TANGENT AND COTANGENT BUNDLES

In this note we discuss a special smooth manifolds which play an important role on the modern differential geometry, namely the tangent and cotangent bundles of a smooth manifold.

DEFINITION 1.1 Let M be an n-dimensional smooth manifold, then the set $TM = \cup_{p\in M} T_pM$ is called the **tangent bundle** of M .

REMARK. If $v \in TM$ we write $v = (p, X_p)$ for some $p \in M$, and $X_p \in T_pM$.

THEOREM 1.1 Let M be an n-dimensional smooth manifold, then the tangent bundle TM is a $2n$-dimensional smooth manifold.

Proof We give the proof on four steps

[1] TM `is a topological space`

Let $S = \{(U_\alpha, \phi_\alpha)\ \alpha \in \Lambda\ \}$ be the differentiable structure of M, for a chart $(U, \phi) \in S$ with local coordinates $x^1, x^2, ..., x^n$ define $\bar{\Phi} : TU \to R^{2n}$ by

$$\begin{aligned}\bar{\Phi}(p,\ X_p) &= (x^1(p), x^2(p), ..., x^n(p), dx^1_p(X_p), ..., dx^n_p(X_p)) \\ &= (\phi(p), X_p(x^1), ..., X_p(x^n))\end{aligned}$$

where $TU = \cup_{p\in U} T_pM$, then clearly $\bar{\Phi}$ is well defined.

Let $(p, X_p), (q, y_q) \in TU$, be such that $\bar{\Phi}\ (p,\ X_p) = \bar{\Phi}(q,\ y_q)$

$\Rightarrow \quad \phi(p) = \phi(q)$ and $\quad X_p(x^i) = y_q(x^i) \quad i = 1, 2, ..., n$

$\Rightarrow \quad p = q \quad$ (since ϕ is a homeomorphism)

and since $X_p = \sum_i X_p(x^i)(\frac{\partial}{\partial x^i})_p$ and $\quad y_q = \sum_i y_q(x^i)(\frac{\partial}{\partial x^i})_q$

$\Rightarrow \quad X_p = y_q \quad \Rightarrow \quad (p, X_p) = (q, y_q) \quad \Rightarrow \quad \bar{\Phi}$ is one-to-one map.

Consider the collection

$$\beta = \{\bar{\Phi}^{-1}_\alpha(w) : w \text{ is open in } R^{2n},\ (U_\alpha, \phi_\alpha) \in S\}$$

of subsets of TM. Note that

i) $\forall\ (p, X_p) \in TM$, as $p \in M \Rightarrow$ there exists $(U_\alpha, \phi_\alpha) \in S$ such that $p \in U_\alpha$, i.e. $(p, X_p) \in TU_\alpha$, and we have $TU_\alpha = \bar{\Phi}^{-1}_\alpha(R^{2n}) \in \beta$.

ii) If we define $F_\alpha : T_pM \to R^n$ by $F_\alpha(X_p) = (X_p(x^1), X_p(x^2), ..., X_p(x^n))$ where $x^1, x^2, ..., x^n$ are local coordinates on (U_α, ϕ_α), then clearly F_α is an isomorphism, so $\bar{\Phi}_\alpha\ (p, X_p) = (\phi_\alpha(p), F_\alpha(\ X_p))$, and $\bar{\Phi}^{-1}_\alpha = (\phi^{-1}_\alpha, F^{-1}_\alpha)$.

Now take $\bar{\Phi}^{-1}_\alpha(U), \bar{\Phi}^{-1}_\beta(V) \in \beta$ and suppose $(p,\ X_p) \in \bar{\Phi}^{-1}_\alpha(U) \cap \bar{\Phi}^{-1}_\beta(V)$ for some U, V open in R^{2n}.

$\Rightarrow \quad \bar{\Phi}_\alpha(p,\ X_p) \in U \quad$ and $\quad \bar{\Phi}_\beta(p,\ X_p) \in V$

Take $U = U^* \times U^{**}$ and $V = V^* \times V^{**}$ where U^*, U^{**}, V^*, V^{**} are open sets in R^n.Clearly $\phi_\alpha(p) \in U^*, F_\alpha(\ X_p) \in U^{**}, \phi_\beta(p) \in V^*$, and $F_\beta(X_p) \in V^{**}$.

