TANGENT AND COTANGENT BUNDLES

In this note we discuss a special smooth manifolds which play an important role on the modern differential geometry, namely the tangent and cotangent bundles of a smooth manifold.

**DEFINITION 1.1** Let $M$ be an $n$-dimensional smooth manifold, then the set $TM = \bigcup_{p \in M} T_p M$ is called the **tangent bundle** of $M$.

**REMARK.** If $v \in TM$ we write $v = (p, X_p)$ for some $p \in M$, and $X_p \in T_p M$.

**THEOREM 1.1** Let $M$ be an $n$-dimensional smooth manifold, then the tangent bundle $TM$ is a $2n$-dimensional smooth manifold.

**Proof** We give the proof on four steps

1. $TM$ is a topological space

   Let $S = \{(U_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ be the differentiable structure of $M$, for a chart $(U, \phi) \in S$ with local coordinates $x^1, x^2, \ldots, x^n$ define $\Phi : TU \to \mathbb{R}^{2n}$ by
   
   $$
   \Phi(p, X_p) = (x^1(p), x^2(p), \ldots, x^n(p), dx^1_p(X_p), \ldots, dx^n_p(X_p))
   $$
   
   where $TU = \bigcup_{p \in U} T_p M$, then clearly $\Phi$ is well defined.

   Let $(p, X_p), (q, y_q) \in TU$, be such that $\Phi(p, X_p) = \Phi(q, y_q)$

   \[ \Rightarrow \quad \phi(p) = \phi(q) \quad \text{and} \quad X_p(x^i) = y_q(x^i) \quad i = 1, 2, \ldots, n \]

   \[ \Rightarrow \quad p = q \quad \text{(since $\phi$ is a homeomorphism)} \]

   and since $X_p = \sum_i X_p(x^i)(\frac{\partial}{\partial x^i})_p$ and $y_q = \sum_i y_q(x^i)(\frac{\partial}{\partial x^i})_q$

   \[ \Rightarrow \quad X_p = y_q \quad \Rightarrow \quad (p, X_p) = (q, y_q) \quad \Rightarrow \quad \Phi \text{ is one-to-one map.} \]

   Consider the collection

   $$
   \beta = \{ \Phi^{-1}_\alpha(w) : w \text{ is open in } \mathbb{R}^{2n}, \ (U_\alpha, \phi_\alpha) \in S \}
   $$

   of subsets of $TM$. Note that

   i) $\forall \ (p, X_p) \in TM$, as $p \in M \Rightarrow$ there exists $(U_\alpha, \phi_\alpha) \in S$ such that $p \in U_\alpha$, i.e. $(p, X_p) \in TU_\alpha$ and we have $TU_\alpha = \Phi^{-1}_\alpha(R^{2n}) \in \beta$.

   ii) If we define $F_\alpha : T_p M \to \mathbb{R}^n$ by $F_\alpha(X_p) = (X_p(x^1), X_p(x^2), \ldots, X_p(x^n))$ where $x^1, x^2, \ldots, x^n$ are local coordinates on $(U_\alpha, \phi_\alpha)$, then clearly $F_\alpha$ is an isomorphism, so $\Phi_\alpha(p, X_p) = (\phi_\alpha(p), F_\alpha(X_p))$, and $\Phi^{-1}_\alpha = (\phi^{-1}_\alpha, F^{-1}_\alpha)$.

   Now take $\Phi^{-1}_\alpha(U), \Phi^{-1}_\beta(V) \in \beta$ and suppose $(p, X_p) \in \Phi^{-1}_\alpha(U) \cap \Phi^{-1}_\beta(V)$ for some $U, V$ open in $\mathbb{R}^{2n}$.

   \[ \Rightarrow \quad \Phi^{-1}_\alpha(p, X_p) \in U \quad \text{and} \quad \Phi_{\beta}(p, X_p) \in V \]

   Take $U = U^* \times U^{**}$ and $V = V^* \times V^{**}$ where $U^*, U^{**}, V^*, V^{**}$ are open sets in $\mathbb{R}^n$. Clearly $\phi_\alpha(p) \in U^*, F_\alpha(X_p) \in U^{**}, \phi_\beta(p) \in V^*$, and $F_\beta(X_p) \in V^{**}$.

