M-570, II-MIDTERM EXAMINATION

I-Semester, 1428-1429

Department of Mathematics (Women's Section)

College of Science, King Saud University

Time 2 Hours **Max. Marks-20**

Note: *Attempt all questions*

Q.1. (a) Let $f : X \to Y$ be a quotient map. Define a relation $\backsim$ on X by $x_1 \backsim x_2$ if $f(x_1) = f(x_2)$. Show that $\backsim$ is an equivalence relation and that the set $X^* = \{[x] : x \in X\}$ can be given a topology so that X^* is homeomorphic to Y.

(b) Give an example of a quotient map $f : X \to Y$ such that X is Hausdorff and Y is not Hausdorff, explain your answer in detail.

Q.2. Let $f : R^2 \to R$ be $f(x,y) = y^2 - x$. Show that this map is onto. Assume that f is a quotient map and define the relation $\backsim$ on R^2 as in Q.1(a). What is $[(1,1)]$? Let Z be the set of all equivalence classes under this relation. Find the map $\varphi : Z \to R$ that gives the hoemorphism of Q.1(a).

Q.3. Show that the real projective space RP^2 is a compact connected Hausdorff space.

Q.4. Let $f, g : R^n \to R^m$ be differentiable functions at $p \in R^n$. Define $f + g : R^n \to R^m$ by

$$(f+g)(X) = f(X) + g(X)$$

and show that $f + g$ is also differntiable at p, find $(D_p(f+g))(X)$. Use this to show that the function $h : R^4 \to R^4$, $h(X) = X + X^2$ is differentiable at any $A \in R^4$, where R^4 is treated as set of 2×2 real matrices. Find the kernel of the linear map $D_A h$ when $A = I$ the identity matrix.

Q.5. (a) Let $U = R^2 - \{(0,0)\}$ and $f : U \to R^2$ be

$$f(x,y) = \left(\frac{1}{2}\left(x^2 - y^2\right), xy\right)$$

Is f a diffeomorphism? Show that at each point $p \in U$, there is a neighbourhood V of p in U and a neighbourhood W of $f(p)$ such that $f : V \to W$ is a diffeomorphism.

(b) Show that the map $f : R \to R^{n^2}$ defined by $f(t) = e^{tA}$, where $A \in R^{n^2}$ is a fixed element, is a diffentiable map at $t = 0$ and find $(D_0 f)(t)$.