From the definition of open set there exist neighborhoods $W^*_1, W^*_2, W^{**}_1, W^{**}_2$ of $\phi_\alpha(p), \phi_\beta(p), F_\alpha(\ X_p)$ and $F_\beta(X_p)$ respectively such that

$\phi_\alpha(p) \in W_1^* \subset U^*$ $\qquad$ $\phi_\beta(p) \in W_2^* \subset V^*$

$F_\alpha(X_p) \in W_1^{**} \subset U^{**}$ $\qquad$ $F_\beta(X_p) \in W_2^{**} \subset V^{**}$

Take $W^* = W_1^* \cap (\phi_\alpha \circ \phi_\beta^{-1})(W_2^*)$ and $W^{**} = W_1^{**} \cap (F_\alpha \circ F_\beta^{-1})(W_2^{**})$, then $\phi_\alpha(p) \in W^*$ and $F_\alpha(X_p) \in W^{**}$, where W^*, W^{**} are both open in R^n.

$\Rightarrow \quad (\phi_\alpha(p), F_\alpha(X_p)) \in W^* \times W^{**} = W$ open in R^{2n}.

$\Rightarrow \quad \bar{\Phi}_\alpha(p, X_p) \in W$

$\Rightarrow \quad (p, X_p) \in \bar{\Phi}_\alpha^{-1}(W) = (\phi_\alpha^{-1}, F_\alpha^{-1})(W^* \times W^{**}) = (\phi_\alpha^{-1}(W^*), F_\alpha^{-1}(W^{**})) = (\phi_\alpha^{-1}(W_1^*) \cap \phi_\beta^{-1}(W_2^*), F_\alpha^{-1}(W_1^{**}) \cap F_\beta^{-1}(W_2^{**})) \subset (\phi_\alpha^{-1}(U^*) \cap \phi_\beta^{-1}(V^*), F_\alpha^{-1}(U^{**}) \cap F_\beta^{-1}(V^{**})) = (\phi_\alpha^{-1}, F_\alpha^{-1})(U) \cap (\phi_\beta^{-1}, F_\beta^{-1})(V) = \bar{\Phi}_\alpha^{-1}(U) \cap \bar{\Phi}_\beta^{-1}(V)$, so β is a basis for a topology on TM. i.e. TM is a topological space.

[2] <u>TM `is a Hausdorff space`</u>

Let $(p, X_p), (q, y_q) \in TM$, then $p, q \in M$ and since M is a Hausdorff space then there exist U, V open in M such that $U \cap V = \phi$ and $p \in U, q \in V$. Since $U, V \neq \phi \Rightarrow U, V$ are n-dimensional smooth manifolds, then there exist charts $(U_1, \phi), (V_1, \psi)$ around p and q in U and V respectively. $U \cap V = \phi$ $\Rightarrow \quad U_1 \cap V_1 = \phi$, now $TU_1, TV_1 \in \beta$ i.e. open neighborhoods of (p, X_p) and (q, y_q) in TM, and $TU_1 \cap TV_1 = \phi$, so TM is a Hausdorff space.

[3] <u>TM `is a topological manifold`</u>

For $(p, X_p) \in TM$, as $p \in M$, there exists a chart (U, ϕ) with local coordinates $x^1, x^2, ..., x^n$ in M around p. Define $\bar{\Phi} : TU \rightarrow R^{2n}$ as before, since the coordinate functions $x^1, x^2, ..., x^n$ are smooth and the differential dx_p^i are linear transformations, then $\bar{\Phi}$ is continuous map. Moreover, for $(y^1, y^2, ..., y^{2n}) \in \bar{\Phi}(TU) \subset R^{2n}$ we have $\bar{\Phi}^{-1}(y^1, y^2, ..., y^{2n}) = (p, X_p)$, where $\phi(p) = (y^1, y^2, ..., y^n)$ and $X_p(x^i) = y^{i+n} \quad i = 1, 2, ..., n$.

Thus $p = \phi^{-1}(y^1, y^2, ..., y^n)$, also $X_p = \sum_{i=1}^n y^{i+n}(\frac{\partial}{\partial x^i})_p$

$\Rightarrow \quad \bar{\Phi}^{-1}(y^1, y^2, ..., y^{2n}) = (\phi^{-1}(y^1, y^2, ..., y^n), \sum_{i=1}^n y^{i+n}(\frac{\partial}{\partial x^i})_p)$

Note that ϕ^{-1} is continuous map (ϕ is homeomorphism), and the map $F : R^n \rightarrow T_pM$ defined by $F(y^{1+n}, , ..., y^{2n}) = \sum_{i=1}^n y^{i+n}(\frac{\partial}{\partial x^i})_p$ is linear transformation and therefore is continuous.

$\Rightarrow \quad \bar{\Phi}^{-1}$ is continuous.

$\Rightarrow \quad \bar{\Phi}\ TU \rightarrow \bar{\Phi}(TU) \subset R^{2n}$ is a homeomorphism.