   From the definition of open sets there exist neighborhoods $W^*_1, W^*_2, W^{**}_1, W^{**}_2$ of $\phi_\alpha(p), \phi_\beta(p), F_\alpha(X_p)$ and $F_\beta(X_p)$ respectively such that
\[ \phi_\alpha(p) \in W_1^* \subset U^* \quad \text{and} \quad \phi_\beta(p) \in W_2^* \subset V^* \]

\[ F_\alpha(X_p) \in W_1^{**} \subset U^{**} \quad F_\beta(X_p) \in W_2^{**} \subset V^{**} \]

Take \( W^* = W_1^* \cap (\phi_\alpha \circ \phi_\beta^{-1})(W_2^*) \) and \( W^{**} = W_1^{**} \cap (F_\alpha \circ F_\beta^{-1})(W_2^{**}) \), then \( \phi_\alpha(p) \in W^* \) and \( F_\alpha(X_p) \in W^{**} \), where \( W^*, W^{**} \) are both open in \( \mathbb{R}^n \).

\[ \Rightarrow (\phi_\alpha(p), F_\alpha(X_p)) \in W^* \times W^{**} = W \quad \text{open in} \quad \mathbb{R}^{2n}. \]

\[ \Rightarrow \quad \Phi_\alpha(p, X_p) \in \mathbb{R}^{2n}. \]

\[ \Rightarrow \quad (p, X_p) \in \Phi_\alpha^{-1}(W) = (\phi_\alpha^{-1}, F_\alpha^{-1})(W^* \times W^{**}) = (\phi_\alpha^{-1}(W^*), F_\alpha^{-1}(W^{**})) = \langle \phi_\alpha^{-1}(W_1^*), \phi_\alpha^{-1}(W_2^*), F_\alpha^{-1}(W_1^{**}) \cap F_\alpha^{-1}(W_2^{**}) \rangle = (\phi_\alpha^{-1}(U^*), F_\alpha^{-1}(U^{**}) \cap F_\alpha^{-1}(V^{**})). \]

\[ \Phi_\alpha^{-1}(V^*) = (\phi_\alpha^{-1}(U), F_\alpha^{-1}(U) \cap F_\alpha^{-1}(V)) = (\phi_\alpha^{-1}(U), F_\alpha^{-1}(V)) = \Phi_\alpha^{-1}(U) \cap \Phi_\alpha^{-1}(V), \text{so} \ \beta \text{is a basis for a topology on} \ TM. \quad \text{i.e.} \ TM \text{is a topological space.} \]

\[ 2 \quad \text{TM is a Hausdorff space} \]

Let \((p, X_p), (q, y_q) \in TM\), then \( p, q \in M \) and since \( M \) is a Hausdorff space then there exist \( U, V \) open in \( M \) such that \( U \cap V = \emptyset \) and \( p \in U, q \in V \). Since \( U, V \neq \emptyset \Rightarrow U, V \) are \( n \)-dimensional smooth manifolds, then there exist charts \((U_1, \phi_1), (V_1, \psi_1)\) around \( p \) and \( q \) in \( U \) and \( V \) respectively. \( U \cap V = \emptyset \Rightarrow U_1 \cap V_1 = \emptyset \) \( \text{i.e.} \) open neighborhoods of \((p, X_p)\) and \((q, y_q)\) in \( TM \), and \( TU_1 \cap TV_1 = \emptyset \), so \( TM \) is a Hausdorff space.

\[ 3 \quad \text{TM is a topological manifold} \]

For \((p, X_p) \in TM\), as \( p \in M \), there exists a chart \((U, \phi)\) with local coordinates \( x^1, x^2, \ldots, x^n \) in \( M \) around \( p \). Define \( \Phi : TU \to \mathbb{R}^{2n} \) as before, since the coordinate functions \( x^1, x^2, \ldots, x^n \) are smooth and the differential \( dx^i_p \) are linear transformations, then \( \Phi \) is continuous map. Moreover, for \((y^1, y^2, \ldots, y^n) \in \Phi(TU) \subset \mathbb{R}^{2n}\) we have \( \Phi^{-1}(y^1, y^2, \ldots, y^n) = (p, X_p) \), where \( \phi(p) = (y^1, y^2, \ldots, y^n) \) and \( X_p(x^i) = y^{i+n}, \quad i = 1, 2, \ldots, n \).

Thus \( p = \phi^{-1}(y^1, y^2, \ldots, y^n) \), also \( X_p = \sum_{i=1}^{n} y^{i+n}(\frac{\partial}{\partial x^i})_p \).

\[ \Rightarrow \quad \Phi^{-1}(y^1, y^2, \ldots, y^n) = (\phi^{-1}(y^1, y^2, \ldots, y^n), \sum_{i=1}^{n} y^{i+n}(\frac{\partial}{\partial x^i})_p) \]

Note that \( \phi^{-1} \) is continuous map (\( \phi \) is homeomorphism), and the map \( F : \mathbb{R}^n \to T_p M \) defined by \( F(y^{1+n}, \ldots, y^{2n}) = \sum_{i=1}^{n} y^{i+n}(\frac{\partial}{\partial x^i})_p \) is linear transformation and therefore is continuous.

\[ \Rightarrow \quad \Phi^{-1} \text{ is continuous.} \]

\[ \Rightarrow \quad \Phi(TU) \subset \mathbb{R}^{2n} \quad \text{is a homeomorphism.} \]

\[ \Rightarrow \quad (TU, \Phi) \text{ is a chart around } (p, X_p). \]

\[ \Rightarrow \quad TM \text{ is a } 2n \text{-dimensional topological manifold.} \]

\[ 4 \quad TM \text{ is a } 2n \text{-dimensional smooth manifold} \]

Consider \( S = \{ (TU, \Phi)(U, \phi) \in S \} \); then

i) clearly \( \cup_{(U, \phi) \in S} TU = TM \).

ii) take \((U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta) \in S \) with local coordinates \( x^1, x^2, \ldots, x^n, y^1, \ldots, y^n \) respectively, such that \( U_\alpha \cap U_\beta \neq \emptyset \Rightarrow TU_\alpha \cap TU_\beta \neq \emptyset \).

