M-573, II-MIDTERM EXAMINATION
SEMESTER-II, 1429H.
DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE
KING SAUD UNIVERSITY

Time: 2-hours Max. Marks-20

Note: *Attempt all questions*

Q.1 (a) Let $\{\phi_t\}$ be a one-parameter group of transformations of a smooth manifold M and X be the vector field induced by $\{\phi_t\}$. If $f \in C^\infty(p)$, show that

$$(Xf)(p) = \lim_{t \to 0} \frac{f(\phi_t(p)) - f(p)}{t}$$ [2]

(b) Show that $\{\phi_t\}$ defined by $\phi_t(x,y) = (xe^t, ye^{-t})$, $(x,y) \in R^2$ is a one-parameter group of transformations on R^2 and find the vector field $\xi \in \mathfrak{X}(R^2)$ induced by $\{\phi_t\}$. [2]

(c) Consider the function $f(x,y) = xy$ and $p = (1,1) \in R^2$ to find $(\xi f)(p)$ for the vector field ξ described in (b). [2]

Q.2 Let $\{\phi_t\}$ be a one-parameter group of transformations of a smooth manifold M and X be the vector field induced by $\{\phi_t\}$.

(a) If $f : M \to N$ is a diffeomorphism and $\psi_t = f \circ \phi_t \circ f^{-1}$, show that $\{\psi_t\}$ is a one-parameter group of transformations on N. [2]

(b) If $Y \in \mathfrak{X}(N)$ is induced by $\{\psi_t\}$, then show that $df(X)(p) = Y(f(p))$, $p \in M$. [2]

(c) Let $N = S^2 - \{(0,0,1)\}$ and $f : R^2 \to N$ be the diffeomorphism

$$f(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right)$$

and $\{\phi_t\}$ be as in Q.1(b). Find $\{\psi_t\}$ as in Q.2(a) and the vector field induced by $\{\psi_t\}$. [4]

Q.3 (a) Let $f : R^3 \to R^3$ be $f(x,y,z) = (x+y, x-y, z)$ and $\omega \in \Lambda^2(R^3)$ be $\omega = z dx \wedge dy$. Find $f^*(\omega)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. [2]

(b) If $\alpha = x dy \wedge dz - y dx \wedge dy \in \Lambda^2(R^3)$, find $d\alpha$. Is α a closed form? [2]

(c) If $\alpha = dx \wedge dy \in \Lambda^2(R^2)$, find a 1-form $\beta \in \Lambda^1(R^2)$ satisfying $\alpha = d\beta$. [2]