$\Rightarrow \quad (TU, \bar{\Phi})$ is a chart around (p, X_p).

$\Rightarrow \quad TM$ is a $2n$-dimensional topological manifold.

[4] <u>TM `is a` $2n$`-dimensional smooth manifold`</u>

Consider $\bar{S} = \{ (TU, \bar{\Phi})(U, \phi) \in S\}$, then

i) clearly $\cup_{(U,\phi)\in S} TU = TM$.

ii) take $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta) \in S$ with local coordinates $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n$ respectively, such that $U_\alpha \cap U_\beta \neq \phi \Rightarrow TU_\alpha \cap TU_\beta \neq \phi$

Now for $(z^1, z^2, ..., z^{2n}) \in R^{2n}$ we have

$(\bar{\Phi}_\alpha \circ \bar{\Phi}_\beta^{-1})(z^1, z^2, ..., z^{2n}) = \bar{\Phi}_\alpha(p, X_p)$ where $\bar{\Phi}_\beta(p, X_p) = (z^1, z^2, ..., z^{2n})$, that is $\phi_\beta(p) = (z^1, z^2, ..., z^n)$ and $X_p(y^i) = z^{i+n}, i = 1, 2, .., n$

Now $(\bar{\Phi}_\alpha \circ \bar{\Phi}_\beta^{-1})(z^1, z^2, ..., z^{2n}) = (\phi_\alpha(p), X_p(x^1), ..., X_p(x^n)) = (\phi_\alpha \circ \phi_\beta^{-1}(z^1, z^2, ..., z^n), X_p(x^1), ..., X_p(x^n))$ but $X_p(x^i) = dx_p^i(X_P) = dx_p^i(\sum_{\alpha=1}^n z^{\alpha+n}(\frac{\partial}{\partial y^\alpha})_p) =$

$\sum_{\alpha=1}^{n} z^{\alpha+n} dx_p^i((\frac{\partial}{\partial y^\alpha})_p) = \sum_{\alpha=1}^{n} z^{\alpha+n} \frac{\partial\ x^i}{\partial\ y^\alpha}(p) = \sum_{\alpha=1}^{n} z^{\alpha+n} \frac{\partial}{\partial u^\alpha}(x^i \circ \phi_\beta^{-1})(\phi_\beta(p))$
thus $\bar{\Phi}_\alpha \circ \bar{\Phi}_\beta^{-1}$ is smooth since the first component given by $\phi_\alpha \circ \phi_\beta^{-1}$ is smooth and other n-components are smooth.

Similarly we can show that $\bar{\Phi}_\beta \circ \bar{\Phi}_\alpha^{-1}$ is also smooth.

$\Rightarrow$ $\bar{S}$ is a differentiable structure on TM.

$\Rightarrow$ TM is a $2n$-dimensional smooth manifold.

LEMMA 1.1 For a smooth map $f : M \to N$ the map $df : TM \to TN$ define by $df(p, X_p) = (f(p), df_P\ (X_p))$ is smooth.

LEMMA 1. 2 The natural projection map $\pi : TM \to M$ given by $\pi(p, X_p) = p$ is smooth submersion.

Cotangent bundle

In this section we develop the theory of cotangent bundle.

DEFINITION 2.1 For a smooth manifold M we define $TM^* = \cup_{p\in M} T_p^* M$ and call it **the cotangent bundle** of M.

REMARK An element of TM^* is written as (p, w_p) where $w_p \in T_p^* M$, $p \in M$.

THEOREM 2.2 The cotangent bundle TM^* for a n-dimensional smooth manifold M is $2n$-dimensional smooth manifold and for a chart (U, ϕ) with local coordinates $x^1, x^2, ..., x^n$ on M the corresponding chart is $(TU, \bar{\Phi})$ where $TU = \cup_{p\in U} T_p^* M$ and $\bar{\Phi} : TU \to R^{2n}$ is defined by

$$\bar{\Phi}(p,\ w_p) = (x^1(p), x^2(p), ..., x^n(p), w_p(\frac{\partial}{\partial x^1})_p, w_p(\frac{\partial}{\partial x^2})_p, ..., w_p(\frac{\partial}{\partial x^n})_p)$$

Proof similar to the proof of theorem (1.2.1)

LEMMA 2.3 The natural projection map $\pi : TM^* \to M$ given by $\pi(p, w_p) = p$ is smooth submersion.

VECTOR FIELDS AND SMOOTH FORMS

In this section we introduce vector fields, one parameter groups of transformations, complete vector fields. We also study smooth forms and the exterior differential operator.