Now for \((z^1, z^2, \ldots, z^{2n}) \in \mathbb{R}^{2n}\) we have

\[ (\Phi_\alpha \circ \Phi_\beta^{-1})(z^1, z^2, \ldots, z^{2n}) = \Phi_\alpha(p, X_p) \quad \text{where} \quad \Phi_\beta(p, X_p) = (z^1, z^2, \ldots, z^{2n}), \]

that is \( \phi_\beta(p) = (z^1, z^2, \ldots, z^n) \) and \( X_p(y^i) = z^{i+n}, \quad i = 1, 2, \ldots, n \)

Now \( (\Phi_\alpha \circ \Phi_\beta^{-1})(z^1, z^2, \ldots, z^{2n}) = (\phi_\alpha(p), X_p(x^1), \ldots, X_p(x^n)) = (\phi_\alpha(p), \Phi_\beta^{-1}(z^1, z^2, \ldots, z^n), X_p(x^1), \ldots, X_p(x^n)) \) but \( X_p(x^i) = dx^i_p(X_p) = dx^i_p(\sum_{\alpha=1}^{n} z^{\alpha+n}(\frac{\partial}{\partial x^i})_p) = \)

2
\[ \sum_{\alpha=1}^{n} z^{\alpha+n} dx_{\mu}^{\alpha} (\frac{\partial}{\partial y^\mu})_p = \sum_{\alpha=1}^{n} z^{\alpha+n} \frac{\partial}{\partial y^\mu} (x^{\alpha})_p = \sum_{\alpha=1}^{n} z^{\alpha+n} \frac{\partial}{\partial u^\alpha} (\phi_\alpha^{-1}) (\phi_\beta (p)) \]

Thus \( \Phi_\beta \circ \Phi_\alpha^{-1} \) is smooth since the first component given by \( \phi_\alpha \circ \phi_\beta^{-1} \) is smooth and other \( n \)-components are smooth.

Similarly we can show that \( \Phi_\beta \circ \Phi_\alpha^{-1} \) is also smooth.

\[ \Rightarrow \quad \tilde{S} \text{ is a differentiable structure on } TM. \]

\[ \Rightarrow \quad TM \text{ is a } 2n \text{-dimensional smooth manifold.} \]

**Lemma 1.1** For a smooth map \( f : M \to N \) the map \( df : TM \to TN \) define by \( df(p, X_p) = (f(p), df_{X_p} (X_p)) \) is smooth.

**Lemma 1.2** The natural projection map \( \pi : TM \to M \) given by \( \pi(p, X_p) = p \) is smooth submersion.

### Cotangent bundle

In this section we develop the theory of cotangent bundle.

**Definition 2.1** For a smooth manifold \( M \) we define \( TM^* = \cup_{p \in M} T^*_p M \) and call it the cotangent bundle of \( M \).

**Remark** An element of \( TM^* \) is written as \( (p, \omega_p) \) where \( \omega_p \in T^*_p M \), \( p \in M \).

**Theorem 2.2** The cotangent bundle \( TM^* \) for an \( n \)-dimensional smooth manifold \( M \) is \( 2n \)-dimensional smooth manifold and for a chart \((U, \phi)\) with local coordinates \( x^1, x^2, \ldots, x^n \) on \( M \) the corresponding chart is \((TU, \tilde{\phi})\) where \( TU = \cup_{p \in U} T^*_p M \) and \( \tilde{\phi} : TU \to \mathbb{R}^{2n} \) is defined by

\[ \tilde{\phi}(p, \omega_p) = (x^1(p), x^2(p), \ldots, x^n(p), \omega_p(\frac{\partial}{\partial x^1})_p, \omega_p(\frac{\partial}{\partial x^2})_p, \ldots, \omega_p(\frac{\partial}{\partial x^n})_p) \]

**Proof** similar to the proof of theorem (1.2.1)

**Lemma 2.3** The natural projection map \( \pi : TM^* \to M \) given by \( \pi(p, \omega_p) = p \) is smooth submersion.

### Vector Fields and Smooth Forms

In this section we introduce vector fields, one parameter groups of transformations, complete vector fields. We also study smooth forms and the exterior differential operator.

