

QUOTIENT SPACES

Given two topological spaces X and Y and a surjective function $f : X \rightarrow Y$, we are familiar with the terms ‘continuous’, as well as ‘open’, ‘closed’ maps. We shall introduce a concept which is stronger than ‘continuity’ and weaker than ‘homeomorphism’, such onto functions will be called as ‘Quotient maps’. Formally we define

Definition: Let X and Y be two topological spaces and $p : X \rightarrow Y$ be a surjective map. The map p is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X .

Since the map $p : X \rightarrow Y$ is surjective for any $B \subset Y$, we have

$$p^{-1}(Y - B) = X - p^{-1}(B)$$

and consequently we get the equivalent form of the definition of a quotient map that requires ‘subset U of Y is closed in Y if and only if $p^{-1}(U)$ is closed in X ’.

Remark: It is clear from the definition that a quotient map is certainly continuous, and that the projections $p_i : X_1 \times X_2 \rightarrow X_i$ are quotient maps, where $X_1 \times X_2$ is the product space of the topological spaces X_1 and X_2 .

Remark: We shall consider a topological space X and a set Y . Suppose we are given a surjective map $p : X \rightarrow Y$. Then we can define a collection $\tau_Y = \{U \subset Y : p^{-1}(U) \text{ is open in } X\}$ which can be easily shown to define a topology on Y with respect to which $p : X \rightarrow Y$ becomes a quotient map. It is a straight forward to check the following:

Exercise1. If τ is any topology on Y such that $p : X \rightarrow Y$ is continuous with respect to τ , then $\tau \subset \tau_Y$.

Exercise2. The topology τ_Y is unique, that is if τ is a topology on Y such that $p : X \rightarrow Y$ is a quotient map with respect to τ , then $\tau = \tau_Y$.

Exercise3. If $f : X \rightarrow Y$ is an open(or closed) continuous surjective map, then f is a quotient map. However the converse is not true.

Propositon3 below gives the condition under which the converse is true.

Some properties of a quotient map: We summarize some important properties shared by a quotient map in the following propositions:

Proposition1: Let $f : X \rightarrow Y$ be a continuous map. If there exists a continuous function $g : Y \rightarrow X$ such that $f \circ g = i_Y$ is the identity map of Y then f is a quotient map.

Proposition2: Composition of two quotient maps is a quotient map.

Proposition3: Let $p : X \rightarrow Y$ be a quotient map. Then p is an open(respectively closed) map if and only if for each open(respectively closed) set $U \subset X$, the set $p^{-1}(p(U))$ is open(respectively closed) set in X .

Proposition4: Let $p : X \rightarrow Y$ be a quotient map and $f : Y \rightarrow Z$ be any function. Then the function f is continuous if and only if $f \circ p$ is continuous.

Remark: Given a topological space X and a set Y , we saw in Exercise2 above that a surjective map $p : X \rightarrow Y$ defines a unique topology τ_Y on Y with respect to which p is a quotient map. This topology τ_Y on Y defined by a surjective map p and is called the *Quotient topology* on Y and Y with this topology is called the *Quotient space*. In light of this definition we see that Proposition4 above gives a characterization for a continuous function on a quotient space.

Exercise set-1

1. Give examples for each of the following:
 - (a) A continuous map which is not a quotient map.
 - (b) A quotient map which is not open
 - (c) A quotient map which is not closed
 - (d) A quotient map which is neither closed nor open
 - (e) An open map which is not a homeomorphism
 - (f) A closed map which is not a homeomorphism

2. Show that a continuous map from a compact space onto a Hausdorff space is a quotient map. Show that the condition 'Hausdorff' can not be omitted.

3. Let $X = [0, 1]$ be subspace of R and $S^1 = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$ be the unit sphere as subspace of R^2 . Define $f : [0, 1] \rightarrow S^1$ by $f(t) = e^{2\pi ti}$. Show that f is a quotient map.

4. Let (X, d) and (Y, d') be two metric spaces and $f : X \rightarrow Y$ be a distance preserving function, that is, $d'(f(x), f(y)) = d(x, y)$ holds. Show that f is a quotient map.