DEFINITION 3.1 Let M be an n-dimensional smooth manifold. **A smooth vector field** (or vector field) X on M is a smooth map $X : M \to TM$ satisfying $\pi \circ X = id_M$, i.e. $\forall p \in M$, $X(p) \in T_p M$.

EXAMPLE 3.1 For a chart (U, ϕ) on M with local coordinates $x^1, x^2, ..., x^n$ define $\frac{\partial}{\partial\, x^i} : U \to TU = \pi^{-1}(U) \subset TM$ by

$$\frac{\partial}{\partial\, x^i}(p) = (\frac{\partial}{\partial\, x^i})_p \qquad i = 1, 2, ..., n$$

Then clearly $\{\frac{\partial}{\partial\, x^i} \quad , \quad i = 1, ..., n\}$ are smooth vector fields on U. Moreover for a vector field X we can write X locally on U as

$$X = \sum_{i=1}^{n} X(x^i) \frac{\partial}{\partial\, x^i}$$

DEFINITION 3.2 We define $\mathfrak{X}(M)$ to be the **set of all smooth vector fields on** M.

REMARKS

(1) Since on R^n there is one chart (R^n, id) with local coordinates $x^1, x^2, ..., x^n$ then the vector fields $\{\frac{\partial}{\partial\, x^i} \quad , \quad i = 1, ..., n\}$ will be defined globally and $\forall \quad X \in \mathfrak{X}(R^n)$ we write $X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial\, x^i}$, $f^i \in C^\infty(R^n)$, $f^i = X(x^i)$, $i = 1, ..., n$ and we say $\mathfrak{X}(R^n)$ has finite dimension.

(2) For $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$

i) Define $fX : M \to TM$ by $(fX)(p) = f(p)X(p)$, then it is easy to see that $\mathfrak{X}(M)$ is a module over $C^\infty(M)$.

ii) Define a smooth function$X(f) : M \to R$ by $X(f)(p) = X(p)(f)$ then it is suitable to define a vector field X as

$X : C^\infty(M) \to C^\infty(M)$ which satisfies

$X(\lambda f + \mu g) = \lambda\ X(f) + \mu\ X(g)$

$X(fg) = f\ X(g) +\ X(f)g \qquad f, g \in C^\infty(M)$

We call $X(f)$ **the derivative of** f **with respect to** the vector field X, and it can be shown that the two definitions of a smooth vector field are equivalent.

DEFINITION 3.3 For $X, Y \in \mathfrak{X}(M)$ then **the bracket** of X and Y, $[X, Y]$, is a vector field on M defined by $[X, Y] : C^\infty(M) \to C^\infty(M)$

$$[X, Y](f) = X(Y(f)) - Y(\ X(f))$$

The bracket operation has the following properties

THEOREM 3.1 If X, Y and $Z \in \mathfrak{X}(M)$, $a, b \in R$, and $f, g \in C^\infty(M)$ then

(1) $[X, Y] = -[Y, X]$

(2) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$

(3) $[fX, gY] = fg[X, Y] + f\ X(g)Y - gY(f)X$

(4) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (**Jacobi identity**)

(5) $[X, X] = 0$

REMARK In above theorem (4)and (5) makes $\mathfrak{X}(M)$ a lie-algebra.

DEFINITION 3.4 Let $\alpha : (a,b) \to M$ be a smooth curve, for $t \in (a,b)$ with $\alpha(t) = p \in M$,we define $\dot{\alpha}(t) = d\alpha_t(\frac{d}{dt})_t \in T_pM$, and we call it **tangent vector to the curve** α at t.

REMARK For a chart (U, ϕ) around $\alpha(t) \in M$ with local coordinates $x^1, x^2, ..., x^n$, we have

$$\dot{\alpha}(t) = \sum_{i=1}^{n} \frac{d\ x^i}{d\ t}(t)(\frac{\partial}{\partial\ x^i})_{\alpha(t)}$$

where $x^i = x^i \circ \alpha$

DEFINITION 1.3.5 Let M be a smooth manifold. A smooth curve $\alpha(a,b) \to M$ is said to be **an integral curve** of $X \in \mathfrak{X}(M)$ if $\dot{\alpha}(t) = X(\alpha(t))$ $\forall t \in (a,b)$

EXAMPLE 3.2 Consider $X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \in \mathfrak{X}(R^2)$ and a smooth curve $\alpha : R \to R^2$ defined by $\alpha(t) = (1+t, 1-t)$ then $x^1 = x \circ \alpha = 1+t$, $x^2 = y \circ \alpha = 1-t \Rightarrow \dot{\alpha}(t) = \sum_{i=1}^{2} \frac{d\ x^i}{d\ t}(t)(\frac{\partial}{\partial\ x^i})_{\alpha(t)} = (\frac{\partial}{\partial\ x})_{\alpha(t)} - (\frac{\partial}{\partial\ y})_{\alpha(t)} = X(\alpha(t))$, so α is an integral curve of X.

