M-573 FIRST MIDTERM EXAMINATION
II-Semester, 1429
Department of Mathematics (Women’s Section)
College of Science, King Saud University

Time 2 Hours

Note: Attempt all questions

Q.1. (a) Let $M$ be an $n$-dimensional smooth manifold and $p \in M$. If the functions $f_1, \ldots, f_n \in C^\infty(p)$ are such that $\{(df_1)_p, \ldots, (df_n)_p\}$ is a basis of the cotangent space $T^*_p M$, then show that there is a chart $(U, \phi)$ around $p$ with local coordinates $f_1, \ldots, f_n$.

(b) Use (a) to show that there is a chart around $p = (1, -1, 1) \in \mathbb{R}^3$ with local coordinates $\rho, \varphi, \theta$ defined by
\[
\rho = \sqrt{x^2 + y^2 + z^2}, \quad \varphi = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \quad \theta = \tan^{-1} \frac{y}{x}.
\]

Q.2. (a) Let $\alpha : (-\frac{\pi}{2}, \frac{\pi}{2}) \to S^2$ be the smooth curve defined by $\alpha(t) = (\frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}})$. Find the expression for the tangent vector to the curve $\beta = \pi \circ \alpha$ in $RP^2$ at the point $t = \frac{\pi}{2}$, using the local coordinates in $RP^2$ around the point $[(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})] \in RP^2$.

(b) Show that the projection map $\pi : TM \to M$ is a submersion for a smooth manifold $M$.

(c) Show that a smooth vector field $X \in \mathfrak{X}(M)$ gives a smooth immersion $X : M \to TM$.

Q.3. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be $f(x, y, z) = (xy, yz)$ and $p = (1, 1, 1)$. Find the matrix of the differential $df_p$ of $f$ at $p$. Also for the smooth curve $\alpha : \mathbb{R} \to \mathbb{R}^3$, $\alpha(t) = (1 + t^2, 1 - t^2, 1 - 2t)$. Find the tangent vector $\dot{\alpha}(t)$ to the curve $\beta(t) = f(\alpha(t))$ and for a smooth function $g(x, y) = x^2 + y^2$ on $\mathbb{R}^2$ find the number $\dot{\beta}(0)(g)$.

(b) Let $h : S^2 \to R$ be the smooth function $h(x, y, z) = z$ and $X = x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2}, \quad Y = \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1}$ be smooth vector fields on $S^2$ expressed locally in a neighbourhood of the point $p = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}) \in S^2$. Find $[X, Y](h)$.

(c) Find an integral curve of the vector field $Y$ in (b) passing through the point $p = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}) \in S^2$.