle = -\rho.$$

**Proof of the Theorem**

Using the equations in (1.7), we compute the Hessian of the function $\rho_i$ as

$$H_{\rho_i} (X, Y) = -g(\vec{\nabla}_X A)(Y), t_i) - g(AX, \phi_i Y) - g(AY, \phi_i X) - \rho_i g(AX, AY).$$

From the definition of mean curvature $\alpha$ and the Codazzi equation for the hypersurface $M$ in $R^4$, we have

$$3X. \alpha = \sum_{i=1}^{3} g(\vec{\nabla}_i A)(e_i), X), \quad X \in \mathfrak{X}(M),$$
where \( \{e_1, e_2, e_3\} \) is a local orthonormal frame on \( M \). Thus, using the adapted frame \( \{e, \phi_i e, \xi_i\} \) and equation (1.10) in (1.9) to compute the Laplacian of \( \rho_i \), we get

\[
\Delta \rho_i = -3t_i, \alpha - \rho_i \text{tr}. A^2.
\]

From (1.7) we find \( \text{div} t_i = 3 \rho_i, \alpha \) and consequently \( \text{div} \alpha t_i = t_i, \alpha + 3 \rho_i, \alpha^2 \). Thus equation (1.11) takes the form

\[
\Delta \rho_i = \rho_i S - 3 \text{div} \alpha t_i,
\]

where \( S = 9 \alpha^2 - \text{tr}. A^2 \) is the scalar curvature of \( M \).

Next we use the second equation in (1.7) to find \( \text{div}(\rho_i \alpha t_i) \) and \( \text{grad} \rho_i \) as

\[
\text{div}(\rho_i \alpha t_i) = \alpha t_i, \rho_i + \rho_i \text{div}(\alpha t_i)
\]

\[
= -\alpha g(At_i, t_i) + \alpha g(t_i, \xi_i) + \rho_i \text{div}(\alpha t_i);
\]

\[
\text{grad} \rho_i = -At_i + \xi_i.
\]

Using (1.12), (1.13), and (1.14) in \( \Delta \rho_i^2 = 2 \rho_i \Delta \rho_i + 2 \| \text{grad} \rho_i \|^2 \), we obtain

\[
\Delta \rho_i^2 = 2 \rho_i^2 S - 6 \alpha g(At_i, t_i) + 6 \alpha g(t_i, \xi_i) - 6 \text{div}(\rho_i \alpha t_i)
\]

\[
+ 2 \| At_i \|^2 - 4g(At_i, \xi_i) + 2,
\]

which in light of (1.7) and (1.8) can be rearranged as

\[
\Delta \rho_i^2 = 2 \rho_i^2 S - 2 \text{Ric}(t_i, t_i) - 6 \alpha \rho + 4(\rho_i, \xi_i) - 1) + 2 - 6 \text{div}(\rho_i \alpha t_i).
\]

Since \( \xi_i \) is divergence-free (cf. preliminaries), we have

\[
\text{div}(\rho_i \xi_i) = \xi_i, \rho_i + \rho_i \text{div} \xi_i = d\rho_i(\xi_i).
\]

Thus we have

\[
\Delta \rho_i^2 = 2 \rho_i^2 S - 2 \text{Ric}(t_i, t_i) - 6(1 + \rho \alpha) + 4 + 4 \text{div}(\rho_i, \xi_i) - 6 \text{div}(\rho_i \alpha t_i).
\]

Integrating this equation over \( M \) and using Minkowski's formula (cf. [2]), we get

\[
\int_M \{ \rho_i^2 S - \text{Ric}(t_i, t_i) + 2\} \, dv = 0, \quad i = 1, 2, 3.
\]

Define the unit vector fields \( \hat{t}_i \) by \( t_i = \| t_i \| \hat{t}_i \). Then, using \( \| \psi \|^2 = \| J_i \psi \|^2 = \| t_i \|^2 + \rho_i^2 \) in (1.15), we have

\[
\int_M \{ \rho_i^2 (S + \text{Ric}(\hat{t}_i, \hat{t}_i)) + (2 - \| \psi \|^2 \text{Ric}(\hat{t}_i, \hat{t}_i)) \} \, dv = 0.
\]

From the hypothesis of the theorem it follows that the Ricci curvature is nonnegative and so is the scalar curvature \( S \), and thus from the above integral we get

\[
\rho_i^2 (S + \text{Ric}(\hat{t}_i, \hat{t}_i)) = 0 \quad \text{and} \quad \| \psi \|^2 \text{Ric}(\hat{t}_i, \hat{t}_i) = 2.
\]

The second equation gives \( \text{Ric}(\hat{t}_i, \hat{t}_i) > 0 \), and thus from the first equation we have \( \rho_i = 0, \ i = 1, 2, 3 \). Now, using \( \rho_i = 0 \) in the second equation in (1.7),
we get \( A t_i = \xi_i \). Since the \( \xi_i \) are globally defined unit vector fields on \( M \) and \( A \) is a linear operator, it follows that the \( t_i \) are nowhere zero on \( M \) and thus the \( \hat{t}_i \) are defined everywhere on \( M \). From (1.1) and (1.2), it can be easily deduced that \( \{ \hat{t}_1, \hat{t}_2, \hat{t}_3 \} \) is an orthonormal frame on \( M \). Then equation (1.11) together with \( \rho_i = 0 \) ensures that \( \alpha \) is a constant.

Our next aim is to show that the support function \( \rho \) is nowhere zero on \( M \). For this, from \( \| \psi \|^2 \text{Ric}(\hat{t}_1, \hat{t}_1) = 2 \) we observe that \( \text{Ric}(t_1, t_1) > 0 \) and consequently, as \( A t_i = \xi_1, 3\alpha g(A t_1, t_1) - \| A t_1 \|^2 = -3\alpha \rho - 1 > 0 \). This proves \( \rho \) is nowhere zero on \( M \).

We write \( \psi = t + \rho N \) for some \( t \in \mathcal{X}(M) \). Then it is easy to get

\[
\nabla_X t = X + \rho AX, \quad d\rho(X) = -g(AX, t), \quad X \in \mathcal{X}(M).
\]

Using these equations and the fact that \( \alpha \) is a constant, we compute the Laplacian of \( \rho \) as \( \Delta \rho = -3\alpha - \rho \text{tr}. A^2 \). Thus we have

\[
(1.16) \quad \int_M \{3\alpha + \rho \text{tr}. A^2\} \, dv = 0.
\]

As \( \alpha \) is constant, from Minkowski's formula we have

\[
(1.17) \quad \int_M 3\alpha \, dv = -\int_M 3\rho \alpha^2 \, dv.
\]

Equations (1.16) and (1.17) give

\[
(1.18) \quad \int_M \rho (3\alpha^2 - \text{tr}. A^2) \, dv = 0.
\]

The Schwarz inequality states that \( 3\alpha^2 \leq \text{tr}. A^2 \), with equality holding at a point if any only if it is an umbilic point. Since \( M \) is connected and \( \rho \neq 0 \), the integral (1.18) gives \( 3\alpha^2 = \text{tr}. A^2 \), proving that \( M \) is totally umbilic; our theorem then follows from [4, Thm. 5.1, p. 30].

We express our sincere thanks to the referee for many helpful suggestions.

References


Department of Mathematics
College of Science
King Saud University
P.O. Box 2455, Riyadh 11451
Saudi Arabia