5. Is a one to one quotient map a homeomorphism?

6. Show that the restriction of a quotient map to an open subset need not be a quotient map.

7. Let $\pi_1 : R^2 \rightarrow R$ be the projection on the first coordinate and Y be the subspace $(\{0\} \times R) \cup (R \times \{0\})$ of R^2 . Let p be the restriction of π_1 to Y . Show that p is a quotient map. Is p an open map? is it a closed map?

8. Let $\pi_1 : R^2 \rightarrow R$ be the projection on the first coordinate and $\beta = \{\pi_1^{-1}\{(a, b)\} : a, b \in R\}$ be the collection of subsets of R^2 . Show that β is a basis for a topology τ on R^2 . Compare τ with the standard topology on and denote $X = (R^2, \tau)$. Is $\pi_1 : X \rightarrow R$ a quotient map? Is X Hausdorff?

9. Show that the map $p : R^2 \rightarrow R$ defined by $p(x, y) = x^2 + y^2$ is a quotient map.

10. Let Y be the subspace of R^2 as given in exercise 7. Define $p : R^2 \rightarrow Y$ by

$$p(x, y) = \begin{cases} (x, 0), & x \neq 0 \\ (0, y), & x = 0 \end{cases}$$

Is p a quotient map? Is p continuous?

11. Is the quotient space of a locally compact space necessarily locally compact? What happens if 'locally compact' is replaced by 'compact' in this question?

Equivalence relation and quotient map: We will use the properties of a surjection $f : X \rightarrow Y$ to define an equivalence relation \sim on X as follows:

For $x, y \in X$, $x \sim y$ if and only if $f(x) = f(y)$.

It can be easily shown that \sim is an equivalence relation on X and we shall denote by $X^* = \{[x] : x \in X\}$ where $[x] = \{y \in X : y \sim x\}$ is the equivalence class of x . Moreover we have $Y = X^*$ in the sense that if $y = f(x)$ then $y = [x]$, and $f : X \rightarrow X^*$ satisfies $f(x) = [x]$ that is it becomes the natural projection corresponding to the equivalence relation induced by it on X . Conversely given an equivalence

relation \sim on a nonempty set X we have the set X^* of equivalence classes of X and the natural projection $p : X \rightarrow X^*$. Thus an equivalence relation \sim on a topological space X gives the quotient space X^* (the set of equivalence classes of X) and the natural projection $p : X \rightarrow X^*$ becomes the quotient map. Note that X^* is a partition of X and that any partition X^* of X also gives an equivalence relation on X .

Remark: Suppose that we have a partition X^* of a topological space X . Then we can describe the quotient topology on X^* , which is a collection of the subsets $U \subset X^*$ containing those equivalence classes such that the set $p^{-1}(U)$ is just the union of the equivalence classes belonging to U . Thus a typical open set in X^* is a collection of equivalence classes whose union is an open set in X .

Theorem: Let X, Y be topological spaces and $f : X \rightarrow Y$ be continuous surjection. Let X^* be the collection $\{f^{-1}(y) : y \in Y\}$ of subsets (fibers) of X . The following hold for the quotient space X^* :

- (a) If Y is Hausdorff, then so is X^*
- (b) The map $h : X^* \rightarrow Y$ defined by $h([x]) = f(x), x \in X$ is a homeomorphism if and only if f is a quotient map.

Examples: 1. Consider the rectangle $X = [0, 1] \times [0, 1]$ as a subspace of R^2 . Let X^* be the partition of X consisting of the one point sets $\{(x, y)\}, 0 < x < 1, 0 < y < 1$, the two point sets $\{(x, 0), (x, 1)\}, 0 < x < 1$, $\{(0, y), (1, y)\}, 0 < y < 1$ and the four point set $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. If we sketch the figure of the quotient space X^* we see that it is a torus, and describe the quotient topology by showing the typical sets $p^{-1}(U)$ for an open set $U \subset X^*$.

2. Let $X = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\}$ be the closed unit ball with subspace topology. Let X^* be a partition of X consisting of all one point sets $\{(x, y)\}, x^2 + y^2 < 1$ and the set $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. The quotient space X^* is homeomorphic to the unit 2-sphere $S^2 \subset R^3$. Describe a typical open set U on X^* and corresponding $p^{-1}(U)$.