LEMMA 3.1 Let α and β be integral curves of $X \in \mathfrak{X}(M)$ defined on an open intervals I and J respectively, containing 0. If $\alpha(0) = \beta(0)$ then $\alpha(t) = \beta(t) \quad \forall t \in I \cap J$.

REMARK If $\alpha : (a,b) \to M$ is an integral curve of $X \in \mathfrak{X}(M)$ passing through $p = \alpha(t_\circ)$, $t_\circ \in (a,b)$. Let (U, ϕ) be a chart around p with local coordinates $x^1, x^2, ..., x^n$ and write $X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}$ on U. Namely, α^i is a solution of a system of differential equations $\frac{dx^i}{dt} = f^i \quad (i = 1, ..., n)$ subject to the initial condition $x^i(t_\circ) = x^i(p)$, then $\alpha = \phi^{-1}(\alpha^1, ..., \alpha^n)$.

DEFINITION 3.6 **A vector field** X **along a curve** $\alpha : I \to M$ is a smooth map $X : I \to TM$ such that $X(t) \in T_{\alpha(t)}M$, for $t \in I$.

The set of vector fields along α is denoted by $\Gamma_\alpha(TM)$

EXAMPLE 3.3 $\dot{\alpha} = d\alpha(\frac{d}{dt})$ is a vector field along the curve $\alpha : I \to M$ and it is denote some time by $\dot{\alpha} = \frac{d\ \alpha}{d\ t}$.

REMARK Let $\alpha : I \to M$ be a smooth curve, then each $X \in \mathfrak{X}(M)$ gives $X : I \to TM$, $X(t) = X(\alpha(t))$ a smooth vector field along α.

DEFINITION 3.7 Let $f : M \to N$ be a smooth map. If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are such that $df(X) = Y \circ f$, where $df(X)(p) = df_p(X(p))$, then we say X **is** f**-related to** Y.

LEMMA 3.2 Let M and N be two smooth manifold and $f: M \to N$ is a smooth map. If $X_1, Y_1 \in \mathfrak{X}(M)$ are f-related to $X_2, Y_2 \in \mathfrak{X}(N)$ respectively, then $[X_1, Y_1]$ is f-related to$[X_2, Y_2]$.

THEOREM 3.2 If $f : M \to N$ is a diffeomorphism then the map $f_*\mathfrak{X}(M) \to \mathfrak{X}(N)$ define by $f_*(X) = df(X) \circ f^{-1}$ is a lie-Algebra isomorphism.

DEFINITION 3.8 Let M be a smooth manifold, then **a one parameter group of transformation** $\{\phi_t\}$ of M is a smooth map ϕ $R \times M \to M$, $\phi(t,p) = \phi_t(p)$ that satisfies

1) $\forall t \in R, \ \phi_t : M \to M$ is a diffeomorphism.
2) $\phi_{t+s} = \phi_t \circ \phi_s \qquad \forall t, s \in R$.

REMARK From any one parameter group of transformation $\{\phi_t\}$ of M we can induce a vector field $X \in \mathfrak{X}(M)$ as follows

Fix $p \in M$ and define $\sigma_p : R \to M$ by $\sigma_p(t) = \phi_t(p)$, which is a smooth curve passing through p, now define $X : M \to TM$ by $X(p) = \dot{\sigma}_p(0) \in T_pM$, then X is a vector field which is induced by $\{\phi_t\}$, and σ_p is an integral curve of X.

EXAMPLE 3.4 Consider $M = R^2$ and $\phi_t : R^2 \to R^2$ be defined as $\phi_t(x,y) = (x+t, y-t), t \in R$, then$\{\phi_t\}$ is a one parameter group of transformation. Fix $p = (a,b) \in R^2$ and put $\sigma_p(t) = \phi_t(p) = (a+t, b-t)$ $\Rightarrow \dot{\sigma}_p(0) = (\frac{\partial}{\partial x})_p - (\frac{\partial}{\partial y})_p = X(p)$

$\Rightarrow \quad X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \in \mathfrak{X}(R^2)$ is the induced vector field of $\{\phi_t\}$.

DEFINITION 3.9 $X \in \mathfrak{X}(M)$ is said to be **invariant** under a smooth map $f : M \to M$ if X is f-related to it self.

THEOREM 3.3 The vector field $X \in \mathfrak{X}(M)$ which induced by a one parameter group of transformation $\{\phi_t\}$ is invariant with respect to $\phi_t : M \to M$.