**Definition 3.1** Let \( M \) be an \( n \)-dimensional smooth manifold. A smooth vector field (or vector field) \( X \) on \( M \) is a smooth map \( X : M \to TM \) satisfying \( \pi \circ X = id_M \), i.e. \( \forall p \in M, \ X(p) \in T_p M \).
EXAMPLE 3.1 For a chart \((U, \phi)\) on \(M\) with local coordinates \(x^1, x^2, \ldots, x^n\) define \(\frac{\partial}{\partial x^i} : U \to TU = \pi^{-1}(U) \subset TM\) by

\[
\frac{\partial}{\partial x^i}(p) = \left(\frac{\partial}{\partial x^i}\right)_p \quad i = 1, 2, \ldots, n
\]

Then clearly \(\left\{\frac{\partial}{\partial x^i}, \ i = 1, \ldots, n\right\}\) are smooth vector fields on \(U\). Moreover for a vector field \(X\) we can write \(X\) locally on \(U\) as

\[
X = \sum_{i=1}^{n} X(x^i) \frac{\partial}{\partial x^i}
\]

DEFINITION 3.2 We define \(\mathfrak{X}(M)\) to be the set of all smooth vector fields on \(M\).

REMARKS

(1) Since on \(\mathbb{R}^n\) there is one chart \((\mathbb{R}^n, id)\) with local coordinates \(x^1, x^2, \ldots, x^n\) then the vector fields \(\left\{\frac{\partial}{\partial x^i}, \ i = 1, \ldots, n\right\}\) will be defined globally and \(\forall \ X \in \mathfrak{X}(\mathbb{R}^n)\) we write \(X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}, \ f^i \in C^\infty(\mathbb{R}^n), \ f^i = X(x^i), \ i = 1, \ldots, n\) and we say \(\mathfrak{X}(\mathbb{R}^n)\) has finite dimension.

(2) For \(f \in C^\infty(M)\) and \(X \in \mathfrak{X}(M)\)

i) Define \(fX : M \to TM\) by \((fX)(p) = f(p)X(p)\), then it is easy to see that \(\mathfrak{X}(M)\) is a module over \(C^\infty(M)\).

ii) Define a smooth function \(X(f) : M \to \mathbb{R}\) by \(X(f)(p) = X(p)(f)\) then it is suitable to define a vector field \(X\) as

\[
X : C^\infty(M) \to C^\infty(M) \quad \text{which satisfies}
\]

\[
X(\lambda f + \mu g) = \lambda X(f) + \mu X(g)
\]

\[
X(fg) = f X(g) + X(f)g, \ f, g \in C^\infty(M)
\]

We call \(X(f)\) the derivative of \(f\) with respect to the vector field \(X\), and it can be shown that the two definitions of a smooth vector field are equivalent.

DEFINITION 3.3 For \(X, Y \in \mathfrak{X}(M)\) then the bracket of \(X\) and \(Y\), \([X, Y]\), is a vector field on \(M\) defined by \([X, Y] : C^\infty(M) \to C^\infty(M)\)

\[
[X, Y](f) = X(Y(f)) - Y(X(f))
\]

The bracket operation has the following properties

THEOREM 3.1 If \(X, Y\) and \(Z \in \mathfrak{X}(M)\), \(a, b \in \mathbb{R}\), and \(f, g \in C^\infty(M)\) then

(1) \([X, Y] = -[Y, X]\)

(2) \([aX + bY, Z] = a[X, Z] + b[Y, Z]\)

(3) \([fX, gY] = f g[X, Y] + f X(g)Y - g Y(f)X\)

(4) \([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\) (Jacobi identity)

(5) \([X, X] = 0\)

REMARK In above theorem (4) and (5) makes \(\mathfrak{X}(M)\) a lie-algebra.
**DEFINITION 3.4** Let $\alpha : (a, b) \to M$ be a smooth curve, for $t \in (a, b)$ with $\alpha(t) = p \in M$, we define $\dot{\alpha}(t) = d\alpha(t) \in T_p M$, and we call it tangent vector to the curve $\alpha$ at $t$.

**REMARK** For a chart $(U, \phi)$ around $\alpha(t) \in M$ with local coordinates $x^1, x^2, ..., x^n$, we have

$$\dot{\alpha}(t) = \sum_{i=1}^{n} \frac{d}{dt}(\frac{\partial}{\partial x^i})_{\alpha(t)}$$

where $x^i = x^i \circ \alpha$

**DEFINITION 1.3.5** Let $M$ be a smooth manifold. A smooth curve $\alpha : (a, b) \to M$ is said to be an integral curve of $X \in \mathfrak{X}(M)$ if $\dot{\alpha}(t) = X(\alpha(t)) \quad \forall t \in (a, b)$

**EXAMPLE 3.2** Consider $X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$ and a smooth curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ defined by $\alpha(t) = (1 + t, 1 - t)$ then $x^1 = x \circ \alpha = 1 + t, \quad x^2 = y \circ \alpha = 1 - t \Rightarrow \dot{\alpha}(t) = \sum_{i=1}^{2} \frac{dx^i}{dt}(\frac{\partial}{\partial x^i})_{\alpha(t)} = (\frac{\partial}{\partial x})_{\alpha(t)} - (\frac{\partial}{\partial y})_{\alpha(t)} = X(\alpha(t))$, so $\alpha$ is an integral curve of $X$.