3. Consider $X = [0, 1] \times [0, 1]$ as a subspace of R^2 and the partition X^* of X consisting of all singletons $\{(x, y)\}, 0 < x < 1, 0 < y < 1$, and the two point sets $\{(0, y), (1, 1 - y)\}$. The quotient space X^* is the Mobius band. Draw the figure and analyze the space.

4. Consider $X = [0, 1] \times [0, 1]$ as a subspace of R^2 and define an equivalence relation \sim on X by

(i) $(x, y) \sim (x, y)$, $x \in (0, 1), y \in (0, 1)$ (ii) $(x, 0) \sim (x, 1)$,
 $x \in [0, 1]$ (iii) $(0, y) \sim (1, 1-y)$, $y \in [0, 1]$. The resulting quotient space X^* is called the Klein bottle. Draw the figure, note that first two steps give a cylinder and then opposite circumferences of this cylinder are joined after giving a twist to this cylinder.

5. Consider the subspace $X = R^n - \{0\}$ of R^n and define an equivalence relation \sim on X by $x \sim y$ if and only if $x = ty$, $x, y \in X$, where t is a nonzero real number. The resulting quotient space is denoted by RP^n and is called the n -dimensional real projective space. If we replace R^n by C^n , where C is the set of complex numbers, the resulting quotient space is denoted by CP^n and is called the n -dimensional complex projective space.

Exercise set-2

1. Consider $S^1 = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$ as subspace of R^2 . Use above theorem to show that $f : R \rightarrow S^1$ defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$ is a quotient map.

2. Define a relation \sim on R^2 by requiring that any two points will be related under \sim if they have the same first coordinate. Show that \sim is an equivalence relation and the resulting quotient space is homeomorphic to R .

3. Define a relation \sim on $X = R$ by requiring that $x \sim y$ if $x-y =$ an integer. Show that the quotient space X^* is homeomorphic to S^1 .

4. Define a relation \sim on $X = R^2$ by requiring that $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + y_1^2 = x_2 + y_2^2$. Show that \sim is an equivalence relation and describe the quotient space X^* by showing that it is homeomorphic to a well known space. What is it?

5. Let $A = (0, 1)$ be the open interval in $X = R$ and X^* be the partition of X consisting of all singletons $\{x\}, x \in R - (0, 1)$ and the set A . Describe the quotient topology on X^* . Show that any open set in X^* containing either 0 or 1 must contain the set A . Is the set $\{A\}$ open in X^* ? Is X^* Hausdorff?

6. Let $X = [0, 1]$ be the subspace of R and define \sim on X by $x \sim y$ if $x - y =$ a rational number. Show that the quotient space X^* is not Hausdorff.

7. Let $p : X \rightarrow Y$ be a quotient map. Let Z be a topological space and $h : X \rightarrow Z$ be a continuous map that is constant on each fiber $p^{-1}(\{y\})$, $y \in Y$. Then show that h induces a continuous map $f : Y \rightarrow Z$ such that $f \circ p = h$.

8. Let X^* be the partition of $X = \mathbb{R}^2$ consisting of all concentric circles. Show that the quotient space X^* is homeomorphic to the subspace $[0, \infty)$ of \mathbb{R} .

9. Let $X = \bigcup_{n=0}^{\infty} (R \times \{n\})$ be the subspace of \mathbb{R}^2 where n is a positive integer and L'_n be the straight line in \mathbb{R}^2 passing through origin and having slope n . Let $Y = \bigcup_{n=0}^{\infty} L'_n$ be a subspace of \mathbb{R}^2 . Consider the map $f : X \rightarrow Y$ defined by $f(x, n) = (x, nx)$. Show that f maps a line $L_n = R \times \{n\}$ on to the line L'_n and is a continuous surjection. Show that f is not a quotient map by showing the quotient topology on Y is not equal to its subspace topology. (Compare with exercise-6 in Exercise set-1)

10. Let \sim be an equivalence relation on the topological space X . If the quotient space X^* and each equivalence class $[x], x \in X$ are connected then show that X is connected.