THEOREM 3.4 If $\phi : R \times M \to M$ and $\psi : R \times M \to M$ are two one parameter groups of transformations on M which induce the same vector field, then $\phi = \psi$.

DEFINITION 3.10 If $X \in \mathfrak{X}(M)$ is induced by a one parameter group of transformation $\{\phi_t\}$ then we say X is **complete** and write $\phi_t = \exp tX$

PROPOSITION 3.1 If $f : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$ is complete, then the f-related vector field $Y \in \mathfrak{X}(N)$ to X is also complete.

LEMMA 3.3 Let $X \in \mathfrak{X}(M)$ be induced by $\{\phi_t\}$, then

(1) $\forall f \in C^\infty(M) \qquad Xf = \lim_{t\to 0} \frac{1}{t}\{f \circ \phi_t - f\}$

(2) For $Y \in \mathfrak{X}(M) \qquad [X,Y] = \lim_{t\to 0} \frac{1}{t}\{d\phi_{-t}(Y) \circ \phi_t - Y\}$

THEOREM 3.5 If $X, Y \in \mathfrak{X}(M)$ are complete vector fields corresponding to a one parameter groups of transformations $\{\phi_t\}$ and $\{\psi_t\}$ respectively, then $[X, Y] = 0 \Longleftrightarrow \phi_t$ and ψ_t commute for each $t \in R$.

DEFINITION 3.11 Let U and V be two nonempty open subsets of a smooth manifold M. A diffeomorphism $\phi : U \to V$ is called **a local transformation** of M.

DEFINITION 3.12 **A local one parameter group of local transformations** of M is a set $\{U_\alpha, \in_\alpha, \phi_t^{(\alpha)}\}_{\alpha\in\Lambda}$ where U_α is an open set of M, $\in_\alpha$a positive number, and $\phi_t^{(\alpha)}$ are a local transformation of M for each t, $|t| <\in_\alpha$, satisfying the following conditions

1) $\{U_\alpha\}$ is an open cover of M.

2) The domain of $\phi_t^{(\alpha)}(\ |t| <\in_\alpha)$ contains U_α and $\phi_0^{(\alpha)}$ is the identity transformation on U_α, the map $(t, p) \to \phi_t^{(\alpha)}(p)$ is a smooth from $(-\in_\alpha, \in_\alpha)\times U_\alpha \to M$.

3) If $|t|, |s|, |s+t| <\in_\alpha$, then $\phi_t^{(\alpha)} \circ \phi_s^{(\alpha)}$ is define and its domain contains U_α and $(\phi_t^{(\alpha)} \circ \phi_s^{(\alpha)})(q) = \phi_{t+s}^{(\alpha)}(q)$ holds for $q \in U_\alpha$.

4) If $U_\alpha \cap U_\beta \neq \phi$, then for each $p \in U_\alpha \cap U_\beta$, we can choose $\in < \min\{\in_\alpha, \in_\beta\}$ such that for $|t| <\in$, $\phi_t^{(\alpha)}$ and $\phi_t^{(\beta)}$ agree on a sufficiently small neighborhood of p.

REMARK Let $G = \{U_\alpha, \in_\alpha, \phi_t^{(\alpha)}\}_{\alpha\in\Lambda}$ be a local one parameter group of local transformations on M, then for fix α and a point $p \in U_\alpha$ define $\sigma_p^{(\alpha)} : (-\in_\alpha, \in_\alpha) \to M$ by $\sigma_p^{(\alpha)}(t) = \phi_t^{(\alpha)}(p)$ which is a smooth curve passing through p. Now let $X : M \to TM$ defined by $X(p) = \dot{\sigma}_p^{(\alpha)}(0)$ then X is a vector field induced by G.

DEFINITION 3.13 Let $G_1 = \{U_\alpha, \in_\alpha, \phi_t^{(\alpha)}\}_{\alpha\in\Lambda}$, $G_2 = \{V_i, \eta_i, \psi_t^{(i)}\}_{i\in I}$ be two local one parameter groups of local transformations, then we say G_1 and G_2 are **equivalent** and write $G_1 \sim G_2$ if the following conditions holds

"When ever $U_\alpha \cap V_i \neq \phi$, then for any $p \in U_\alpha \cap V_i$ there is $0 < \delta < \min\{\in_\alpha, \eta_i\}$ such that $\phi_t^{(\alpha)} = \psi_t^{(i)}$ on a sufficiently small neighborhood of p for $|t| < \delta$."

THEOREM 3.6 Let $X \in \mathfrak{X}(M)$, then there exists a local one parameter group of local transformation which induces X, and two local one parameter groups of local transformation inducing X are equivalent.

THEOREM 3.7 A vector field on M is complete if and only if there is a $G = \{U\alpha, \in_\alpha, \phi_t^{(\alpha)}\}_{\alpha\in\Lambda}$ such that $\inf_\alpha \in_\alpha > 0$.

COROLLARY 3.1 If M is compact then every vector field on M is complete.

Now by using a cotangent bundle TM^*, we can similarly define another type of smooth map as follows

DEFINITION 3.14 A smooth map $W : M \rightarrow TM^*$ satisfying $\pi \circ W = id_M$ is called **a smooth 1-form.**

DEFINITION 3.15 The **set of all smooth 1-form on** M is denoted by $\Lambda^1(M)$.

LEMMA 3.4 $\Lambda^1(M)$ is a module over $C^\infty(M)$ and is a vector space over R.

EXAMPLE 3.5 For $f \in C^\infty(M)$ we know $\forall p \in M$, $df_p : T_pM \rightarrow R$, that is $df_p \in T_p^*M$. Now define $df : M \rightarrow TM^*$ by $df(p) = df_p$, it is easy to show that $df \in \Lambda^1(M)$.

REMARKS

(1) For $W \in \Lambda^1(M)$ and $X \in \mathfrak{X}(M)$, we define a smooth map $W(X) : M \rightarrow R$ by $W(X)(p) = W_p(X_P)$, where $W_P = W(p)$, $X_p = X(p)$, thus we can write a smooth 1-form W as a map $W : \mathfrak{X}(M) \rightarrow C^\infty(M)$.

(2) Each $f \in C^\infty(M)$ is also called a smooth 0-form and we some times denote $C^\infty(M)$ by $\Lambda^0(M)$.

Now to define the higher order differential forms we need some algebraic preparation

DEFINITION 3.16

1) Let M be an n-dimensional smooth manifold, take $p \in M$, then a map

$$f \quad : \quad \underbrace{T_pM \times \cdots \times T_pM}_{k-copies} \rightarrow R$$

is said to be **multi-linear** if $f(X_p^1, ..., X_p^k)$ is linear in each slot.

2) We denote by $T^k(T_pM)$ the **set of all multi-linear maps**

$$f \quad : \quad \underbrace{T_pM \times \cdots \times T_pM}_{k-copies} \rightarrow R$$

which is a vector space over R.

3) For $S \in T^k(T_pM)$ and $T \in T^L(T_pM)$, we define **the tensor product** $S \otimes T \in T^{k+L}(T_pM)$ by

$$(S \otimes T)(\ X_p^1, ..., X_p^k, X_p^{k+1}, ..., X_p^{k+L}) = S(X_p^1, ..., X_p^k).T(X_p^{k+1}, ..., X_p^{k+L})$$

4) An element $S \in T^k(T_pM)$ is said to be **alternating** if $S(X_p^1, ..., X_p^k) = 0$ whenever $X_p^i = X_p^j$ for any i, j such that $1 \le i, j \le k$.

5) If $S \in T^k(T_pM)$ is alternating then

$$S(X_p^1, ..., X_p^i, ..., X_p^j, ..., X_p^k) = -S(\ X_p^1, ..., X_p^j, ..., X_p^i, ..., X_p^k).$$

6) The **set of all alternating elements in** $T^k(T_pM)$ is denoted by $\Gamma^k(T_pM)$.

7) For $S \in T^k(T_pM)$ we define **AltS** $\in \Gamma^k(T_pM)$ by

$$(AltS)(X_p^1, ..., X_p^k) = \frac{1}{k!} \sum_{\sigma \in s_k} (-1)^\sigma S(X_p^{\sigma(1)}, ..., X_p^{\sigma(k)})$$

where S_K is the permutation group, and $(-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \texttt{ is even} \\ 2 & \text{if } \sigma \texttt{ is odd} \end{cases}$

8) An element $W_p \in \Gamma^k(T_pM)$ is called **a** k**-form** and $\Gamma^k(T_pM)$ is called a space of k-forms at $p \in M$.

9) For $W_p \in \Gamma^k(T_pM)$ and $\eta_p \in \Gamma^L(T_pM)$ we define the **wedge product** $W_p \wedge \eta_p \in \Gamma^{k+L}(T_pM)$ by

$$W_p \wedge \eta_p = \frac{(k+L)!}{k!L!} \sum_{\sigma \in S_{K+l}} (-1)^\sigma Alt(W_p \otimes \eta_p) \circ \sigma.$$

EXAMPLE 3.6 Let $W_p, \eta_p \in \Gamma^1(T_pM)$, then $W_p \wedge \eta_p \in \Gamma^2(T_pM)$ and
$W_p \wedge \eta_p = \frac{2!}{1!1!} \sum_{\sigma \in S_2} (-1)^\sigma Alt(W_p \otimes \eta_p) \circ \sigma$ so
$(W_p \wedge \eta_p)(X_p, Y_p) = 2\{Alt(W_p \otimes \eta_p)(X_p, Y_p) - Alt(W_p \otimes \eta_p)(Y_p, X_p)\}$
$= 2\{\frac{1}{2}[W_p(X_p)\eta_p(Y_p) - W_p(Y_p)\eta_p(X_p)] - \frac{1}{2}[W_p(Y_p)\eta_p(X_p) - W_p(X_p)\eta_p(Y_p)]\}$
$= 2\{W_p(X_p)\eta_p(Y_p) - W_p(Y_p)\eta_p(X_p)\}$

LEMMA 3.5 Let M be an n-dimensional smooth manifold, then $\Gamma^{n+k}(T_pM) = \{0\}$ for $k \ge 1$.

LEMMA 3.6 For $W_p, W_p^i \in \Gamma^k(T_pM)$, $\eta_p \in \Gamma^L(T_pM)$, $\alpha_p \in \Gamma^m(T_pM)$, $i = 1, 2$ then

1) $(W_p^1 + W_p^2) \wedge \eta_p = W_p^1 \wedge \eta_p + W_p^2 \wedge \eta_p$
2) $W_p \wedge \eta_p = (-1)^{kL} \eta_p \wedge W_p$
3) $W_p \wedge (\eta_p \ \wedge \alpha_p) = (\ W_p \wedge \eta_p) \ \wedge \alpha_p$

COROLLARY 3.2 If k is odd then $W_p \wedge \ W_p = 0$

DEFINITION 3.17 Let M be an n-dimensional smooth manifold, then we define $\Gamma^k(M) = \cup_{p \in M} \Gamma^k(T_pM)$ and call it the **bundle of** k**-forms.**

DEFINITION 3.18 A smooth map $W : M \to \ \Gamma^k(M)$ satisfying $\pi \circ W = id_M$ is called **a smooth** k**-form**.

REMARKS

1) For $p \in M$, $W(p) \in \Gamma^k(T_pM)$, that means $W(p)$ is a k-form at $p \in M$.
2) The set of all smooth k-forms on M is denoted by $\Lambda^k(M)$.
3) $\Lambda^k(M)$ is a module over $C^\infty(M)$.
4) For $W \in \Lambda^k(M)$ and $X^1, ..., X^K \in \mathfrak{X}(M)$ we define
$W : (X^1, ..., X^K)M \to R$ by

$$W(X^1, ..., X^K)(p) = W(p)(X^1(p), ..., X^K(p))$$

and it is easy to show that $W(X^1, ..., X^K) \in C^\infty(M)$, thus we can write a smooth k-form as a map

$W : \underbrace{\mathfrak{X}(M) \times \mathfrak{X}(M) \times .... \times \mathfrak{X}(M)}_{k\text{-copies}} \to C^\infty(M)$ which is multi-linear alternating.

DEFINITION 3.19 For $W \in \Lambda^k(M), k > 0$, we define $dW \in \Lambda^{k+1}(M)$ as

$dW : \underbrace{\mathfrak{X}(M) \times \mathfrak{X}(M) \times .... \times \mathfrak{X}(M)}_{(k+1)\text{-copies}} \to C^\infty(M)$ by

$$\begin{aligned} dW(X_1, ..., X_{k+1}) &= \sum_i (-1)^{i+1} X_i(W(X_1, ..., \hat{X}_i..., X_{k+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} W([X_i, X_j], X_1, ..., \hat{X}_i..., \hat{X}_j., X_{k+1}) \end{aligned}$$

and for $k = 0$ we define $dW(X) = X(W)$.

This operator $d\Lambda^k(M) \to \Lambda^{k+1}(M)$ called the **exterior differential operator** and it has some properties which given in this lemma

LEMMA 3.7 For $W, W^1, W^2 \in \Lambda^k(M), \eta \in \Lambda^L(M)$

1) $d(W^1 + W^2) = dW^1 + dW^2$
2) $d(W \wedge \eta) = dW \wedge \eta + (-1)^K W \wedge d\eta$
3) $d^2 = d \circ d = 0$

EXAMPLE 3.7 For $\eta \in \Lambda^1(M)$ then $d\eta \in \Lambda^2(M)$ and

$$\begin{aligned} d\eta(X, Y) &= (-1)^2 X(\eta(Y)) + (-1)^3 Y(\eta(X)) + (-1)^3 \eta([X, Y]) \\ &= X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \end{aligned}$$