Model Answers
M-374, II-MIDTERM EXAMINATION,
SEMESTER-II, 1429H.

Q.1 (a) Let $\{\phi_t\}$ be a one-parameter group of transformations of a smooth manifold M and X be the vector field induced by $\{\phi_t\}$. If $f \in C^\infty(p)$, show that

$$(Xf)(p) = \lim_{t \to 0} \frac{f(\phi_t(p)) - f(p)}{t}$$

Answer: Note that the vector field X is definded by $X_p = \dot{\sigma}(0)$, $p \in M$, where the curve $\sigma : R \to M$ is definde by $\sigma(t) = \varphi_t(p)$. Consequently for a $f \in C^\infty(p)$, we have

$$(Xf)(p) = \dot{\sigma}(0)(f) = \left(\frac{d}{dt}\right)_0 (f(\sigma(t))) = \lim_{t \to 0} \frac{f(\phi_t(p)) - f(p)}{t}$$

(b) Show that $\{\phi_t\}$ defined by $\phi_t(x,y) = (xe^t, ye^{-t})$, $(x,y) \in R^2$ is a one-parameter group of transformations on R^2 and find the vector field $\xi \in \mathfrak{X}(R^2)$ induced by $\{\phi_t\}$.

Answer: The component functions of φ_t are smooth consequently φ_i is smooth. Moreover for fixed point (x,y) of R^2, the map $\phi_t(x,y) : t \to (xe^t, ye^{-t})$ also smooth. Also we have $(ae^t, be^{-t}) = (ce^t, de^{-t})$ implies $a = c$ and $b = d$ that is φ_t is one-to-one. Next for any $(u,v) \in R^2$, we have $(ue^{-t}, ve^t) \in R^2$ which satisfies $\varphi_t(ue^{-t}, ve^t) = (u,v)$ that is the map φ_t is also on-to and the inverse map $\varphi_t^{-1}(x,y) = (xe^{-t}, ye^t)$ is also smooth, that is $\varphi_t : R^2 \to R^2$ is a diffeomorphism for each t. Next we observe that $\varphi_0(x,y) = (x,y)$ that is φ_0 is an identity map. Also we have $\varphi_{t+s}(x,y) = (xe^{t+s}, ye^{-t-s}) = \varphi_t((xe^s, ye^{-s})) = \varphi_t\varphi_s(x,y)$ that is $\varphi_{t+x} = \varphi_t \circ \varphi_s$. This proves that $\{\varphi_t\}$ is a one-parameter group of transformations of R^2. To find the vector field ξ induced by $\{\varphi_t\}$, fix a point $p = (a,b) \in R^2$ and consider the curve $\sigma(t) = \varphi_t(p) = (ae^t, be^{-t})$, which has tangent vector $\dot{\sigma}(0) = (a, -b) = a\left(\frac{\partial}{\partial x}\right)_p - b\left(\frac{\partial}{\partial y}\right)_p$. Thus the vector field generated by $\{\varphi_t\}$ is given by

$$\xi = x\left(\frac{\partial}{\partial x}\right) - y\left(\frac{\partial}{\partial y}\right)$$

(c) Consider the function $f(x,y) = xy$ and $p = (1,1) \in R^2$ to find $(\xi f)(p)$ for the vector field ξ described in (b).

Answer: We have $a = b = 1$ and consequently

$$\xi_p = \left(\frac{\partial}{\partial x}\right)_p - \left(\frac{\partial}{\partial y}\right)_p$$

and $\left(\frac{\partial}{\partial x}\right)_p(f) = 1$, $\left(\frac{\partial}{\partial y}\right)_p(f) = 1$. Thus we have $(\xi f)(p) = \xi_p(f) = 1 - 1 = 0$

Q.2 Let $\{\phi_t\}$ be a one-parameter group of transformations of a smooth manifold M and X be the vector field induced by $\{\phi_t\}$.

(a) If $f : M \to N$ is a diffeomorphism and $\psi_t = f \circ \phi_t \circ f^{-1}$, show that $\{\psi_t\}$ is a one-parameter group of transformations on N.