11. Show that a quotient space X^* of a locally connected space X is locally connected.

Hausdorff criterion: In exercise-6 above, we have seen that though X is Hausdorff the quotient space X^* is not Hausdorff. This leads to the important question of finding the criterion for the quotient space of a Hausdorff space to be Hausdorff. For this first we introduce some concepts.

Definition: An equivalence relation \sim on a topological space X is said to be open if whenever a subset $A \subset X$ is open then the set $[A] = \bigcup_{a \in A} [a]$ is open in the space X . Similarly we define a closed relation.

Lemma: An equivalence relation \sim on a topological space X is open(closed) if and only if $p : X \rightarrow X^*$ is an open(closed) mapping.

Proof: Suppose that p is an open mapping and $A \subset X$ be an open subset. Then as $[A] = p^{-1}(p(A))$, and $p(A)$ being an open subset of X^* , we see that $[A]$ is an open subset of X . Thus the relation \sim is open. Conversely, suppose that the relation \sim is an open relation and A be an open subset of X . Then as $[A]$ is open and $[A] = p^{-1}(p(A))$, and p is a quotient map we get that $p(A)$ is open proving that p is an open mapping. The proof for closed relation is similar.

Theorem: Let \sim be an open equivalence relation on a topological space X . Then $R = \{(x, y) : x \sim y\}$ is a closed subspace of $X \times X$ if and only if the quotient space X^* is Hausdorff.

Proof: Suppose the quotient space X^* is Hausdorff and that $x, y \in X$ are such that x is not related to y that is $(x, y) \notin R$. Then as $p(x) \neq p(y)$, where $p : X \rightarrow X^*$ is the projection, there exist disjoint open sets $U, V \subset X^*$ containing $p(x)$ and $p(y)$ respectively. Let $U^* = p^{-1}(U)$, and $V^* = p^{-1}(V)$. Clearly $x \in U^*, y \in V^*$. If the open set $U^* \times V^*$ containing the point (x, y) intersects R , then it will contain a point (a, b) of R which satisfies $a \sim b \Rightarrow p(a) = p(b)$, which will imply that $p(a) = p(b) \in U \cap V$ contrary to our assumption. Hence $U^* \times V^*$ does not intersect R . Thus $X \times X - R$ is open and consequently the set R is closed in $X \times X$.

Conversely suppose that R is closed in $X \times X$. Then given any distinct pair of points $p(x), p(y) \in X^*$, there is an open set of the form $U^* \times V^*$ containing the point (x, y) and having no point from R (as $p(x) \neq p(y)$, that is $(x, y) \notin R$ and $X \times X - R$ is open). Put $U = p(U^*), V = p(V^*)$, which by hypothesis together with above lemma imply that they are open subsets of X^* containing $p(x)$ and $p(y)$ respectively. Suppose $[a] \in U \cap V$, which will imply the existence of $b \in U^*, c \in V^*$ such that $p(b) = p(c) = [a] \Rightarrow (b, c) \in R$. This contradicts the choice of $U^* \times V^*$. Hence $U \cap V$ is empty and this proves that X^* is Hausdorff.

Remark: Note that the quotient space X^* could be Hausdorff space without X being Hausdorff. For example $X = (R^2, \tau)$ in exercise-8 in Exercise set-1 is not Hausdorff yet the quotient space X^* given by the quotient map $\pi_1 : X \rightarrow R$ is homeomorphic to R and is therefore Hausdorff. In above Theorem we don't require X to be Hausdorff. The condition on the relation in above theorem can not be replaced by the closed relation see exercise-1 below.

Exercise set-3

1. Consider the projection $\pi_1 : R^2 \rightarrow R$ on the first copy, and the closed set $F = \{(x, y) \in R^2 : xy = 1\}$ in R^2 . Show that the union $\bigcup_{x \in R} \pi_1^{-1}(x)$, such that $\pi_1^{-1}(x) \cap F$ is nonempty, is not closed in R^2 . (Note that $\pi_1^{-1}(x)$ are equivalence classes) This shows that the relation induced by the function π_1 on R^2 is not closed. (Compare with exercise-8 in Exercise set-1, and above theorem)

2. Show that the n -dimensional real projective space RP^n and the n -dimensional complex projective space CP^n are Hausdorff spaces.