AN INTEGRAL FORMULA FOR COMPACT HYPERSURFACES IN A EUCLIDEAN SPACE AND ITS APPLICATIONS

by SHARIEF DESHMUKH

1. Introduction. Let M be a compact hypersurface in a Euclidena space $\mathbb{R}^{n+1}$. The support function ρ of M is the component of the position vector field of M in $\mathbb{R}^{n+1}$ along the unit normal vector field to M, which is a smooth function defined on M. Let S be the scalar curvature of M. The object of the present paper is to prove the following theorems.

THEOREM 1. *Let M be a compact hypersurface of $\mathbb{R}^{n+1}$ with non-negative Ricci curvature. Then*

$$\mathrm{Av}(S) \geq n(n-1)/\mathrm{diam}^2(M),$$

where $\mathrm{Av}(S)$ is the average scalar curvature of M given by the Einstein functional $\mathrm{Av}(S) = 1/\mathrm{vol}(M) \int_M S \, dv$, and $\mathrm{diam}(M)$ is the diameter of M.

THEOREM 2. *Let M be a compact hypersurface of $\mathbb{R}^{n+1}$ with non-negative Ricci curvature. If M is centrally symmetric and R is the radius of the escribed sphere, then $R^2 \geq n(n-1)/\mathrm{Av}(S)$.*

THEOREM 3. *Let M be a compact and connected hypersurface of positive Ricci curvature in $\mathbb{R}^{n+1}$. If the support function ρ of M satisfies $\rho^2 \leq n(n-1)/S$, then ρ is a constant and M is a sphere of radius ρ.*

THEOREM 4. *Let M be a compact and connected hypersurface of non-negative Ricci curvature in $\mathbb{R}^{n+1}$. If M is contained in a closed ball of radius R centered at origin in $\mathbb{R}^{n+1}$ and the scalar curvature S of M satisfies $\sup S = n(n-1)R^{-2}$, then M is the sphere of radius R.*

All above theorems are consequences of an integral formula which we prove in Section 2. We observe that Theorem 4 generalizes Theorem 1 in [1] for hypersurfaces of non-negative Ricci curvature in a Euclidean space. We also get the following corollary to Theorem 1 which generalizes the result of Jacobwitz [2] for non-immersibility of a compact Riemannian manifold of non-negative Ricci curvature into a closed ball in a Euclidean space.

COROLLARY. *Let M be a compact n-dimensional Riemannian manifold of non-negative Ricci curvature whose average scalar curvature satisfies $\mathrm{Av}(S) < n(n-1)R^{-2}$. Then no isometric immersion of M into $\mathbb{R}^{n+1}$ is contained in a closed ball of radius R in $\mathbb{R}^{n+1}$.*

We express our sincere thanks to Referee for many helpful suggestions.

2. Integral formula. Let M be a compact hypersurface in $\mathbb{R}^{n+1}$ and N be the globally defined unit normal vector field on M. We denote by g, ∇ and A, the induced metric, the covariant derivative operator with respect to the induced Riemannian

connection and the shape operator on M. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \bar{\nabla}_X N = -AX, \qquad X, Y \in \mathscr{X}(M), \qquad (2.1)$$

where $\bar{\nabla}$ is the covariant derivative operator with respect to the Euclidean connection on $\mathbb{R}^{n+1}$ and $\mathscr{X}(M)$ is the Lie algebra of vector fields on M. Let T be the position vector field on $\mathbb{R}^{n+1}$. Then the smooth function $\rho: M \to R$ defined by $\rho = \langle T|_M, N \rangle$ is called the support function of the hypersurface M, where $\langle, \rangle$ is the Euclidean metric on $\mathbb{R}^{n+1}$. We have $T|_M = \xi + \rho N$, $\xi \in \mathscr{X}(M)$. Then since $\bar{\nabla}_X T = X$ holds for any $X \in \mathscr{X}(M)$, using (2.1), we have

$$\nabla_X \xi = X + \rho AX, \qquad d\rho(X) = -g(AX, \xi), \qquad X \in \mathscr{X}(M). \qquad (2.2)$$

From the equation of Gauss for hypersurface M in $\mathbb{R}^{n+1}$, we get the following expressions for Ricci curvature and scalar curvature S of M (cf. [3])

$$\text{Ric}(X, Y) = n\alpha g(AX, Y) - g(AX, AY), \qquad X, Y \in \mathscr{X}(M), \qquad (2.3)$$

$$S = n^2\alpha^2 - \text{tr } A^2, \qquad (2.4)$$

where $\alpha = 1/n \sum g(Ae_i, e_i)$ is the mean curvature of M and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

LEMMA 2.1.

$$\int_M \{\text{Ric}(\xi, \xi) + n(n-1) - \rho^2 S\} \, dv = 0.$$

Proof. We use equation (2.2) to compute the Laplacian of the support function ρ and the divergence of the vector field ξ, and obtain

$$\Delta\rho = -n \, d\alpha(\xi) - n\alpha - \rho \, \text{tr } A^2, \qquad \text{div } \xi = n(1 + \rho\alpha). \qquad (2.5)$$

Integrating second equation over M we get

$$\int_M (1 + \rho\alpha) \, dv = 0. \qquad (2.6)$$

Also we have

$$-n\rho \, d\alpha(\xi) = n\alpha \, \text{div}(\rho\xi) - \text{div}(n\alpha\rho\xi) = n\alpha \, d\rho(\xi) + n\rho\alpha \, \text{div } \xi - \text{div}(n\alpha\rho\xi).$$

Using second equation in (2.2) and above equation in (2.5), we find

$$\rho \, \Delta\rho = -n\alpha g(A\xi, \xi) + n^2\rho\alpha + n^2\rho^2\alpha^2 - n\rho\alpha - \rho^2 \, \text{tr } A^2 - \text{div}(n\alpha\rho\xi). \qquad (2.7)$$

We find $\text{grad } \rho = -A\xi$ from equation (2.2) and use it together with equation (2.7) in $\frac{1}{2}\Delta\rho^2 = \rho \, \Delta\rho + \|\text{grad } \rho\|^2$ to obtain

$$\frac{1}{2}\Delta\rho^2 = -\text{Ric}(\xi, \xi) - n(n-1) + n(n-1)(1 + \rho\alpha) + \rho^2 S - \text{div}(n\alpha\rho\xi),$$

where we have also used equation (2.3) and (2.4). Integrating above equation over M and using integral formula (2.6), we get the desired result.

3. Proofs of theorems. Theorems 1 and 2 follow directly from Lemma 2.1, as their hypotheses give

$$\text{diam}^2(M) \int_M S \, dv \geq \int_M \rho^2 S \, dv \geq n(n-1)\text{vol}(M).$$

For Theorem 3, we observe that Lemma 2.1 gives $\xi = 0$, which when combined with equation (2.2) gives ρ is a constant. That ρ is a non-zero constant is guarnateed by integral formula (2.6). Thus in this case from first equation in (2.2), we have that M is a totally umbilic and hence a sphere of radius ρ (cf. [3], p. 30).

The hypothesis of Theorem 4 confirms that $R^2 S \leq n(n-1)$. Now using a unit vector field $t = \xi/\|\xi\|$, defined on the open subset of M where ξ is non-zero, and $\|T|_M\|^2 = \|\xi\|^2 + \rho^2$ in Lemma 2.1, we obtain

$$\int_M \{\|\xi\|^2 (\mathrm{Ric}(t, t) + S) + (n(n-1) - \|T|_M\|^2 S)\} \, dv = 0.$$

Since M lies in a closed ball of radius R in $\mathbb{R}^{n+1}$, we may assume that $\|T|_M\|^2 \leq R^2$ and consequently we have

$$n(n-1) - \|T|_M\|^2 S \geq n(n-1) - R^2 S \geq 0,$$

where we have used $S \geq 0$ which follows from the hypothesis. Thus the above integral together with the inequality above gives $\xi = 0$ and $\|T|_M\|^2 = R^2 = \rho^2$. This proves the theorem.

REFERENCES

1. L. Coghlan and Y. Itokawa, On the sectional curvature of compact hypersurfaces, *Proc. Amer. Math. Soc.* **109**(1) (1990), 215–221.

2. H. Jacobowitz, Isometric embedding of a compact Riemannian manifold into Euclidean space, *Proc. Amer. Math. Soc.* **40**(1) (1973), 245–246.

3. S. Kobayashi and K. Nomizu, *Foundations of differential geometry vol. II* (Interscience Publ., 1969).

DEPARTMENT OF MATHEMATICS
COLLEGE OF SCIENCE
KING SAUD UNIVERSITY
P.O. BOX 2455
RIYADH 11451
SAUDI ARABIA