**LEMMA 3.1** Let $\alpha$ and $\beta$ be integral curves of $X \in \mathfrak{X}(M)$ defined on an open intervals $I$ and $J$ respectively, containing $0$. If $\alpha(0) = \beta(0)$ then $\alpha(t) = \beta(t) \quad \forall t \in I \cap J$.

**REMARK** If $\alpha : (a, b) \to M$ is an integral curve of $X \in \mathfrak{X}(M)$ passing through $p = \alpha(t_0), \quad t_0 \in (a, b)$. Let $(U, \phi)$ be a chart around $p$ with local coordinates $x^1, x^2, ..., x^n$ and write $X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}$ on $U$. Namely, $\alpha^i$ is a solution of a system of differential equations $\frac{dx^i}{dt} = f^i \quad (i = 1, ..., n)$ subject to the initial condition $x^i(t_0) = x^i(p)$, then $\alpha = \phi^{-1}(\alpha^1, ..., \alpha^n)$.

**DEFINITION 3.6** A vector field $X$ along a curve $\alpha : I \to M$ is a smooth map $X : I \to TM$ such that $X(t) \in T_{\alpha(t)}M$, for $t \in I$.

The set of vector fields along $\alpha$ is denoted by $\Gamma_\alpha(TM)$

**EXAMPLE 3.3** $\dot{\alpha} = d\alpha(\frac{d}{dt})$ is a vector field along the curve $\alpha : I \to M$ and it is denote some time by $\dot{\alpha} = \frac{d}{dt}$.

**REMARK** Let $\alpha : I \to M$ be a smooth curve, then each $X \in \mathfrak{X}(M)$ gives $X : I \to TM, \quad X(t) = X(\alpha(t))$ a smooth vector field along $\alpha$.

**DEFINITION 3.7** Let $f : M \to N$ be a smooth map. If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are such that $df(X) = Y \circ f$, where $df(X)(p) = df_p(X(p))$, then we say $X$ is $f$-related to $Y$. 

5
LEMMA 3.2 Let $M$ and $N$ be two smooth manifold and $f : M \to N$ is a smooth map. If $X_1, Y_1 \in \mathfrak{X}(M)$ are $f$-related to $X_2, Y_2 \in \mathfrak{X}(N)$ respectively, then $[X_1, Y_1]$ is $f$-related to $[X_2, Y_2]$.

THEOREM 3.2 If $f : M \to N$ is a diffeomorphism then the map $f_* \mathfrak{X}(M) \to \mathfrak{X}(N)$ defined by $f_* (X) = df(X) \circ f^{-1}$ is a Lie-Algebra isomorphism.

DEFINITION 3.8 Let $M$ be a smooth manifold, then a one parameter group of transformation $\{\phi_t\}$ of $M$ is a smooth map $\phi : R \times M \to M$, $\phi(t, p) = \phi_t(p)$ that satisfies
1) $\forall t \in R, \phi_t : M \to M$ is a diffeomorphism.
2) $\phi_{t+s} = \phi_t \circ \phi_s \ \forall t, s \in R$.

REMARK From any one parameter group of transformation $\{\phi_t\}$ of $M$ we can induce a vector field $X \in \mathfrak{X}(M)$ as follows
Fix $p \in M$ and define $\sigma_p : R \to M$ by $\sigma_p(t) = \phi_t(p)$, which is a smooth curve passing through $p$, now define $X : M \to T_p M$ by $X(p) = \sigma_p(0) \in T_p M$, then $X$ is a vector field which is induced by $\{\phi_t\}$, and $\sigma_p$ is an integral curve of $X$.

EXAMPLE 3.4 Consider $M = R^2$ and $\phi_t : R^2 \to R^2$ be defined as $\phi_t(x, y) = (x + t, y - t), t \in R$, then $\{\phi_t\}$ is a one parameter group of transformation. Fix $p = (a, b) \in R^2$ and put $\sigma_p(t) = \phi_t(p) = (a + t, b - t)$

$\Rightarrow \sigma_p(0) = \left(\frac{\partial}{\partial x}\right)_p - \left(\frac{\partial}{\partial y}\right)_p = X(p)$

$\Rightarrow X = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \in \mathfrak{X}(R^2)$ is the induced vector field of $\{\phi_t\}$.

DEFINITION 3.9 $X \in \mathfrak{X}(M)$ is said to be invariant under a smooth map $f : M \to M$ if $X$ is $f$-related to itself.

