M-573 FIRST MIDTERM EXAMINATION

II-Semester, 1429

Department of Mathematics (Women's Section)

College of Science, King Saud University

Time 2 Hours **Max. Marks-20**

Note: *Attempt all questions*

Q.1.(a) Let M be an n-dimensional smooth manifold and $p \in M$. If the functions $f_1, .., f_n \in C^\infty(p)$ are such that $\left\{(df_1)_p, .., (df_n)_p\right\}$ is a basis of the cotangent space T_p^*M, then show that there is a chart (U, φ) around p with local coordinates $f_1, .., f_n$.

(b) Use (a) to show that there is a chart around $p = (1, -1, 1) \in R^3$ with local coordinates ρ, φ, θ defined by

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \varphi = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \quad \theta = \tan^{-1} \frac{y}{x}$$

Q.2. (a) Let $\alpha : (-\frac{\pi}{2}, \frac{\pi}{2}) \to S^2$ be the smooth curve defined by $\alpha(t) = (\frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}})$. Find the expression for the tangent vector to the curve $\beta = \pi \circ \alpha$ in RP^2 at the point $t = \frac{\pi}{4}$, using the local coordinates in RP^2 around the point $[(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})] \in RP^2$.

(b) Show that the projection map $\pi : TM \to M$ is a submersion for a smooth manifold M.

(c) Show that a smooth vector field $X \in \mathfrak{X}(M)$ gives a smooth immersion $X : M \to TM$.

Q.3. (a) Let $f : R^3 \to R^2$ be $f(x, y, z) = (xy, yz)$ and $p = (1, 1, 1)$. Find the matrix of the differential df_p of f at p. Also for the smooth curve $\alpha : R \to R^3$, $\alpha(t) = (1 + t^2, 1 - t^2, 1 - 2t)$. Find the tangent vector $\dot{\beta}(0)$ to the curve $\beta(t) = f(\alpha(t))$ and for a smooth function $g(x, y) = x^2 + y^2$ on R^2 find the number $\dot{\beta}(0)(g)$.

(b) Let $h : S^2 \to R$ be the smooth function $h(x, y, z) = z$ and $X = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$, $Y = \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1}$ be smooth vector fields on S^2 expressed locally in a neighbourhood of the point $p = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}) \in S^2$. Find $[X, Y](h)$.

(c) Find an integral curve of the vector field Y in (b) passing through the point $p = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}) \in S^2$.