4) Consider the subsets $U_i = \{(x_1, x_2, x_3) \in S^2 : x_i > 0\}$ and $V_i = \{(x_1, x_2, x_3) \in S^2 : x_i < 0\}$ and the maps $\varphi_i : U_i \to B_1(0)$, $\psi_i : V_i \to B_1(0)$, $i = 1, 2, 3$ defined by

$$\varphi_i(x) = \widehat{x},\ x \in U_i \text{ and } \psi_i(x) = \widehat{x},\ x \in V_i$$

where $B_1(0) \subset R^2$ is the open ball of radius 1 with center origin and $\widehat{x}$ mean ith coordinate is removed from x to get an element of R^2 (for example $\varphi_1(x_1, x_2, x_3) = (x_2, x_3)$. Show that each pair (U_i, φ_i), (V_i, ψ_i), $i = 1, 2, 3$ is a chart on S^2 and the collection

$$\{(U_i, \varphi_i), (V_i, \psi_i),\ i = 1, 2, 3\}$$

gives a differentiable structure S on S^2.

Solution: We note that the sets U_i and V_i, $i = 1, 2, 3$ are open subsets of S^2. For example consider $V_2 = \{(x_1, x_2, x_3) \in S^2 : x_2 < 0\}$, then we have the continuous map $\pi^2 : R^3 \to R$ and $V_2 = S^2 \cap \left(\pi^2\right)^{-1}(-\infty, 0)$. Since $(-\infty, 0)$ is open in R and $\left(\pi^2\right)^{-1}(-\infty, 0)$ is open in R^3, we get V_2 is open in S^2. Now we shall show that the map $\psi_2 : V_2 \to B_1(0)$, defined by

$$\psi_2(x_1, x_2, x_3) = (x_1, x_3)$$

is a homeomorphism. Note that components being continuous functions, ψ_2 is a continous function.

ψ_2 **is one-one**: Suppose $\psi_2(x, y, z) = \psi_2(a, b, c)$, where $(x, y, z), (a, b, c) \in V_2$ that is $y < 0$ and $b < 0$ and $x^2 + y^2 + z^2 = 1, a^2 + b^2 + c^2 = 1$. Thus we have by definition of ψ_2 that $(x, z) = (a, c)$ that is $x = a$ and $z = c$ then we have $y^2 = 1 - x^2 - z^2 = 1 - a^2 - c^2 = d^2$ that is $y = \pm b$, but both y and b are negative hence $y = b$. This proves that ψ_2 is one-one.

ψ_2 **is on-to:** Note that for $(u, v) \in B_1(0)$ as $u^2 + u^2 < 1$, we see that $(u, -\sqrt{1 - u^2 - v^2}, v) \in V_2$ and that

$$\psi_2(u, -\sqrt{1 - u^2 - v^2}, v) = (u, v)$$

which proves that ψ_2 is on-to and that

$$(\psi_2)^{-1}(u,v) = (u, -\sqrt{1-u^2-v^2}, v)$$

which is also continuous. Hence ψ_2 is a homeomorphism and (V_2, ψ_2) is a chart on S^2. In the similar way we prove that all (U_i, φ_i), (V_i, ψ_i), $i = 1, 2, 3$ are charts on S^2.

Now we prove that the collection $\{(U_i, \varphi_i), (V_i, \psi_i),\ i = 1, 2, 3\}$ gives a differentiable structure on S^2. First it is clear that

$$S^2 = \left(\underset{i=1}{\overset{3}{U}} U_i\right) \cup \left(\underset{i=1}{\overset{3}{U}} V_i\right)$$

Now clearly $U_2 \cap V_1 \neq \phi$ (because $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ belongs to it. Then we have to show that the maps

$$\varphi_2 \circ \psi_1^{-1} : \psi_1 : \psi_1(U_2 \cap V_1) \to \varphi_2(U_2 \cap V_1)$$

and

$$\psi_1 \circ \varphi_2^{-1} : \varphi_2(U_2 \cap V_1) \to \psi_1(U_2 \cap V_1)$$

are smooth maps. Note that

$$\varphi_2 \circ \psi_1^{-1}(u,v) = \varphi_2((-\sqrt{1-u^2-v^2}, u, v)) = (-\sqrt{1-u^2-v^2}, v)$$

which is a smooth map. Similarly

$$\psi_1 \circ \varphi_2^{-1}(u,v) = \psi_1(u, \sqrt{1-u^2-v^2}, v) = (\sqrt{1-u^2-v^2}, v)$$

which is also smooth. Similary we can show that when ever any two of neighbourhood intersect corresponding transition maps are smooth. Hence the collection $\{(U_i, \varphi_i), (V_i, \psi_i),\ i = 1, 2, 3\}$ gives a differentiable structure S on S^2.