Answer: Since all the three maps f, ϕ_t, f^{-1} are smooth the map $\psi_t : M \to M$ is smooth. Also as f, φ_t are diffeomorphism, $\psi_t^{-1} = f \circ \phi_t^{-1} \circ f^{-1}$ also smooth and thus $\psi_t : M \to M$ is a diffeomorphism. Also as φ_t being smooth in t, we get that ψ_t is smooth in the variable t. Now $\psi_0 =$ is identity map of M as φ_0 is and that $\psi_{t+s} = f \circ \phi_{t+s} \circ f^{-1} = \psi_t = f \circ \phi_t \circ \varphi_s \circ f^{-1} = \psi_t \times \psi_s$, which proves that $\{\psi_t\}$ is a one parameter group of transformation of M.

(b) If $Y \in \mathfrak{X}(N)$ is induced by $\{\psi_t\}$, then show that $df(X)(p) = Y(f(p))$, $p \in M$.

Answer: We have $Y_{f(p)} = \dot{\rho}(0)$, where $\rho(t) = \psi_t(f(p))$ is the smooth curve. We have $\rho(t) = f(\sigma(t))$, where $\sigma(t) = \varphi_t(p)$ and $X_p = \dot{\sigma}(0)$, $p \in M$. Thus we have

$$Y_{f(p)} = \dot{\rho}(0) = df_{\sigma(0)}(\dot{\sigma}(0)) = df_p(X_p) = df(X)(p)$$

(c) Let $N = S^2 - \{(0,0,1)\}$ and $f : R^2 \to N$ be the diffeomorphism

$$f(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right)$$

and $\{\phi_t\}$ be as in Q.1(b). Find $\{\psi_t\}$ as in Q.2(a) and the vector field induced by $\{\psi_t\}$.

Answer: Following (b) to find the vector field induced by $\{\psi_t\}$, we need to find df_p for a $p \in R^2$. We have the local coordinates x,y on the chart (R^2, id) and the chart (N, ϕ) on N with $\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ with local coordinates y^1, y^2. We have for the components f^1, f^2 given by

$$f^1 = y^1 \circ f = \frac{\frac{2x}{1+x^2+y^2}}{1 - \frac{x^2+y^2-1}{1+x^2+y^2}} = x,\ f^2 = y$$

consequently we get $df_p = I : R^2 \to R^2 = T_pN$ and we get the vector field Y induced by $\{\psi_t\}$ as

$$Y = \left(\frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}\right) \circ f^{-1}$$

Q.3 (a) Let $f : R^3 \to R^3$ be $f(x,y,z) = (x+y, x-y, z)$ and $\omega \in \Lambda^2(R^3)$ be $\omega = z dx \wedge dy$. Find $f^*(\omega)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

Answer: We have

$$df_p = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and consequently $df\left(\frac{\partial}{\partial x}\right)(p) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(p)$, $df\left(\frac{\partial}{\partial y}\right)(p) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(p)$. Thus we have

$$f^*(\omega)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})(p) = zf^*(dx)\Lambda f^*(dy)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})(p)$$

As $f^*(dx)(\frac{\partial}{\partial x}) = dx\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) = 1$, $f^*(dx)(\frac{\partial}{\partial y}) = dx\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) = 1$, $f^*(dy)(\frac{\partial}{\partial x}) = dy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) = 1$ and $f^*(dy)(\frac{\partial}{\partial y}) = dy\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) = -1$

$$f^*(\omega)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})(p) = zf^*(dx)\Lambda f^*(dy)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})(p) = \frac{1}{2}z(1+1) = z$$

(b) If $\alpha = xdy \wedge dz - ydx \wedge dy \in \Lambda^2(R^3)$, find $d\alpha$. Is α a closed form?

Answer: We have

$$d\alpha = dx\Lambda dy\Lambda dx - dy\Lambda dx\Lambda dy = dx\Lambda dy\Lambda dz$$

where we used $d^2 f = 0$ and $\beta\Lambda\beta = 0$ for a 1-form β. Since $dx\Lambda dy\Lambda dz$ is a basis element of $\Lambda^1(M)$ which has dimension one, $d\alpha \neq 0$ that is the form α is not closed.

(c) If $\alpha = dx \wedge dy \in \Lambda^2(R^2)$, find a 1-form $\beta \in \Lambda^1(R^2)$ satisfying $\alpha = d\beta$.

Answer: If we choose any two smooth functioins f, g on R^2 and $\beta = fdx + gdy$, then the equation $\alpha = d\beta$. gives $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1$. Thus choosing functions f, g satisfying above differential equation, we get the required form β, for instance we have

$$\beta = \frac{1}{2}(xdy - ydx)$$