Model Answers
M-374, II-MIDTERM EXAMINATION,
SEMESTER-II, 1429H.

Q.1 (a) Let \( \{ \phi_t \} \) be a one-parameter group of transformations of a smooth manifold \( M \) and \( X \) be the vector field induced by \( \{ \phi_t \} \). If \( f \in C^\infty(p) \), show that

\[
(Xf)(p) = \lim_{t \to 0} \frac{f(\phi_t(p)) - f(p)}{t}
\]

**Answer:** Note that the vector field \( X \) is defined by \( X_p = \dot{\sigma}(0), \ p \in M \), where the curve \( \sigma : R \to M \) is define by \( \sigma(t) = \varphi_t(p) \). Consequently for a \( f \in C^\infty(p) \), we have

\[
(Xf)(p) = \dot{\sigma}(0)(f) = \left( \frac{d}{dt} \right)_0 (f(\sigma(t))) = \lim_{t \to 0} \frac{f(\varphi_t(p)) - f(p)}{t}
\]

(b) Show that \( \{ \phi_t \} \) defined by \( \phi_t(x,y) = (xe^t, ye^{-t}) \), \( (x,y) \in R^2 \) is a one-parameter group of transformations on \( R^2 \) and find the vector field \( \xi \in \mathfrak{X}(R^2) \) induced by \( \{ \phi_t \} \).

**Answer:** The component functions of \( \varphi_t \) are smooth consequently \( \varphi_t \) is smooth. Moreover for fixed point \( (x,y) \) of \( R^2 \), the map \( \phi_t(x,y) : t \to (xe^t, ye^{-t}) \) is also smooth. Also we have \( (ae^t, be^{-t}) = (ce^t, de^{-t}) \) implies \( a = c \) and \( b = d \) that is \( \varphi_t \) is one-to-one. Next for any \( (u,v) \in R^2 \), we have \( (ue^{-t}, ve^t) \in R^2 \) which satisfies \( \varphi_t(u) = (u,v) \) that is the map \( \varphi_t \) is also on-to and the inverse map \( \varphi_t^{-1}(x,y) = (xe^{-t}, ye^t) \) is also smooth, that is \( \varphi_t : R^2 \to R^2 \) is a diffeomorphism for each \( t \). Next we observe that \( \varphi_0(x,y) = (x,y) \) that is \( \varphi_0 \) is an identity map. Also we have \( \varphi_{t+s}(x,y) = (xe^{t+s}, ye^{t-s}) = \varphi_t(\varphi_s(x,y)) = \varphi_{t+s}(x,y) \) that is \( \varphi_t \circ \varphi_s = \varphi_{t+s} \). This proves that \( \{ \varphi_t \} \) is a one-parameter group of transformations of \( R^2 \). To find the vector field \( \xi \) induced by \( \{ \varphi_t \} \), fix a point \( p = (a,b) \in R^2 \) and consider the curve \( \sigma(t) = \varphi_t(p) = (ae^t, be^{-t}) \), which has tangent vector \( \dot{\sigma}(0) = (a,-b) = a \left( \frac{\partial}{\partial x} \right) + b \left( \frac{\partial}{\partial y} \right) \). Thus the vector field generated by \( \{ \varphi_t \} \) is given by

\[
\xi = x \left( \frac{\partial}{\partial x} \right) - y \left( \frac{\partial}{\partial y} \right)
\]

(c) Consider the function \( f(x,y) = xy \) and \( p = (1,1) \in R^2 \) to find \( (\xi f)(p) \) for the vector field \( \xi \) described in (b).

**Answer:** We have \( a = b = 1 \) and consequently

\[
\xi_p = \left( \frac{\partial}{\partial x} \right)_p - \left( \frac{\partial}{\partial y} \right)_p
\]
and \( (\frac{\partial}{\partial t})_p (f) = 1, \) \( (\frac{\partial}{\partial t})_p (f) = 1 \). Thus we have \( (\xi f)_p (p) = \xi_p (f) = 1 - 1 = 0 \)

**Q.2** Let \( \{\phi_t\} \) be a one-parameter group of transformations of a smooth manifold \( M \) and \( X \) be the vector field induced by \( \{\phi_t\} \).

(a) If \( f : M \to N \) is a diffeomorphism and \( \psi_t = f \circ \phi_t \circ f^{-1} \), show that \( \{\psi_t\} \) is a one-parameter group of transformations on \( N \).