THEOREM 3.3 The vector field $X \in \mathfrak{X}(M)$ which induced by a one parameter group of transformation $\{\phi_t\}$ is invariant with respect to $\phi_t : M \to M$.

THEOREM 3.4 If $\phi : R \times M \to M$ and $\psi : R \times M \to M$ are two one parameter groups of transformations on $M$ which induce the same vector field, then $\phi = \psi$.

DEFINITION 3.10 If $X \in \mathfrak{X}(M)$ is induced by a one parameter group of transformation $\{\phi_t\}$ then we say $X$ is complete and write $\phi_t = \exp tX$.

PROPOSITION 3.1 If $f : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$ is complete, then the $f$-related vector field $Y \in \mathfrak{X}(N)$ to $X$ is also complete.

LEMMA 3.3 Let $X \in \mathfrak{X}(M)$ be induced by $\{\phi_t\}$, then
1) $\forall f \in C^\infty(M) \ X f = \lim_{t \to 0} \frac{1}{t} (f \circ \phi_t - f)$
2) For $Y \in \mathfrak{X}(M)$ $[X, Y] = \lim_{t \to 0} \frac{1}{t} [d\phi_t(Y) \circ \phi_t - Y]$
**Theorem 3.5** If \(X, Y \in \mathfrak{X}(M)\) are complete vector fields corresponding to a one parameter groups of transformations \(\{\phi_t\}\) and \(\{\psi_t\}\) respectively, then \([X, Y] = 0 \iff \phi_t\) and \(\psi_t\) commute for each \(t \in R\).

**Definition 3.11** Let \(U\) and \(V\) be two nonempty open subsets of a smooth manifold \(M\). A diffeomorphism \(\phi : U \to V\) is called a local transformation of \(M\).

**Definition 3.12** A local one parameter group of local transformations of \(M\) is a set \(\{U_\alpha, \epsilon_\alpha, \phi_t^{(\alpha)}\}_{\alpha \in A}\) where \(U_\alpha\) is an open set of \(M\), \(\epsilon_\alpha\) a positive number, and \(\phi_t^{(\alpha)}\) are a local transformation of \(M\) for each \(t\), \(|t| < \epsilon_\alpha\), satisfying the following conditions

1) \(\{U_\alpha\}\) is an open cover of \(M\).
2) The domain of \(\phi_t^{(\alpha)}(p) < \epsilon_\alpha\) contains \(U_\alpha\) and \(\phi_0^{(\alpha)}(p)\) is the identity transformation on \(U_\alpha\), the map \((t, p) \to \phi_t^{(\alpha)}(p)\) is a smooth from \((-\epsilon_\alpha, \epsilon_\alpha) \times U_\alpha \to U_\alpha\).
3) If \(|t|, |s|, |s + t| < \epsilon_\alpha\), then \(\phi_t^{(\alpha)} \circ \phi_s^{(\alpha)}\) is define and its domain contains \(U_\alpha\) and \((\phi_t^{(\alpha)} \circ \phi_s^{(\alpha)})(q) = \phi_t^{(\alpha)}(\phi_s^{(\alpha)}(q))\) holds for \(q \in U_\alpha\).
4) If \(U_\alpha \cap U_\beta \neq \emptyset\), then for each \(p \in U_\alpha \cap U_\beta\), we can choose \(\epsilon < \epsilon_{\min}\{\epsilon_\alpha, \epsilon_\beta\}\) such that for \(|t| < \epsilon\), \(\phi_t^{(\alpha)}\) and \(\phi_t^{(\beta)}\) agree on a sufficiently small neighborhood of \(p\).

**Remark** Let \(G = \{U_\alpha, \epsilon_\alpha, \phi_t^{(\alpha)}\}_{\alpha \in A}\) be a local one parameter group of local transformations on \(M\), then for fix \(\alpha\) and a point \(p \in U_\alpha\) define \(\sigma_p^{(\alpha)} : (-\epsilon_\alpha, \epsilon_\alpha) \to M\) by \(\sigma_p^{(\alpha)}(t) = \phi_t^{(\alpha)}(p)\) which is a smooth curve passing through \(p\). Now let \(X : M \to TM\) defined by \(X(p) = \sigma_p^{(\alpha)}(0)\) then \(X\) is a vector field induced by \(G\).

**Definition 3.13** Let \(G_1 = \{U_\alpha, \epsilon_\alpha, \phi_t^{(\alpha)}\}_{\alpha \in A}\), \(G_2 = \{V_i, \eta_i, \psi_t^{(i)}\}_{i \in I}\) be two local one parameter groups of local transformations, then we say \(G_1\) and \(G_2\) are equivalent and write \(G_1 \sim G_2\) if the following conditions hold

"When ever \(U_\alpha \cap V_i \neq \emptyset\), then for any \(p \in U_\alpha \cap V_i\) there is \(0 < \delta < \min\{\epsilon_\alpha, \eta_i\}\) such that \(\phi_t^{(\alpha)} = \psi_t^{(i)}\) on a sufficiently small neighborhood of \(p\) for \(|t| < \delta"."

