FINAL EXAMINATION
M-570, II-SEMESTER (1423-1424H)
DEPARTMENT OF MATHEMATICS
Womens Section
KING SAUD UNIVERSITY

Time: 3 hours — Max. Marks: 50

Note: *Attempt all questions*

Q.1 (a) Let X be a topological space in which any two point set $\{x, y\} \subset X$ is connected. Show that X is connected.

(b) (i) Give an example of two connected subsets A, B of a topological space X where $A \cap B$ is nonempty and not connected.

(ii) Let $X = R^2$, $C = B_\epsilon(0), 0 < \epsilon < \frac{1}{2}$ and $Y = ([-1,1] \times [-1,1]) \setminus C$. Is Y connected? Is $X \setminus Y$ connected? What are the components of $X \setminus Y$? Is $X \setminus C$ connected?

(c) Is a path connected space locally path connected? Explain your answer either by giving a proof or by giving an example.

Q.2 (a) Show that S^1 is not homeomorphic to S^2.

(b) Consider the map $f : R \to Q$ defined by

$$f(x) = \begin{cases} x & x \in Q \\ 0 & x \in R \setminus Q \end{cases}$$

Give the topology to Q so that f becomes a quotient map. Is this quotient space Q a Hausdorff space?

(c) Explain what do you mean by a real projective space RP^n and show that the quotient topology on RP^n is Hausdorff.

Q.3 (a) Let $f : R^4 \to R$ be $f(X) = \det X$. Show that f is differentiable at any point $A \in R^4$ and verify that

$$(D_P f)(X) = \det \begin{bmatrix} a_1 & a_2 \\ x_3 & x_4 \end{bmatrix} + \det \begin{bmatrix} x_1 & x_2 \\ a_3 & a_4 \end{bmatrix}$$

where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$.

(b) Let $g : R \to R$ be $g(x) = e^x$ and $T \in L(R^n, R)$. Show that $f : R^n \to R$ defined by $f = g \circ T$ is differentiable at any point $P = (a_1, .., a_n)$ and find $(D_P f)((X)$ for $X \in R^n$.

(c) Consider the function $f : R^2 \to R^2$, $f(x, y) = \left(\frac{1}{2}(x^2 - y^2, xy\right)$. Is this function continuously differentiable on R^2? Show that for any point

$p \in R^2 \setminus \{0\}$, there exists a neighbourhood U of p such that $f : U \to f(U)$ is a bijection. Is f^{-1} differentiable on U? If it is differentiable find $D_p f^{-1}$ for $p = (1, 1)$.

Q.4 (a) Show that

$$SL(2, R) = \{A \in R^4 : \det A = 1\}$$

is a 3-dimensional smooth manifold.

(b) For a smooth manifold M, explain what do you mean by the ring $C^\infty(p)$ for $p \in M$ and show that for a $f \in C^\infty(p)$,

$$f = f(p) + \sum_{i=1}^{n} (x^i - x^i(p)) g \circ \varphi,$$

where (U, φ) is a chart around p with local coordinates $x^1, .., x^n$ and $g : \varphi(U) \to R$ is a smooth function.

(c) Show that for a diffeomorphism $f : M \to N$, $df_p : T_pM \to T_{f(p)}N$ is an isomorphism for any $p \in M$. Is the converse true? State what happens if $df_p : T_pM \to T_{f(p)}N$ is an isomorphism for a $p \in M$ and $f : M \to N$ is a smooth map.