**Answer:** Since all the three maps \( f, \phi_t, f^{-1} \) are smooth the map \( \psi_t : M \to M \) is smooth. Also as \( f, \varphi_t \) are diffeomorphism, \( \psi_t = f \circ \phi_t \circ f^{-1} \) also smooth and thus \( \psi_t : M \to M \) is a diffeomorphism. Also as \( \varphi_t \) being smooth in \( t \), we get that \( \psi_t \) is smooth in the variable \( t \). Now \( \psi_0 = \text{id} \) is identity map of \( M \) as \( \varphi_0 \) is and that \( \psi_{t+s} = f \circ \phi_{t+s} \circ f^{-1} \) also smooth in \( t \) is smooth in the variable \( t \). Now \( \psi_0 = \text{id} \) is identity map of \( M \) as \( \varphi_0 \) is and that \( \psi_{t+s} = f \circ \phi_{t+s} \circ f^{-1} = \psi_t = f \circ \phi_t \circ \varphi_s \circ f^{-1} = \psi_t \circ \psi_s \), which proves that \( \{\psi_t\} \) is a one parameter group of transformation of \( M \).

(b) If \( Y \in \mathfrak{X}(N) \) is induced by \( \{\psi_t\} \), then show that \( df(X)(p) = Y(f(p)) \), \( p \in M \).

**Answer:** We have \( Y_{f(p)} = \dot{\rho}(0) \), where \( \rho(t) = \psi_t(f(p)) \) is the smooth curve.

We have \( \rho(t) = f(\sigma(t)) \), where \( \sigma(t) = \varphi_t(p) \) and \( X_p = \sigma(0), p \in M \). Thus we have

\[
Y_{f(p)} = \dot{\rho}(0) = df_{\sigma(0)}(\dot{\sigma}(0)) = df_p(X_p) = df(X)(p)
\]

(c) Let \( N = S^2 - \{(0,0,1)\} \) and \( f : R^2 \to N \) be the diffeomorphism

\[
f(x,y) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}\right)
\]

and \( \{\phi_t\} \) be as in Q.1(b). Find \( \{\psi_t\} \) as in Q.2(a) and the vector field induced by \( \{\psi_t\} \).

**Answer:** Following (b) to find the vector field induced by \( \{\psi_t\} \), we need to find \( df_p \), for a \( p \in R^2 \). We have the local coordinates \( x,y \) on the chart \( (R^2, \text{id}) \) and the chart \( (N, \phi) \) on \( N \) with \( \phi(x, y, z) = \left(\frac{x}{\sqrt{1-z}}, \frac{y}{\sqrt{1-z}}\right) \) with local coordinates \( y^1, y^2 \). We have for the components \( f^1, f^2 \) given by

\[
f^1 = y^1 \circ f = \frac{2x}{1 + x^2 + y^2} = x, \quad f^2 = y
\]

consequently we get \( df_p = I : R^2 \to R^2 = T_p N \) and we get the vector field \( Y \) induced by \( \{\psi_t\} \) as

\[
Y = \left(\frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}\right) \circ f^{-1}
\]

**Q.3** (a) Let \( f : R^3 \to R^3 \) be \( f(x, y, z) = (x + y, x - y, z) \) and \( \omega \in \Lambda^2(R^3) \) be \( \omega = zd\alpha \wedge dy \). Find \( f^*(\omega)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \).
Answer: We have
\[ df_p = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
and consequently \( df \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)(p) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)(p). \)
Thus we have
\[ f^*(\omega) \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)(p) = z f^*(dx) \Lambda f^*(dy) \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)(p) \]
As \( f^*(dx)(\frac{\partial}{\partial x}) = dx \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) = 1, \)
\[ f^*(dx)(\frac{\partial}{\partial x}) = dx \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) = 1, \]
\[ f^*(dy)(\frac{\partial}{\partial x}) = dy \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) = 1 \]
and
\[ f^*(dy)(\frac{\partial}{\partial x}) = dy \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) = -1 \]
\[ f^*(\omega) \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)(p) = z f^*(dx) \Lambda f^*(dy) \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)(p) = \frac{1}{2} z(1 + 1) = z \]

(b) If \( \alpha = xdy \wedge dz - ydx \wedge dy \in \Lambda^2(R^3), \) find \( d\alpha. \) Is \( \alpha \) a closed form?

Answer: We have
\[ d\alpha = dx \wedge dy \wedge dz - dy \wedge dx \wedge dy = dx \wedge dy \wedge dz \]
where we used \( d^2 f = 0 \) and \( \beta \Lambda \beta = 0 \) for a 1-form \( \beta. \) Since \( dx \wedge dy \wedge dz \) is a basis element of \( \Lambda^3(M) \) which has dimension one, \( d\alpha \neq 0 \) that is the form \( \alpha \) is not closed.

(c) If \( \alpha = dx \wedge dy \in \Lambda^2(R^2), \) find a 1-form \( \beta \in \Lambda^1(R^2) \) satisfying \( \alpha = d\beta. \)

Answer: If we choose any two smooth functioins \( f, g \) on \( R^2 \) and \( \beta = f dx + gdy, \) then the equation \( \alpha = d\beta. \) gives
\[ \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} = 1. \]
Thus choosing functions \( f, g \) satisfying above differential equation, we get the required form \( \beta, \) for instance we have
\[ \beta = \frac{1}{2}(xdy - ydx) \]