**Theorem 3.6** Let \(X \in \mathfrak{X}(M)\), then there exists a local one parameter group of local transformation which induces \(X\), and two local one parameter groups of local transformation inducing \(X\) are equivalent.

**Theorem 3.7** A vector field on \(M\) is complete if and only if there is a \(G = \{U_\alpha, \epsilon_\alpha, \phi_t^{(\alpha)}\}_{\alpha \in A}\) such that \(\inf \epsilon_\alpha > 0\).

**Corollary 3.1** If \(M\) is compact then every vector field on \(M\) is complete.
Now by using a cotangent bundle $TM^*$, we can similarly define another type of smooth map as follows

**Definition 3.14** A smooth map $W : M \to TM^*$ satisfying $\pi \circ W = id_M$ is called a **smooth 1-form**.

**Definition 3.15** The set of all smooth 1-form on $M$ is denoted by $\Lambda^1(M)$.

**Lemma 3.4** $\Lambda^1(M)$ is a module over $C^\infty(M)$ and is a vector space over $R$.

**Example 3.5** For $f \in C^\infty(M)$ we know $\forall p \in M$, $df_p : T_pM \to R$, that is $df_p \in T^*_pM$. Now define $df : M \to TM^*$ by $df(p) = df_p$, it is easy to show that $df \in \Lambda^1(M)$.

**Remarks**

1) For $W \in \Lambda^1(M)$ and $X \in \mathfrak{X}(M)$, we define a smooth map $W(X) : M \to R$ by $W(X)(p) = W_p(X_p)$, where $W_p = W(p)$, $X_p = X(p)$, thus we can write a smooth 1-form $W$ as a map $W : \mathfrak{X}(M) \to C^\infty(M)$.

2) Each $f \in C^\infty(M)$ is also called a smooth 0-form and we some times denote $C^\infty(M)$ by $\Lambda^0(M)$.

Now to define the higher order differential forms we need some algebraic preparation

**Definition 3.16**

1) Let $M$ be an $n$-dimensional smooth manifold, take $p \in M$, then a map

$$f : \underbrace{T_pM \times \cdots \times T_pM}_{k - \text{copies}} \to R$$

is said to be **multi-linear** if $f(X^1_p, \ldots, X^k_p)$ is linear in each slot.

2) We denote by $T^k(T_pM)$ the set of all multi-linear maps

$$f : \underbrace{T_pM \times \cdots \times T_pM}_{k - \text{copies}} \to R$$

which is a vector space over $R$.

3) For $S \in T^k(T_pM)$ and $T \in T^L(T_pM)$, we define the **tensor product** $S \otimes T \in T^{k+L}(T_pM)$ by

$$(S \otimes T)(X^1_p, \ldots, X^k_p, X^{k+1}_p, \ldots, X^{k+L}_p) = S(X^1_p, \ldots, X^k_p).T(X^{k+1}_p, \ldots, X^{k+L}_p)$$
4) An element \( S \in T^k(T_pM) \) is said to be alternating if \( S(X^1_p, \ldots, X^k_p) = 0 \) whenever \( X^i_p = X^j_p \) for any \( i, j \) such that \( 1 \leq i, j \leq k \).

5) If \( S \in T^k(T_pM) \) is alternating then

\[
S(X^1_p, \ldots, X^i_p, \ldots, X^j_p, \ldots, X^k_p) = -S(X^1_p, \ldots, X^j_p, \ldots, X^i_p, \ldots, X^k_p).
\]

6) The set of all alternating elements in \( T^k(T_pM) \) is denoted by \( \Gamma^k(T_pM) \).

7) For \( S \in T^k(T_pM) \) we define \( \text{Alt}S \in \Gamma^k(T_pM) \) by

\[
(\text{Alt}S)(X^1_p, \ldots, X^k_p) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma S(X^1_p, \ldots, X^k_p)
\]

where \( S_k \) is the permutation group, and \((-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ 2 & \text{if } \sigma \text{ is odd} \end{cases}\)

8) An element \( W_p \in \Gamma^k(T_pM) \) is called a \( k \)-form and \( \Gamma^k(T_pM) \) is called a space of \( k \)-forms at \( p \in M \).

