COMPACT SPACES

Recall that a topological space X is said to compact if every open cover of it has a finite subcover and a subset of X is compact if it is compact with respect to the subspace topology on it. Those bubsets of R^n are compact which are closed and bounded. Here we shall be interested in local compactness and one point compactification.

Definition: A topological space X is said to be locally compact at $x \in X$ if there is some compact subset C of X that contains a neighbourhood of x. If X is locally compact at each of its points, then X is said to be locally compact space.

Remark: From the definition it follows that a compact space is locally compact. The Real line R is also locally compact. Indeed R^n is locally compact. As we will see below that the subspace Q of R is not locally compact.

We have the following properties of the locally compact spaces:

Proposition 1. Let X be a Hausdorff space. Then X is locally compact at x if and only if for every neighbourhood U of x, there is a neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

The subspace of a locally compact space need not be locally compact as suggested by the subspace Q of R. However under some suitable conditions we have

Proposition 2. Let X be a locally compact Hausdorff space and Y be a subspace of X. If Y is closed in X or open in X, then is locally compact.

For proof see Munkres p.185-186.

Exercise set-1

1) Let X be a Hausdorff space and D be a dense locally compact subspace of X. Show that D is open.

2) Show that the set of rationals Q as subspace of R is not locally compact.

3) Show that the continuous image of a locally compact space need not be locally compact. What happens if the map is continuous open?

4) Let X be a locally compact space and $f : X \to Y$ be a closed continuous surjective map such that each fiber is compact subset of X. Then show that Y is locally compact.

5) Prove that in a locally compact Hausdorff space, the intersection of an open set with a closed set is locally compact.

6) Let Y be a locally compact subspace of a Hausdorff space X. Show that there exist an open set G and a closed set F in X such that $Y = F \cap G$.

7) Is R_f (R with cofinite topology) locally compact?

8) Show that R_l (R with lower limit topology) is not locally compact?

One point compactification.: The locally compact spaces which we have studied in previous section are important in the study of what we call as compactification of a topological space. Since compactness is a very important property, we shall be looking at the question: Given a non-compact topological space can we find a compact topological space such that our topological space could sit in it as a subspace or equivalently homeomorphic to its subspace? This process is called the compactification of the given non-compact space. Simplest among compactifications is the one-point compactification.

Definition: Let X be a locally compact Hausdorff space. Take some point say ∞ out side X and form the set $Y = X \cup \{\infty\}$ Topologize Y by defining the collection of open sets in Y to be the all sets of the following types:

(i) U, where U is open in X.

(ii) $Y - C$, where C is a compact subset of X.

The topological space Y is called the one-point compactification of X.

Remark: Note that it is not difficult to see that the above mentioned collection of subsets of Y is indeed topology on Y. The empty set is of type (i), where as Y is in type (ii). We have to verify several cases for checking that the intersection of finite number of open sets is open and that the union of arbitrary number of open sets is open. Now we state the following important property of the one point compactification of a locally compact Hausdorff space.

Theorem: Let X be a locally compact Hausdorff space which is not compact. Let Y be a one-point compactification of X. Then Y is a compact Hausdorff space; X is a subspace of Y; and $\overline{X} = Y$.

For proof see Munkres p.184.

Examples: **1)** Let $X = (0, 1]$, $Y = [0, 1]$ be the subspaces of R. Then Y is the one-point compactification of X with the point ∞ is taken as 0. Note that for an open set G in the one-point compactification Y, if $0 \in G$, then $Y - G$ is a compact subset of X and if $0 \notin G$, then G is an open subset of X. Conversely if U is an open subset of X, then U is an open subset of Y, and if C is a compact subset of X, then $Y - C$ is an open subset of Y.

2) For the subspace $X = (0, 1)$ of R, it can be shown that the circle $S^1 = \left\{x \in R^2 : \|x\|^2 = 1\right\}$ is the one-point compactification of X. For the removal of one point from S^1 leaves a homeomorphic copy of X.

3) We know that the set N of natural numbers is non-compact subspace of R, and it is locally compact Hausdorff space. It can be shown that the subspace $M = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ of R is one-point compactification of N.

4) Consider the space R, we shall show that its one-point compactification is S^1. Define a function $f : R \to S^1$ by $f(t) = \left(\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2}\right)$, which is a one-to-one function and that $f(R) = S^1 - \{(0, 1)\}$. Let $Y = R \cup \{\infty\}$ be the one-point compactification of R. Now we extend $f : Y \to S^1$ by defining $f(\infty) = (0, 1)$. Note that f is continuous on R and the extension to Y can be shown to be continuous using the topology of Y. Moreover f is a bijection from a compact space Y to the Hausdorff space S^1 which is continuous, it must be a homeomorphism. Hence the one-point compactification of R is S^1.

Exercise set-2

1) Let $f : X \to Y$ be a bijection of the Hausdorff space X on to the compact space Y. Suppose $x \in X$, $Z = Y - \{f(x)\}$ and that $f^{-1} : Z \to X$ is continuous. Then show that f is continuous at x.

2) Show that the result in above exercise is false if the assumption that Y is compact is dropped.

3) Let Y be a compact Hausdorff space and $x \in Y$ be a nonisolated point of Y. Then show that Y is one-point compactification of $X = Y - \{x\}$, with $x = \infty$.

4) Give an example of a subspace X of a topological space Y such that the one-point compactification of X is not the subspace of the one-point compactification of Y.

5) Show that the one-point compactification of R^n is the n-sphere S^n.

6) Let Y be one-point compactification of X. Show that Y is connected if and only if X has no compact open component. Use this to show that the one-point compactification of Q is connected.

7) Let X be locally connected and Y be one-point compactification of X. Show that Y is connected if and only X has no compact components.

8) Show that above theorem characterizes one-point compactification up to a homeomorphism.