3) Let $f, g : R^n \to R$ be two functions differentiable at $p \in R^n$. Then show that $f+g, f.g : R^n \to R$ defined by $(f+g)(x) = f(x) + g(x)$ and $(f.g)(x) = f(x)g(x)$, $x \in R^n$ are differentiable at p and show that

$$D_p(f+g) = D_p f + D_p g$$

and

$$D_p f.g = g(p)D_p f + f(p)D_p g$$

Solution:

1) Given that $f : R^n \to R$ and $g : R^n \to R$ are differentiable at $p \in R^n$ we have to show that $f + g : R^n \to R$ defined by $(f + g)(x) = f(x) + g(x)$ is differentiable at p. Since f, g are differentiable there exist $T_1, T_2 \in L(R^n, R)$ such that

$$\lim_{X\to 0} \frac{|f(p + X) - f(p) - T_1(X)|}{\|X\|} = 0, \quad \lim_{X\to 0} \frac{|g(p + X) - g(p) - T_2(X)|}{\|X\|} = 0$$

Define $T = T_1 + T_2$, then it is easy to show that $T(\lambda X + \mu Y) = \lambda T(X) + \mu T(Y)$, which proves $T \in L(R^n, R)$ and we have

$$\begin{aligned}
& \lim_{X\to 0} \frac{|(f + g)(p + X) - (f + g)(p) - T(X)|}{\|X\|} \\
= & \lim_{X\to 0} \frac{|f(p + X) - f(p) - T_1(X) + g(p + X) - g(p) - T_2(X)|}{\|X\|} \\
\leq & \lim_{X\to 0} \frac{|f(p + X) - f(p) - T_1(X)|}{\|X\|} + \frac{|g(p + X) - g(p) - T_2(X)|}{\|X\|} \\
= & \ 0
\end{aligned}$$

Hence $f + g$ is differentiable at p and

$$D_p(f + g) = T = T_1 + T_2 = D_p f + D_p g$$

2) Now we shall show that fg is differentiable. Note that if $D_p f = T_1$ and $D_p g = T_2$. Then we define $T = f(p)T_2 + g(p)T_1$, where $f(p), g(p) \in R$, it is easy to see that

$$\begin{aligned}
T(\lambda X + \mu Y) &= (f(p)T_2 + g(p)T_1)(\lambda X + \mu Y) \\
&= f(p)T_2(\lambda X + \mu Y) + g(p)T_1(\lambda X + \mu Y) \\
&= f(p)T_2(\lambda X) + f(p)T_2(\mu Y) + g(p)T_1(\lambda X) + g(p)T_2(\mu Y) \\
&= \lambda(f(p)T_2 + g(p)T_1)(X) + \mu(f(p)T_2 + g(p)T_1)(Y) \\
&= \lambda T(X) + \mu T(Y)
\end{aligned}$$

that is $T \in L(R^n, R)$ and we have

$$\begin{aligned}
& f(X+p)g(X+p) - f(p)g(p) - T(X) \\
= \;& f(X+p)g(X+p) - f(p)g(p) - f(p)T_2(X) - g(p)T_1(X) \\
= \;& f(X+p)g(X+p) - f(X+p)g(p) - f(X+p)T_2(X) + f(X+p)g(p) \\
& + f(X+p)T_2(X) - f(p)g(p) - f(p)T_2(X) - g(p)T_1(X) \\
= \;& f(X+p)\left(g(X+p) - g(p) - T_2(X)\right) + g(p)\left(f(X+p) - f(p) - T_1(X)\right) \\
& + f(X+p)T_2(X) - f(p)T_2(X)
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \lim_{X \to 0} \frac{|f(X+p)g(X+p) - f(p)g(p) - T(X)|}{\|X\|} \\
\leq \;& \lim_{X \to 0} \frac{|(f(X+p)\left(g(p+X) - g(p) - T_2(X)\right)|}{\|X\|} \\
& + \lim_{X \to 0} \frac{|(g(p)\left(f(p+X) - f(p) - T_1(X)\right)|}{\|X\|} \\
& + \lim_{X \to 0} \frac{|f(p+X)T_2(X) - f(p)T_2(X)|}{\|X\|} \qquad \text{(A)}
\end{aligned}$$

Note that first two limits are zero and for the last limit we use $\|T_2(X)\| \leq M\|X\|$, which gives

$$\lim_{X \to 0} \frac{|f(p+X)T_2(X) - f(p)T_2(X)|}{\|X\|} \leq \lim_{X \to 0} M\,|f(p+X) - f(p)| = 0$$

Thus the equation (A) gives

$$\lim_{X \to 0} \frac{|f(X+p)g(X+p) - f(p)g(p) - T(X)|}{\|X\|} \leq 0$$

that is

$$\lim_{X \to 0} \frac{|f(X+p)g(X+p) - f(p)g(p) - T(X)|}{\|X\|} = 0$$

Hence fg is differentiable at p and

$$D_p fg = f(p)T_2 + g(p)T_1 = f(p)D_p g + g(p)D_p f$$