9) For \( W_p \in \Gamma^k(T_pM) \) and \( \eta_p \in \Gamma^L(T_pM) \) we define the wedge product \( W_p \wedge \eta_p \in \Gamma^{k+L}(T_pM) \) by

\[
W_p \wedge \eta_p = \frac{(k + L)!}{k! L!} \sum_{\sigma \in S_{k+i}} (-1)^\sigma \text{Alt}(W_p \otimes \eta_p) \circ \sigma.
\]

**EXAMPLE 3.6** Let \( W_p, \eta_p \in \Gamma^1(T_pM) \), then \( W_p \wedge \eta_p \in \Gamma^2(T_pM) \) and

\[
W_p \wedge \eta_p = \frac{2}{1!} \sum_{\sigma \in S_2} (-1)^\sigma \text{Alt}(W_p \otimes \eta_p) \circ \sigma \quad \text{so}
\]

\[
(W_p \wedge \eta_p)(X_p, Y_p) = 2\{\text{Alt}(W_p \otimes \eta_p)(X_p, Y_p) - \text{Alt}(W_p \otimes \eta_p)(Y_p, X_p)\}
\]

\[
= 2\{\frac{1}{2}[W_p(X_p)\eta_p(Y_p) - W_p(Y_p)\eta_p(X_p)] - \frac{1}{2}[W_p(Y_p)\eta_p(X_p) - W_p(X_p)\eta_p(Y_p)]\}
\]

\[
= 2\{W_p(X_p)\eta_p(Y_p) - W_p(Y_p)\eta_p(X_p)\}
\]

**LEMMA 3.5** Let \( M \) be an \( n \)-dimensional smooth manifold, then \( \Gamma^{n+k}(T_pM) = \{0\} \) for \( k \geq 1 \).

**LEMMA 3.6** For \( W_p, W_i \in \Gamma^k(T_pM), \eta_p \in \Gamma^L(T_pM), \alpha_p \in \Gamma^m(T_pM), \)

\( i = 1, 2 \) then

1) \( (W^1_p + W^2_p) \wedge \eta_p = W^1_p \wedge \eta_p + W^2_p \wedge \eta_p \)

2) \( W_p \wedge \eta_p = (-1)^{kL} \eta_p \wedge W_p \)

3) \( W_p \wedge (\eta_p \wedge \alpha_p) = (W_p \wedge \eta_p) \wedge \alpha_p \)

**COROLLARY 3.2** If \( k \) is odd then \( W_p \wedge W_p = 0 \)

**DEFINITION 3.17** Let \( M \) be an \( n \)-dimensional smooth manifold, then we define \( \Gamma^k(M) = \bigcup_{p \in M} \Gamma^k(T_pM) \) and call it the bundle of \( k \)-forms.

**DEFINITION 3.18** A smooth map \( W : M \to \Gamma^k(M) \) satisfying \( \pi \circ W = id_M \) is called a smooth \( k \)-form.
REMARKS
1) For \( p \in M \), \( W(p) \in \Gamma^k(T_pM) \), that means \( W(p) \) is a \( k \)-form at \( p \in M \).
2) The set of all smooth \( k \)-forms on \( M \) is denoted by \( \Lambda^k(M) \).
3) \( \Lambda^k(M) \) is a module over \( C^\infty(M) \).
4) For \( W \in \Lambda^k(M) \) and \( X^1, \ldots, X^K \in \mathfrak{X}(M) \) we define
\[ W : (X^1, \ldots, X^K)M \to R \]
by
\[ W(X^1, \ldots, X^K)(p) = W(p)(X^1(p), \ldots, X^K(p)) \]
and it is easy to show that \( W(X^1, \ldots, X^K) \in C^\infty(M) \), thus we can write a smooth \( k \)-form as a map
\[ W : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \to C^\infty(M) \]
which is multi-linear \( k \)-copies and alternating.

DEFINITION 3.19 For \( W \in \Lambda^k(M) \), \( k > 0 \), we define \( dW \in \Lambda^{k+1}(M) \) as
\[ dW : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \to C^\infty(M) \]
by \( (k + 1) \)-copies
\[ dW(X_1, \ldots, X_{k+1}) = \sum_i (-1)^{i+1} X_i(W(X_1, \ldots, X_i, \ldots, X_{k+1})) \]
\[ + \sum_{i<j} (-1)^{i+j} W([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, X_{k+1}) \]
and for \( k = 0 \) we define \( dW(X) = X(W) \).
This operator \( d : \Lambda^k(M) \to \Lambda^{k+1}(M) \) called the exterior differential operator and it has some properties which given in this lemma

LEMMA 3.7 For \( W, W^1, W^2 \in \Lambda^k(M), \eta \in \Lambda^k(M) \)
1) \( d(W^1 + W^2) = dW^1 + dW^2 \)
2) \( d(W \wedge \eta) = dW \wedge \eta + (-1)^k W \wedge d\eta \)
3) \( d^2 = d \circ d = 0 \)

EXAMPLE 3.7 For \( \eta \in \Lambda^1(M) \) then \( d\eta \in \Lambda^2(M) \) and
\[ d\eta(X, Y) = (-1)^2 X(\eta(Y)) + (-1)^3 Y(\eta(X)) + (-1)^3 \eta([X, Y]) \]
\[ = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \]